COMPOSITIO MATHEMATICA

K. W. ROGGENKAMP

Projective modules over clean orders

Projective modules over clean orders

by

K. W. Roggenkamp

1. Introduction

We shall use the following notation:

K = algebraic number field
R = Dedekind domain with quotient field K
A = finite dimensional semisimple K-algebra
G = R-order in A.

A *G-lattice* is a unitary left G-module, which is finitely generated and torsion-free over R.

$C(G)$ = category of G-lattices.
$M \vee N$ means, that $M, N \in C(G)$ lie in the same genus (cf. [7])
$V(M) = \{N : N \vee M\}, M \in C(G)$.

Following Strooker [12], we call a projective G-lattice P *faithfully projective*, if G is a direct summand of P^n, the direct sum of n copies of P, for some natural number n. And we shall say that G is a *clean R-order*, if every special projective G-lattice (i.e., P is special projective, if KