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SOME PROPERTIES OF SIMPLE I-REGULAR SEMIGROUPS

by

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Let S be a regular semigroup and let E_S denote its set of idempotents. As usual, E_S is partially ordered in the following manner: if $e, f \in E_S$, $e \leqq f$ if and only if $ef = fe = e$. We then say that E_S is under or assumes its natural order. Let I denote the integers. If E_S, under the natural order, is order isomorphic to I under the reverse of the usual order, we call S an I-regular semigroup. We determined the structure of I-regular semigroups mod groups in [10].

In section 1, we develop the ideal extension theory of simple I-regular semigroups. In section 2, we obtain the maximal group homomorphic image of a simple I-regular semigroup including the defining homomorphism. In section 3, we determine the nature of the congruences admitted by a simple I-regular semigroup, and we describe the idempotents separating congruences.

In the special case S is bisimple, the results of this paper reduce to the corresponding results for I-bisimple semigroups (bisimple semigroups S such that E_S is order isomorphic to I under the reverse of the usual order) [6, 7].

Unless otherwise specified, we utilize the definitions, terminology, and notation of [1].

1. Ideal extension theory

In this section, we determine the translational hull $\bar{S}$ of a simple I-regular semigroup S. All ideal extensions of S by a semigroup T with zero, o, can then be described if one knows the structure of T and the partial homomorphisms θ of $T^* = T\setminus 0$ into $\bar{S}$ such that $AB = 0$ in T implies that $A\theta B\theta \in S$ [1]. This determination is carried out if T is a completely 0-simple (Brandt) semigroup. We also completely determine the extensions of a Brandt semigroup with finite index set by a simple I-regular semigroup (with zero appended) by specializing our general determination of the extensions of a Brandt semigroup by an arbitrary semigroup [5, theorem 1].

Before commencing, let us state the structure theorem for simple I-regular semigroups.

Let $C_1^* = IxI$ under the multiplication $(a, b)(c, d) = (a+c-\min(b, c), b+d-\min(b, c))$. We called C_1^* the extended bicyclic semigroup in [6].

THEOREM 1.1 (Warne, [10]). *S is a simple I-regular semigroup if and only if* $S = (U(G_j : j = 0, 1, \cdots, d-1)) \times C_1^*$, *where d is a positive integer,* $\{G_j : 0 \leqq j \leqq d-1\}$ *is a collection of pairwise disjoint groups, and* C_1^* *is the extended bicyclic semigroup, under the multiplication*

$$(g_s, (m, n))(h_r, (p, q)) = (t, (m, n)(p, q)) \qquad (*)$$

where

$$g_s \in G_s, g_r \in G_r \quad (0 \leqq r,\ s \leqq d-1) \quad \text{and} \quad t =$$

$$g_s(f_{n-p,p}^{-1} \prod_{j=0}^{s-1} \gamma_j)(h_r \prod_{j=pd+r}^{nd+s-1} \gamma_j)(f_{n-p,q} \prod_{j=0}^{s-1} \gamma_j),$$

$$(f_{p-n,m}^{-1} \prod_{j=0}^{r-1} \gamma_j)(g_s \prod_{j=nd+s}^{pd+r-1} \gamma_j)(f_{p-n,n} \prod_{j=0}^{r-1} \gamma_j)h_r, \qquad \text{or}$$

$$(g_s \prod_{j=s}^{v-1} \gamma_j)(h_r \prod_{j=r}^{v-1} \gamma_j) \quad (v = \max(r, s))$$

according to whether $n > p$, $p > n$, *or* $p = n$ *where* $\gamma_j = \gamma_{j(\text{mod } d)} (j \in I, j \geqq 0)$ *is a homomorphism of* $G_{j(\text{mod } d)}$ *into* $G_{(j+1)\text{mod } d}$. *Juxtaposition denotes multiplication in* C_1^* *and in the appropriate* G_j. *For* $m \in I^0$, *the non-negative integers,* $n \in I, f_{0,n} = k_0$, *the identity of* G_0, *while, for* $m > 0$,

$$f_{m,n} = u_{(n+1)d}(\prod_{j=0}^{d-1} \gamma_j)^{m-1} u_{(n+2)d}(\prod_{j=0}^{d-1} \gamma_j)^{m-2} \cdots u_{(n+(m-1))d}(\prod_{j=0}^{d-1} \gamma_j) u_{(n+m)d}$$

where $\{u_{kd} : k \in I\}$ *is a collection of elements of* G_0 *with* $u_{kd} = k_0$ *for* $k > 0$. *In* $(*)$ $\prod_{j=a}^{a-1} \gamma_j$ *will denote the identity automorphism of* $G_{a(\text{mod } d)}$.

Let S be a simple I-regular semigroup. In connection with theorem 1.1, we write $S = (d, G_0, G_1, \cdots, G_{d-1}, C_1^*, \gamma_0, \gamma_1, \cdots, \gamma_{d-1}, u_{id})$.

For convenience, we write $\alpha_{m,n} = \gamma_m \gamma_{m+1} \cdots \gamma_{n-1}$ if $m < n$ and let $\alpha_{n,n}$ denote the identity automorphism of $G_{n(\text{mod } d)}$.

LEMMA 1.1. *A simple I-regular semigroup is left and right reductive.*

PROOF. This lemma is an immediate consequence of theorem 1.1. We will utilize the multiplication of theorem 1.1 without explicit mention.

THEOREM 1.2 *Let* $S = (d, G_0, G_1, \cdots, G_{d-1}, C_1^*, \gamma_0, \gamma_1, \cdots, \gamma_{d-1}, m_{id})$ *be a simple I-regular semigroup. Let* $W = \{(\theta, p) : \theta : I \to G_0, p \in I,$ *and* $(i+1)\theta = m_{(i+1)d}^{-1}(i\theta \prod_{j=0}^{d-1} \gamma_j)\, m_{(i+p+1)d}$ *for all* $i \in I\}$. *Let* ρ_i $(i \in I)$ *denote the inner right translation of* $(I, +)$ *determined by* $i \cdot W$, *under the multiplication*

$$^*(\theta, w)(\eta, p) = (\theta \circ \rho_w \eta, w+p),$$

where $\circ$ *denotes pointwise multiplication of mappings and juxtaposition denotes iteration of mappings is a group. Let* $\bar{S}$ *be the translational hull of* S. *Then,* $\bar{S} = W \cup S$ $(W \cap S = \square)$, *under the multiplication*

$$(\theta, a) \cdot (\eta, p) = (\theta, a)(\eta, p)$$

$$(g_s, a, b) \cdot (h_r, c, d) = (g_s, a, b)(h_r, c, d)$$

where juxtaposition denotes multiplication in W *and* S *and*

$$(\theta, p) \cdot (g_r, a, b) = ((a-p)\theta \prod_{j=0}^{r-1} \gamma_j g_r, a-p, b)$$

$$(g_r, a, b) \cdot (\theta, p) = (g_r(b\theta \prod_{j=0}^{r-1} \gamma_j), a, b+p).$$

PROOF. Let λ be a left translation of S. Then, if e_0 is the identity of G_0,

$$(e_0, i, i)\lambda = (i\delta, i\delta_1, i+i\rho_1)$$

where $\delta : I \to U(G_j : 0 \leqq j \leqq d-1)$; $\delta_1 : I \to I$; and $\rho_1 : I \to I^0$ the non-negative integers. Since $(e_0, i, i)(e_0, i+1, i+1) = (e_0, i+1, i+1)$, we have the following two possibilities: If $i\rho_1 = 0$,

$$\begin{aligned} (i+1)\delta &= m_{(i\delta_1+1)d}^{-1}(i\delta\alpha_{r,d})m_{(i+1)d} \quad \text{where} \quad i\delta \in G_r, \\ (i+1)\rho_1 &= 0, \quad \text{and} \\ (i+1)\delta_1 &= i\delta_1+1 \end{aligned} \tag{1.1}$$

while, if $i\rho_1 \geqq 1$,

$$\begin{aligned} (i+1)\delta &= i\delta \\ (i+1)\rho_1 &= i\rho_1 - 1 \\ (i+1)\delta_1 &= i\delta_1. \end{aligned} \tag{1.2}$$

Let us first consider the case $i\rho_1 = 0$ for all $i \in I$. In this case it is easily seen that $\lambda | D_0$, where $D_0 = \{(g_0, m, n) : g_0 \in G_0, m, n \in I\}$, is a left translation of D_0. Hence, since D_0 is the *I*-bisimple semigroup $(G_0, C_1^*, \alpha_{0,d}, m_{id})$ [6, theorem 1.2] (notation of [6]),

$$(e_0, i, i)\lambda = (i\delta, i+p, i)$$

where $p \in I$ and δ is a mapping of I into G_0 such that

$$(i+1)\delta = m_{(i+p+1)d}^{-1} i\delta\alpha_{0,d} m_{(i+1)d} \tag{1.3}$$

by virtue of [7, 8] or by [9, proof of theorem 1]. Hence, since $(g_r, i, j) = (e_0, i, i)(g_r, i, j)$,

$$(g_r, i, j)\lambda_{(\delta, p)} = ((i\delta)\alpha_{0,r} g_r, i+p, j) \tag{1.4}$$

where $\lambda = \lambda_{(\delta, p)}$, $p \in I$ and δ is a mapping of I into G_0 satisfying (1.3).

Conversely, (1.3) and (1.4) define a left translation of D_0 by [7] or by [9, proof of theorem 1]. By (1.3),

$$(g_r, a, b)\lambda_{(\delta, p)} = (e_0, a, a)\lambda_{(\delta, p)}(g_r, a, b).$$

Thus,

$$\begin{aligned}((g_r, a, b)(h_s, c, d))\lambda_{(\delta, p)} &= (e_0, a+c-\min(b, c), a+c-\min(b, c))\lambda_{(\delta, p)}(g_r, a, b)(h_s, c, d)\\ &= (e_0, a, a)\lambda_{(\delta, p)}(g_r, a, b)(h_s, c, d) = (g_r, a, b)\lambda_{(\delta, p)}(h_s, c, d).\end{aligned}$$

Hence, $\lambda_{(\delta, p)}$ is a left translation of S.

Next, suppose that there exists $u \in I$ such that $u\rho_1 \neq 0$. Utilizing (1.1) and (1.2), we obtain: $(t+i)\rho_1 = 0$, where t is a unique element in I, for $i \geqq 0$, and $(t+i)\rho_1 = -i$ for $i < 0$; $(t+i)\delta_1 = a+i$, where $a \in I$, for $i \geqq 0$, and $(t+i)\delta_1 = a$ for $i < 0$; and $(t+i)\delta = f_{i,a}^{-1} g_s \alpha_{s,d} \alpha_{0,d}^{i-1} f_{i,t}$ for $i > 0$, and $(t+i)\delta = g_s \in G_s$ for $i \leqq 0$. Since $(e_0, i, i)(e_0, i+n, i) = (e_0, i+n, i)$ for all $n \geqq 0$, we are able to determine $(e_0, i+n, i)\lambda$. Next, since $(g_r, i+n, i+m) = (e_0, i+n, i)(g_r, i, i+m)$ for $i \in I$, $m, n \in I^0$, we are able to determine $(g_r, i+n, i+m)\lambda$. By [3] and theorem 1.1, every element of S may be written in the form $(g_r, i+n, i+m)$ where $g_r \in G_r$, $i \in I$, and $m, n \in I^0$. We let $i = t+q$ and determine $(g_r, t+q+n, t+q+m)\lambda$ in terms of the values of δ, ρ_1, and δ_1 given above. In this calculation, we utilize the identity $f_{m+c,n} f_{c,m+n}^{-1} = f_{m,n} \alpha_{0,d}^c$ for $m, c \in I^0$ and $n \in I$ [10]. (This identity may be developed by a routine calculation.) Finally, if $a_1 = t+q+n$ and $b_1 = t+q+m$, we show that $(g_r, a_1, b_1)\lambda = (g_s, a, t)(g_r, a_1, b_1)$, i.e. λ is an inner left translation. (We omit the details of these calculations as they parallel calculations given in [7] and [9]).

In a similar manner, it may be shown that the semigroup of right translations of S consists of the inner right translations of S and the transformations of S defined by

$$(g_r, i, j)\rho_{(\theta, w)} = (g_r(j\theta\alpha_{0,r}), i, j+w) \tag{1.5}$$

where $w \in I$ and θ is a mapping of I into G_0 such that

$$(i+1)\theta = m_{(i+1)d}^{-1}(i\theta\alpha_{0,d})m_{(i+w+1)d} \quad \text{for all } i \in I. \tag{1.6}$$

It is easily seen that $\rho_{(\theta, w)}$ as defined by (1.5) and (1.6) is not an inner right translation of S, and $\lambda_{(\delta, p)}$ as defined by (1.4) and (1.3) is not an inner left translation of S. Hence, by lemma 1.1 and [7, lemma 1], $\lambda_{(\delta, p)}$ and $\rho_{(g_r, a, b)}$ are not linked and $\lambda_{(g_r, a, b)}$ and $\rho_{(\theta, w)}$ are not linked. Similarly, $\lambda_{(h_r, c, d)}$ and $\rho_{(g_s, a, b)}$ are linked if and only if $(h_r, c, d) = (g_s, a, b)$. Next, suppose that $\rho_{(\theta, w)}$ and $\lambda_{(\delta, p)}$ are linked. Then $\rho_{(\theta, w)}|D_0$ and $\lambda_{(\delta, p)}|D_0$

are linked. Thus, by the proof of [9, theorem 1] or [7], $w = -p$ and $\delta = \rho_{-w}\theta$. By the proof of [9, theorem 1] or [7], $\rho_{(\theta, w)}|D_0$ and $\lambda_{(p_{-w}\theta, -w)}|D_0$ are linked. Thus, $(g_s, a, b)\rho_{(\theta,w)}(h_r, c, d) = ((g_s, a, b)(e_0, b, b))\rho_{(\theta,w)}(h_r, c, d) = (g_s, a, b)((e_0, b, b)\rho_{(\theta,w)}(e_0, c, c))(h_r, c, d) = (g_s, a, b)((e_0, c, c)\lambda_{(\rho_{-w}\theta, -w)}(h_r, c, d)) = (g_s, a, b)((h_r, c, d)\lambda_{(\rho_{-w}\theta, -w)})$. Thus, $\rho_{(\theta, w)}$ and $\lambda_{(\rho_{-w}\theta, -w)}$ are linked. The mapping $\rho \to (\lambda, \rho)$, where ρ is a right translation of S and λ is the left translation of S linked with ρ, is an isomorphism of the semigroup of right translations of S onto $\bar{S}$. If $\rho_{(\theta, q)}, \rho_{(\eta, p)} \in \bar{S}\setminus S$, $\rho_{(\theta, q)}\rho_{(\eta, p)} = \rho_{(\theta \circ p_q \eta, q+p)}$ by (1.5) and (1.6). Hence $\bar{S}\setminus S$ is a semigroup. The mapping $(\theta, p) \to \rho_{(\theta, p)}$ is an isomorphism of W, under the multiplication $*$, onto $\bar{S}\setminus S$. Clearly, W is a group. The remainder of the theorem is a consequence of [1, p. 12, lemma 1.2], (1.4), and (1.5).

REMARK 1.1. In the case $d = 1$, we obtain [7, theorem 1] (see also [8]).

COROLLARY 1.1. *Let S be a weakly reductive semigroup and let $\bar{S}$ be its translational hull. Let T be a* 0*-simple semigroup having proper divisors of zero. If $S = \bar{S}$ or $\bar{S}\setminus S$ is a subsemigroup of S, then every extension of S by T is given by a partial homomorphism* [4].

PROOF. Replace $\mathfrak{D}$ by $\mathfrak{J}$ in the proof of [7, theorem 3].

REMARK 1.2. Let $S = (d, G_0, G_1, \cdots, G_{d-1}, \gamma_0, \gamma_1, \cdots, \gamma_{d-1}, m_{id})$ be a simple I-regular semigroup. S has d $\mathfrak{D}$-classes, $D_0, D_1, \cdots, D_{d-1}$. $D_r = \{(g_r, a, b) : g_r \in G_r : a, b \in I\}$ is the I-bisimple semigroup $(G_r, C_1^*, \alpha_{r, r+d}, m_{id}\alpha_{0,r})$. (Notation of [6]). Let T be a 0-bisimple semigroup. To determine the partial homomorphisms of $T\setminus 0$ into S one must just determine the partial homomorphisms of $T\setminus 0$ into D_r for each $r \in \{0, 1, 2, \cdots, d-1\}$. In the case T is a completely 0-simple semigroup, (a Brandt semigroup), these determinations are given mod groups by [7, theorem 2] ([7, corollary 1]). By lemma 1.1, theorem 1.2, and Corollary 1.1, if T is a 0-simple semigroup with proper divisors of zero, every extension of S by T is given by a partial homomorphism. In particular, this is valid if T is a completely 0-simple semigroup (Brandt Semigroup) with proper divisors of zero.

COROLLARY 1.2. *Let $S = (d, G_0, G_1, \cdots, G_{d-1}, C_1^*, \delta_0, \cdots, \delta_{d-1}, m_{id})$ be a simple I-regular semigroup and let $T = M^0(R; K; \Lambda; P)$ be a completely* 0*-simple semigroup (with zero,* 0'*) without proper divisors of zero. Let V be an extension of S by T. Then, either V is given by a partial homomorphism and an explicit multiplication is thus given by employing remark* 1.2. *(Conversely, every partial homomorphism of $T\setminus 0'$ into S determines an extension of S by T), or $V = (T\setminus 0') \cup S$ under the multiplication*

A) $(a; s, \lambda)^*(g_r, m, n)$

$$= ((m-k_s-i_a-t_\lambda)(\beta_s \circ \rho_{k_s}\theta_a \circ \rho_{k_s+i_a}\gamma_\lambda)\prod_{j=0}^{r-1}\delta_j g_r,\ m-k_s-i_a-t_\lambda,\ n)$$

B) $(g_r, m, n)^*(a; s, \lambda)$

$$= (g_r((n\beta_s \circ \rho_{k_s}\theta_a \circ \rho_{k_s+i_a}\gamma_\lambda)\prod_{j=0}^{r-1}\delta_j),\ m,\ k_s+i_a+t_\lambda+n)$$

where $(g_s, m, n) \in S$ *and* $(a; s, \lambda) \in T\setminus 0'$, $\circ$ *denotes pointwise multiplication of mappings,* $a \to i_a$ *is a homomorphism of* R *into* $(I, +)$, $a \to \theta_a$ *is a mapping of* R *into* $H = \{\beta : (\beta, a) \in W$ *for some* $a \in I\}$ *(see statement of theorem* 1.2*) such that* $\theta_{ab} = \theta_a \circ \rho_{i_a}\theta_b$ *for all* $a, b \in R$, $s \to \beta_s$ *is a mapping of* K *into* H, $s \to k_s$ *is a mapping of* K *into* I, $\lambda \to \gamma_\lambda$ *is a mapping of* Λ *into* H, *and* $\lambda \to t_\lambda$ *is a mapping of* Λ *into* I *such that* $i_{p_{\lambda s}} = t_\lambda + k_s$ *and* $\theta_{p_{\lambda s}} = \gamma_\lambda \circ \rho_{t_\lambda}\beta_s$. *Conversely,* (A) *and* (B) *define an extension of* S *by* T.

PROOF. The proof utilizes theorem 1.1, theorem 1.2, corollary 1.1, and [1, theorem 4.20 and theorem 4.22]. It is similar in nature to the proof of [7, theorem 4] (see also [8]) and [9, theorem 4] and it will be omitted.

REMARK 1.3. In the case $d = 1$, we obtain [7, theorem 4][see also [8]).

REMARK 1.4. In the special case that $T\setminus 0'$ is a group R, V is either given by a partial homomorphism or (A) and (B) become

$$a^*(g_r, m, n) = ((m-i_a)\theta_a \prod_{j=0}^{r-1}\gamma_j g_r,\ m-i_a,\ n)$$

$$(g_r, m, n)^* a = (g_r((n\theta_a)\prod_{j=0}^{r-1}\gamma_j),\ m,\ n+i_a).$$

REMARK 1.5. If T is a 0-simple semigroup without proper divisors of zero, an extension of S by T is either given by a partial homomorphism or by the equations in the above remark with $a \to \theta_a$ a mapping of $T\setminus 0'$ into H and with $a \to i_a$ a homomorphism of $T\setminus 0'$ into $(I, +)$.

We close this section by giving a specialization of [5, theorem 1]. The theorem is obtained by combining theorem 3.1 (below), [5, theorem 1], and [5, lemma 1]. The theorem is quite similar to [9, theorem 7].

In the theorem below, capital roman letters will denote elements of T^*.

THEOREM 1.3. *Let* $S = M^0(G; J; J; \Delta)$, *where* J *is a finite set, be a Brandt semigroup; let* $T^* = (d, U_0, U_1, \cdots, U_{d-1}, C_1^*, \gamma_0, \gamma_1, \cdots, \gamma_{d-1}, m_{id})$ *be a simple I-regular semigroup; and let* V *be an extension of* S *by* T. *Then, there exists a homomorphism* $w : A \to w_A$ *of* T^* *into* H_r, *the full symmetric group on some* r *element subset* Q *of* J. *This homomorphism is explicitly given by theorem* 2.4. *For each* $A \in T^*$, *there exists a mapping* ψ_A *of* Q *into the group* G *such that*

$$(i\psi_A)(iw_A\psi_B) = i\psi_{AB} \quad \textit{for all} \quad i \in Q.$$

The products in V are given by

$$A \circ B = AB \tag{1.7}$$

$$\begin{aligned}(a; i, j) \circ A &= (a(j\psi_A), i, jw_A) \quad \textit{if} \quad j \in Q \\ &= 0', \textit{ the zero of } S, \textit{ if } j \bar{\in} Q \\ 0' \circ A &= 0'\end{aligned} \tag{1.8}$$

$$\begin{aligned}A \circ (a; i, j) &= ((iw_A^{-1}\psi_A)a, iw_A^{-1}, j) \textit{ if } i \in Q \\ &= 0' \quad \textit{if} \quad i \bar{\in} Q \\ A \circ 0' &= 0'.\end{aligned} \tag{1.9}$$

Conversely, let S be a Brandt semigroup and let T^* be a simple I-regular semigroup. If we are given the mappings w and ψ_A described above and define product $\circ$ in the class sum of S and T^* by (1.7)–(1.9), *then V is an extension of S by T.*

2. The maximal group homomorphic image

The major purpose of this section is to determine the maximal group homomorphic image of a simple I-regular semigroup including the defining homomorphism.

To do this, we first determine the homomorphisms of a simple regular ω-semigroup (a simple regular semigroup S such that E_S is order isomorphic to I^0, the non-negative integers, under the reverse of the usual order) into a group (theorem 2.1). Utilizing this result and our determination of the maximal group homomorphic image of an ω-bisimple semigroup (a bisimple semigroup S such that E_S is order isomorphic to I^0 under the reverse of the usual order) [6, theorem 3.4], we determine the maximal group homomorphic image a simple regular ω-semigroup including the defining homomorphism (theorem 2.2). Finally, utilizing theorem 2.1, theorem 2.2, and 'an inverse limit process' and 'an inductive process' (introduced in [6]), we determine the maximal group homomorphic image of a simple I-regular semigroup. We also completely determine the homomorphisms of a simple I-regular semigroup into a group. This result was used in section 1.

The multiplication for a simple regular ω-semigroup S (due to Munn [2]) may be obtained from theorem 1.1 by considering the triples $\{(g_r, m, n) : g_r \in G_r \ (0 \leqq r \leqq d-1), m, n \in I^0\}$. Thus, we may write $S = (d, G_0, G_1, \cdots, G_{d-1}, C_1, \gamma_0, \gamma_1, \cdots, \gamma_{d-1})$ where C_1 is the bicyclic semigroup.

THEOREM 2.1. *Let* $S = (d, G_0, G_1, \cdots, G_{d-1}, C_1, \gamma_0, \gamma_1, \cdots, \gamma_{d-1})$ *be a simple regular* ω*-semigroup and let* H *be a group. For each* $i \in \{0, 1, \cdots, d-1\}$, *let* f_i *be a homomorphism of* G_i *into* H *and let* $z \in H$ *such that*

$$f_{d-1} C_z = \gamma_{d-1} f_0, \quad \textit{where } xC_z = zxz^{-1} \quad \textit{for} \quad x \in H, \tag{2.1}$$

and

$$f_r = \gamma_r f_{r+1} \quad \textit{for} \quad 0 \leqq r \leqq d-2. \tag{2.2}$$

Then,

$$(g_r, m, n)\phi = z^{-m}(g_r f_r)z^n \tag{2.3}$$

is a homomorphism of S *into* H *and, conversely every such homomorphism is obtained in this fashion.*

PROOF. Let ϕ be a homomorphism of S into H. Define $(g_r, 0, 0)\phi = g_r f_r$. Clearly, f_r is a homomorphism of G_r into H. Let $(e_0, 0, 1)\phi = z$, where e_0 is the identity of G_0. Hence $(g_r, m, n)\phi = z^{-m} g_r f_r z^n$ and (2.3) is valid. Since $(g_{d-1}\gamma_{d-1}, 0, 0)(e_0, 0, 1) = (e_0, 0, 1)(g_{d-1}, 0, 0)$, (2.1) is valid. Since, for $0 \leqq r \leqq d-2$, $(g_r, 0, 0)(e_{r+1}, 0, 0) = (g_r \gamma_r, 0, 0)(e_{r+1}, 0, 0)$, (2.2) is valid.

Conversely, let us show that (2.3) subject to the conditions (2.1) and (2.2) defines a homomorphism of S into H. Clearly, ϕ is a well defined mapping of S into H. Form (2.1) and (2.2), we obtain

$$\alpha_{j,d} f_0 = f_j C_z \tag{2.4}$$

By induction, we obtain

$$t^r b_j f_j = b_j \alpha_{j,rd} f_0 z^r \tag{2.5}$$

for each positive integer r and each $b_j \in G_j$ $(0 \leqq j \leqq d-1)$.

Utilizing (2.5) and (2.2), it is easy to show that (2.3) defines a homomorphism of S into H.

REMARK 2.1 In the case $d = 1$, we obtain [6, theorem 3.5].

THEOREM 2.2. *Let* $S = (d, G_0, G_1, \cdots, G_{d-1}, C_1, \gamma_0, \gamma_1, \cdots, \gamma_{d-1})$ *be a simple regular* ω*-semigroup. Let* $N = \{g \in G_0 | g(\gamma_0 \gamma_1 \cdots \gamma_{d-1})^n = k_0$, *the identity of* G_0, *for some* $n \in I^0\}$. *Then,* N *is a normal subgroup of* G_0. *Let* $g \to \bar{g}$ *be the natural homomorphism of* G_0 *onto* G_0/N. *Define* $\bar{x}\theta = \overline{x\gamma_0\gamma_1 \cdots \gamma_{d-1}}$ *for* $x \in G_0$. *Then,* θ *is an endomorphism of* G_0/N. *Define a relation* σ *on* $G_0/N \times (I^0)^2$ *by the rule* $((\bar{g}_0, a, b), (\bar{h}_0, c, d)) \in \sigma$ *if and only if there exist* $x, y \in I^0$ *such that* $x+a = y+c$, $x+b = y+d$ *and* $\bar{g}_0\theta^x = \bar{h}_0\theta^y$. *Define a binary operation on* $V = G_0/N x (I^0)^2/\sigma$ *by the rule*

$$(\bar{g}_0, a, b)_\sigma(\bar{h}_0, c, d)_\sigma = (\bar{g}_0\theta^c\bar{h}_0\theta^b, a+c, b+d)_\sigma.$$

Then, V is a group which is the maximal group homomorphic image of S. The canonical homomorphism of S onto V is given by

$$(g_r, m, n)\zeta = (\overline{g_r \prod_{j=r}^{d-1} \gamma_j}, m+1, n+1)_\sigma \quad where \quad g_r \in G_r$$

PROOF. For simplicity, let $\alpha_{n,m} = \prod_{j=n}^{m-1} \gamma_j$ if $m > n$ and let $\alpha_{n,n}$ denote the identity automorphism of $G_{n(\mathrm{mod}\, d)}$. Let $T = \{(g_0, a, b) : g_0 \in G_0; a, b \in I^0\}$. Then, T is the ω-bisimple semigroup $(G_0, C_1, \alpha_{0,d})$ by theorem 1.1 and [6, theorem 1.1] (notation of [6]). Thus, by [6, theorem 3.4], σ is an equivalence relation and V is a group. By a routine calculation $(\bar{k}_0, 0, 0)_\sigma$ is the identity of V and $(\bar{g}_0^{-1}, b, a)_\sigma$ is the inverse of $(\bar{g}_0, a, b)_\sigma$. We first employ theorem 2.1 to show that ξ is a homomorphism of S into V. Let $z = (\bar{k}_0, 0, 1)_\sigma$ and $g_r f_r = (\overline{g_r \alpha_{r,d}}, 1, 1)_\sigma$ for $0 \leqq r \leqq d-1$. By a straight forward calculation, (2.1) and (2.2) of theorem 2.1 are valid, and $(g_r, m, n)\xi = z^{-m} g_r f_r z^n$.

Since

$$\begin{aligned}(g_0, m, n)\zeta &= (\overline{g_0 \alpha_{0,d}}, m+1, n+1)_\sigma \\ &= (\overline{g_0 \theta}, m+1, n+1)_\sigma \\ &= (\bar{g}_0, m, n)_\sigma,\end{aligned}$$

ξ maps S onto V.

Let δ be a homomorphism of S onto a group X. We show that $\delta|T$ is a homomorphism of T onto X. By theorem 2.1, for each $r \in \{0, \cdots, d-1\}$, there exists a homomorphism δ_r of G_r into X and $ap \in X$ such that (2.1) and (2.2) of theorem 2.1 are valid and

$$(g_r, m, n)\delta = p^{-m} g_r \delta_r p^n$$

where $g_r \in G_r$. Thus, if $x \in X$, there exists $g_r \in G_r$, $a, b \in I^0$, such that

$$x = p^{-a} g_r \delta_r p^b.$$

Hence, utilizing (2.1) and (2.2) of theorem 2.1,

$$\begin{aligned}x &= p^{-a} g_r \gamma_r \delta_{r+1} p^b \\ &= p^{-a} g_r \gamma_r \cdots \gamma_{d-2} \delta_{d-1} p^b \\ &= p^{-(a+1)} g_r \alpha_{r,d} \delta_0 p^{(b+1)} \\ &= (g_r \alpha_{r,d}, a+1, b+1)\delta.\end{aligned}$$

By [6, theorem 3.4], V is the maximal group homomorphic image of T under the homomorphism

$$(g_0, a, b)\phi = (g_0, a, b)_\sigma.$$

Hence, there exists a homomorphism η of V onto X such that $\phi\eta = \delta|T$.

We will show that V is the maximal group homomorphic image of S under the homomorphism ξ. We note that

$$(\bar{g}_0, m, n)_\sigma \eta = (g_0, m, n)\phi\eta \\ = (g_0, m, n)\delta. \tag{2.6}$$

Hence, by (2.6), (2.2), and (2.1),

$$\begin{aligned}(g_r, m, n)\zeta\eta &= (\overline{g_r \alpha_{r,d}}, m+1, n+1)_\sigma \eta \\ &= p^{-(m+1)}(g_r \alpha_{r,d})\delta_0 p^{n+1} \\ &= p^{-m} g_r \delta_r p^n \\ &= (g_r, m, n)\delta.\end{aligned}$$

REMARK 2.2. In the case $d = 1$, we obtain [6, theorem 3.4].

The following remarks will be utilized in giving the canonical homomorphism in theorem 2.3 (below) a convenient form.

Let $S = (d, G_0, G_1, \cdots, G_{d-1}, C_1^*, \gamma_0, \gamma_1, \cdots, \gamma_{d-1}, u_{id})$ be a simple I-regular semigroup. Let $\alpha_{m,n} = \gamma_m \gamma_{m+1} \cdots \gamma_{n-1}$ for $m < n$ let $\alpha_{m,m}$ denote the identity automorphism of $G_{m(\text{mod } d)}$. Let a_1 denote a non-negative integer. Define

$$t_{id,a_1} = \begin{cases} f_{a_1-1,i+1}^{-1} u_{(i+1)d} \alpha_{0,d}^{a_1-2} \cdots u_{(i+1)d} \alpha_{0,d} u_{(i+1)d} & \text{if } a_1 \geqq 2 \\ k_0, \text{ the identity of } G_0, \text{ otherwise.} \end{cases} \tag{2.7}$$

By the proof of [10, theorem 1] *, $S \cong (U(S_{id} : i \in I, i \leqq 0))\lambda$ where S_{id} is the simple regular ω-semigroup $S_{id} = (d, G_0, \cdots, G_{d-1}, C_1, \gamma_{id,0}, \gamma_{id,1} \cdots, \gamma_{id,d-1})_{id}$ the congruence λ defined in [10],

$$\gamma_{id,d-1} = \gamma_{d-1} C_{u_{(i+1)d}^{-1}}, \tag{2.8}$$

and

$$\gamma_{id,s} = \gamma_s \quad \text{for} \quad 0 \leqq s \leqq d-2 \tag{2.9}$$

under an isomorphism Ψ (defined in [10]).

For $g_r \in G_r$ for $0 \leqq r \leqq d-1$,

$$(g_r, m, n)_{(i+1)d}\lambda = ((s_{id}^{-1}\alpha_{id,0,d}^{m-1} \cdots s_{id}^{-1}\alpha_{id,0,d} s_{id}^{-1})\alpha_{id,0,r} g_r \\ ((s_{id} \cdot s_{id}\alpha_{id,0,d} \cdots s_{id}\alpha_{id,0,d}^{n-1})\alpha_{id,0,r}), m+1, n+1)_{id}\lambda \tag{2.10}$$

where if $m = 0$ ($n = 0$) the right (left) multiplier of g_r is k_r, the identity of G_r and

$$s_{id} = u_{(i+2)d}^{-1} u_{(i+1)d}, \tag{2.11}$$

$$g_{d-1}\gamma_{id,d-1} = s_{id}^{-1}(g_{d-1}\gamma_{(i+1)d,d-1})s_{id} \tag{2.12}$$

By the proof of [10, theorem 1], if Ψ_{id} is as in [10],

* In [10], S_{id} is denoted by X_{id}.

$$(g_r, a_1, b_1)_{id} = ((t_{id,a_1}\alpha_{0,r})g_r(t_{id,b_1}^{-1}\alpha_{0,r}), a_1+i, b_1+i)\Psi_{id} \tag{2.13}$$

and

$$(g_r, a_1+i, b_1+i)\Psi_{id} = ((t_{id,a_1}^{-1}\alpha_{0,r})g_r t_{id,b_1}\alpha_{0,r}, a_1, b_1)_{id}. \tag{2.14}$$

THEOREM 2.3. *Let* $S = (d, G_0, G_1, \cdots, G_{d-1}, C_1^*, \gamma_0, \gamma_1, \cdots, \gamma_{d-1}, u_{id})$ *be a simple I-regular semigroup. Let* $N = \{g \in G_0 | g(\gamma_0\gamma_1 \cdots \gamma_{d-1})^n = k_0$, *the identity of* G_0, *for some* $n \in I^0\}$. *Then,* N *is a normal subgroup of* G_0. *Let* $g \to \bar{g}$ *be the natural homomorphism of* G_0 *onto* G_0/N. *Define* $\bar{x}\theta = \overline{x\gamma_0\gamma_1 \cdots \gamma_{d-1}}$ *for* $x \in G_0$. *Then,* θ *is an endomorphism of* G_0/N. *Define a relation* σ *on* $G_0/N x (I^0)^2$ *by the rule* $((\bar{g}_0, a, b), (\bar{h}_0, c, d)) \in \sigma$ *if and only if there exist* $x, y \in I^0$ *such that* $x+a = y+c$, $x+b = y+d$, *and* $\bar{g}_0\theta^x = \bar{h}_0\theta^y$. *Define a binary operation on* $H = G_0/N x(I^0)^2/\sigma$ *by the rule* $(\bar{g}_0, a, b)_\sigma(\bar{h}_0, c, d)_\sigma = (\bar{g}_0\theta^c\bar{h}_0\theta^b, a+c, b+d)_\sigma$. *Then,* H *is a group which is the maximal group homorphic image of* S. *The canonical homorphism of* S *onto* V *is given by*

$$(g_r, a, b)\bar{\phi} = \begin{cases} (x_{id}^{-1}\theta^{a_1-i-1} \cdots x_{id}^{-1}\theta x_{id}^{-1}((t_{id,a-i}^{-1}\alpha_{0,r})g_r(t_{id,b-i}\alpha_{0,r}))\delta_{id} \\ \quad \cdot x_{id} \cdot x_{id}\theta \cdots x_{id}\theta^{b_1-i-1}, a_1-i, b_1-i)_\sigma \textit{ for } i \leqq -1. \\ \textit{If } a = i\ (b = i), \textit{ the corresponding factor is } \bar{k}_0; \\ (\overline{g_r\alpha_{r,d}}, a-i+1, b-i+1)_\sigma \textit{ for } i = 0, \end{cases} \tag{2.15}$$

where $(g_r, a, b) \in (k_0, i, i)S(k_0, i, i)$ *and where*

$x_0 = \bar{k}_0$,

$x_{-d} = \bar{u}_0^{-1}$ *while for* $i \leqq -2$,

$x_{id} = \bar{u}_0^{-1}(\bar{u}_{-d}^{-1}\theta) \cdots \bar{u}_{(i+1)d}^{-1}\theta^{-(i+1)}\bar{u}_{(i+2)d}\theta^{-(i+1)}\bar{u}_{(i+3)d}\theta^{-(i+2)} \cdots \bar{u}_0\theta$

$g_r\delta_0 = \overline{g_r\alpha_{r,d}}$ *while for* $i \leqq -1$

$g_r\delta_{id} = \bar{u}_0^{-1}\bar{u}_{-d}^{-1}\theta \cdots \bar{u}_{(i+1)d}^{-1}\theta^{-(i+1)}\overline{g_r\alpha_{r,d}}\theta^{-i-1}\bar{u}_{(i+1)d}\theta^{-(i+1)} \cdots \bar{u}_{-d}\theta\bar{u}_0$.

PROOF. As our proof parallels that of [6, theorem 3.6], we will just give a sketch of the proof. We first use theorem 2.1 to determine a homomorphism ϕ_{id} of S_{id} into H for each $i \in I$ with $i \leqq 0$. Let x_{id} and δ_{id} be defined as in the statement of the theorem. In the notation of theorem 2.1, let $z_{id} = (x_{id}, 0, 1)_\sigma$, $g_r f_{r,0} = (g_r\delta_0, 1, 1)_\sigma$ and $g_r f_{r,id} = (g_r\delta_{id}, 0, 0)_\sigma$ for $i \leqq -1$ where $g_r \in G_r$ $(0 \leqq r \leqq d-1)$. Utilizing (2.8), we show that (2.1) and (2.2) are valid.

Hence, by (2.3),

$$(g_r, m, n)_{id}\phi_{id} = \begin{cases} (x_{id}^{-1}\theta^{m-1} \cdots x_{id}^{-1}\theta x_{id}^{-1})(g_r\delta_{id})(x_{id} \cdot x_{id}\theta \cdots x_{id}\theta^{n-1}), m, n)_\sigma \text{ if } i \leqq -1. \\ \text{If } m = 0\ (n = 0) \text{ the corresponding factor is } \bar{k}_0; \\ (\overline{g_r\alpha_{r,d}}, m+1, n+1)_\sigma \text{ if } i = 0, \end{cases} \tag{2.16}$$

defines a homomorphism of S_{id} into H.

We note that $(g_r, m, n)_0\phi_0 = \overline{(g_r\alpha_{r,d}, m+1, n+1)}_\sigma$. Hence, by theorem 2.2, ϕ_0 is a homomorphism of S_0 onto H.

Let us define $x\lambda\phi = x\phi_{id}$ if $x \in S_{id}$. We will show that ϕ is a homomorphism of $S\Psi$ onto H. We note that $(g_r, 1, 1)_{id}\phi_{id} = (g_r, 0, 0)_{(i+1)d}\phi_{(i+1)d}$. Utilizing (2.11), we obtain $(s_{id}, 1, 2)_{id}\phi_{id} = (k_0, 0, 1)_{(i+1)d}\phi_{(i+1)d}$. The desired result is then a consequence of (2.10).

Let G^* be an arbitrary group and let ρ be a homomorphism of $S\Psi$ onto G^*. We denote $\lambda\rho|S_{id}$ by ρ_{id}. Thus, ρ_{id} is a homomorphism of S_{id} into G^*. Since H is the maximal group homomorphic image of S_0 under the homomorphism ϕ_0 by virtue of theorem 2.2, there exists a homomorphism γ of H onto the subgroup $S_0\rho_0$ of G^* such that $(g_r, m, n)_0\phi_0\gamma = (g_r, m, n)_0\rho_0$ for all $(g_r, m, n)_0 \in S_0$.

Next suppose that $(g_r, m, n)_{(i+1)d}\phi_{(i+1)d}\gamma = (g_r, m, n)_{(i+1)d}\rho_{(i+1)d}$ where γ is a homomorphism of H onto $S_{(i+1)d}\rho_{(i+1)d}$.

By virtue of theorem 2.1, there exists v_{id} in G^* and a homomorphism $\eta_{r,id}$ of G_r into G^* for each $r \in \{0, 1, 2, \cdots, d-1\}$ such that $v_{id}(g_{d-1}\eta_{d-1,id})v_{id}^{-1} = g_{d-1}\gamma_{id,d-1}\eta_{0,id}$ and $g_r\eta_{r,id} = g_r\gamma_{id,r}\eta_{r+1,id}$ for $0 \leqq r \leqq d-2$. Furthermore $(g_r, m, n)_{id}\rho_{id} = v_{id}^{-m}(g_r\eta_{r,id})v_{id}^n$ for $(g_r, m, n)_{id} \in S_{id}$. Since $(g_r, 0, 0)_{(i+1)d}\lambda = (g_r, 1, 1)_{id}\lambda$, when $g_r \in G_r$, by (2.10), $(g_r, 0, 0)_{(i+1)d}\rho_{(i+1)d} = (g_r, 1, 1)_{id}\rho_{id}$. Thus, $g_r\eta_{r,id} = v_{id}(g_r\eta_{r,(i+1)d})v_{id}^{-1}$. Hence, since $(k_0, 0, 1)_{(i+1)d}\rho_{(i+1)d} = (s_{id}, 1, 2)_{id}\rho_{id}$ by (2.10), $v_{id} = (s_{id}^{-1}\eta_{0,(i+1)d})v_{(i+1)d}$. Thus, $g_r\eta_{r,id} = (s_{id}^{-1}(g_r\alpha_{(i+1)d,r,d})s_{id})\eta_{0,(i+1)d}$. Utilizing (2.8), (2.9), and (2.2), we obtain $\overline{s_{id}^{-1}(g_r\alpha_{(i+1)d,r,d})s_{id}} = \bar{u}_{(i+1)d}^{-1}(g_r\alpha_{r,d})\bar{u}_{(i+1)d}$. Hence, $(s_{id}^{-1}(g_r\alpha_{(i+1)d,r,d})\,s_{id}, 0, 0)_{(i+1)d}\phi_{(i+1)d} = (g_r, 0, 0)_{id}\phi_{id}$ and $(g_r, 0, 0)_{id}\rho_{id} = (g_r, 0, 0)_{id}\phi_{id}\gamma$. We also note that $(s_{id}^{-1}, 0, 1)_{(i+1)d}\phi_{(i+1)d} = (k_0, 0, 1)_{id}\phi_{id}$ by (2.16). Hence, $(k_0, 0, 1)_{id}\rho_{id} = (k_0, 0, 1)_{id}\phi_{id}\gamma$. Thus, $(g_r, m, n)_{id}\phi_{id}\gamma = (g_r, m, n)_{id}\rho_{id}$ for all $(g_r, m, n) \in S_{id}$. Hence, H is the maximal group homomorphic image of $S\Psi$ under the homomorphism ϕ. We put ϕ in the form (2.15) by combining (2.16) and (2.14).

REMARK 2.3. In the case $d = 1$, we obtain [6, theorem 3.6].

The following result is needed to give an explicit determination of the extensions of a Brandt semigroup by a simple I-regular semigroup (theorem 1.3).

THEOREM 2.4. *Let* $S = (d, G_0, G_1, \cdots, G_{d-1}, C_1^*, \gamma_0, \gamma_1, \cdots, \gamma_{d-1}, u_{id})$ *be a simple I-regular semigroup and let* X *be a group. Let* $\{z_{id} : i \in I, i \leqq 0\}$ *be a sequence of elements of* X *and for each* $r \in \{0, 1, \cdots, d-1\}$ *let* $\{f_{id,r} : i \in I, i \leqq 0\}$ *be a sequence of homomorphisms of* G_r *into* X *such that*

$$f_{id,d-1}C_{z_{id}} = \gamma_{d-1}C_{u^{-1}_{(i+1)d}}f_{id,0},$$
$$f_{id,r} = \gamma_r f_{id,r+1} \text{ for } 0 \leqq r \leqq d-2,$$
$$z_{(i+1)d} = z_{id}^{-1}((u^{-1}_{(i+2)d}u_{(i+1)d})f_{0,id})z_{id}^2, \quad \text{and}$$
$$f_{r,(i+1)d} = f_{r,id}Cz_{id}$$

For each $(g_r, a, b) \in (k_0, i, i)S(k_0, i, i)$, *define* $(g_r, a, b)\phi = z_{id}^{-(a-i)}((t^{-1}_{id,a-i}\alpha_{0,r})g_r(t_{id,b-i}\alpha_{0,r}))f_{r,id}z_{id}^{b-i}$. *Then,* ϕ *defines a homomorphism of* S *into* X *and conversely every such homomorphism is defined in this fashion.*

PROOF. We utilize theorem 2.1, (2.14), and the 'inverse limit' process (see [10]).

3. The congruences

In this section, we show that each congruence ρ on a simple I-regular semigroup S is a group congruence (S/ρ is a group), an idempotent separating congruence (each ρ-class contains at most one idempotent) or that S/ρ is a simple I-regular semigroup with fewer $\mathfrak{D}$-classes than S. We determine the idempotent separating congruences in terms of certain normal subgroups of the structure groups of S. The group congruences of S are in a $1-1$ correspondence with the normal subgroups of the maximal group homomorphic image of S.

THEOREM 3.1. *Let* S *be a simple* I*-regular semigroup. Let* ρ *be a congruence on* S*. Then* ρ *is a group congruence,* ρ *is an idempotent separating congruence, or* S/ρ *is a simple* I*-regular semigroup with* t $\mathfrak{D}$*-classes where* $t < d$*, the number of* $\mathfrak{D}$*-classes of* S.

PROOF. Let $\{(f_i, n, n) : 0 \leqq i \leqq d-1, n \in I\}$ denote the set of idempotents of S. Each $D_i = \{(g_i, m, n); g_i \in G_i, m, n \in I\}$ is an I-bisimple semigroup for $0 \leqq i \leqq d-1$. Thus, by [6, theorem 4.2], $\rho|D_i$ is a group congruence or an idempotent separating congruence for $0 \leqq i \leqq d-1$. Suppose that ρ is not an idempotent separating congruence. First suppose that $\rho|D_i$ is a group congruence for some i. Hence, $(f_i, 0, 0)\rho = (f_i, k, k)\rho$ for all $k \in I$. Let $(f_j, n, n) \in E_{D_j}$ and $(f_k, p, p) \in E_{D_k}$ and suppose that $(f_j, n, n) < (f_k, p, p)$. Thus, $(f_i, n+1, n+1) < (f_j, n, n) < (f_k, p, p) < (f_i, p-1, p-1)$. Hence, $(f_i, n, n)\rho = (f_k, p, p)\rho$ and ρ is a group congruence. Next, suppose that $\rho|D_i$ is an idempotent separating congruence for each $0 \leqq i \leqq d-1$. Then, there exist (f_i, n, n), $(f_r, q, q) \in E_S$ such that $(f_i, n, n)\rho = (f_k, q, q)\rho$. Thus, $D_i\rho$ and $D_k\rho$ lie in the same $\mathfrak{D}$-class of S/ρ. Hence, S/ρ is a simple I-regular semigroup with t $\mathfrak{D}$-classes with $t < d$.

REMARK 3.1. In the case $d = 1$, we obtain [6, theorem 4.2].

REMARK 3.2. We may replace 'simple I-regular' by 'simple ω-regular' in theorem 3.1. The proof is analogous.

We next determine the idempotent separating congruences of a simple I-regular semigroup.

Let $G_0, G_1, \cdots, G_{d-1}$ be a collection of disjoint groups and let γ_i be a homomorphism of G_i into G_{i+1} for $0 \leqq i \leqq d-2$ and let γ_{d-1} be a homomorphism of G_{d-1} into G_0. Let V_i be a normal subgroup of G_i for $0 \leqq i \leqq d-1$ such that $V_i\gamma_i \subseteq V_{i+1}$ for $c \leqq i \leqq d-2$ and $V_{d-1}\gamma_{d-1} \subseteq V_0$. Then, $(V_0, V_1, \cdots, V_{d-1})$ will be called a $\gamma_0-\gamma_1 \cdots -\gamma_{d-1}$ invariant d-tuple of $(G_0, G_1, \cdots, G_{d-1})$. Let $(V_0, V_1, \cdots, V_{d-1})$ and $(U_0, U_1, \cdots, U_{d-1})$ be $\gamma_0-\gamma_1- \cdots -\gamma_{d-1}$ invariant d-tuples of $(G_0, G_1, \cdots, G_{d-1})$. Then, we say $(V_0, V_1, \cdots, V_{d-1}) \subseteq (U_0, U_1, \cdots, U_{d-1})$ if and only if $V_i \subseteq U_i$ for $0 \leqq i \leqq d-1$.

In the proof of the following theorem, we will utilize a theorem of Preston [6, theorem 4.3]. We also utilize the notation of this theorem. We will sketch the following proof where it parallels the proof of [6, theorem 4.4].

THEOREM 3.2. *Let* $S = (d, G_0, G_1, \cdots, G_{d-1}, C_1^*, \gamma_0, \gamma_1, \cdots, \gamma_{d-1}, m_{id})$ *be a simple I-regular semigroup. There exists a* $1-1$ *correspondence between the idempotent separating congruences on S and the* $\gamma_0-\gamma_1- \cdots -\gamma_{d-1}$ *invariant d-tuples of* $(G_0, G_1, \cdots, G_{d-1})$. *If* $\rho^{(V_0, V_1, \cdots, V_{d-1})}$ *is the idempotent separating congruence corresponding to the* $\gamma_0-\gamma_1- \cdots -\gamma_{d-1}$ *invariant d-tuple* $(V_0, V_1, \cdots, V_{d-1})$, $(g_r, a, b)\rho^{(V_0, \cdots, V_{d-1})}(h_s, c, d)$ *if and only if* $r = s$, $a = c$, $b = d$ *and* $V_r g_r = V_r h_r$. *If* $(V_0, V_1, \cdots, V_{d-1})$ *and* $(U_0, U_1, \cdots, U_{d-1})$ *are two* $\gamma_0-\gamma_1- \cdots -\gamma_{d-1}$ *invariant d-tuples* $(V_0, V_1, \cdots, V_{d-1}) \subseteq (U_0, U_1, \cdots, U_{d-1})$ *if and only if* $\rho^{(V_0, V_1, \cdots, V_{d-1})} \subseteq \rho^{(U_0, U_1, \cdots, U_{d-1})}$.

PROOF. Let $(V_0, V_1, \cdots, V_{d-1})$ be a $\gamma_0-\gamma_1- \cdots -\gamma_{d-1}$ invariant d-tuple of $(G_0, G_1, \cdots, G_{d-1})$. Let $N_{(k_r, a, a)} = \{(v_r, a, a) : v_r \in V_r\}$ and let $N = U(N_{(k_r, a, a)} : 0 \leqq r \leqq d-1, a \in I)$. By a routine calculation, $N_{(k_r, a, a)}$ is a subgroup of S isomorphic to V_r. By [6, theorem 4.3] ρ_N is an idempotent separating congruences of S. We denote ρ_N by $\rho^{(V_0, V_1, \cdots, V_{d-1})}$.

Let ρ be an idempotent separating congruence of S. Then, by [6, theorem 4.3] $\rho = \rho_N$ where N is given in the statement of [6, theorem 4.3]. $N_{(e_r, a, a)} = \{(v_r, a, a) : v_r \in V_r\}$ where V_r is an invariant subgroup of G_r. Since $(e_{r+1}, 0, 0)(e_r, 0, 0) = (e_{r+1}, 0, 0)$, $(e_{r+1}, 0, 0)(v_r, 0, 0) \in N_{(e_{r+1}, 0, 0)}$. Thus $v_r\gamma_r \subseteq V_{r+1}$ for $0 \leqq r \leqq d-2$. Since $(e_0, 0, 1)(e_{d-1}, 0, 0)(e_0, 1, 0) = (e_0, 0, 0)$, $(e_0, 0, 1)(v_{d-1}, 0, 0)(e_0, 1, 0) \in N_{(e_0, 0, 0)}$. Thus, $v_{d-1}\gamma_{d-1} \in V_0$. Hence, $\rho = \rho^{(V_0, V_1, \cdots, V_{d-1})}$ and we have the desired correspondence.

REMARK. In the case $d = 1$, we obtain [6, theorem 4.4].

REMARK. We may replace 'simple I-regular semigroup' by 'simple regular ω-semigroup' in theorem 3.2. The proof is analogous.

REFERENCES

A. H. CLIFFORD AND G. B. PRESTON
[1] *The algebraic theory of semigroups,* Math. Surveys of the American Math. Soc. 7 (Vol 1), Providence, R.I., 1961.

W. D. MUNN,
[2] *Regular ω-semigroups,* Glasgow Math. J., J., 9 (1968), 44–66.

R. J. WARNE
[3] *Regular d-classes whose idempotents obey certain conditions,* Duke Math. J. 33 (1966), 187–195.

R. J. WARNE
[4] *Extensions of Completely* 0-*simple semigroups by completely* 0-*simple semigroups,* Proc. Amer. Math. Soc. 17 (1966), 524–526.

R. J. WARNE
[5] *Extensions of Brandt semigroups and applications,* Illinois J. Math., 10 (1966), 652–660.

R. J. WARNE
[6] *I-bisimple semigroups,* Trans. Amer. Math. Soc., 130 (1968), 367–386.

R. J. WARNE
[7] *Extensions of I-bisimple semigroups,* Can. J. Math., 19 (1967), 419–426.

R. J. WARNE
[8] Errata to [7].

R. J. WARNE
[9] *Extensions of $\omega^n I$-bisimple semigroups,* Mathematica Japanicae, 13 (1968), 105–121.

R. J. WARNE
[10] *I-regular semigroups,* to appear.

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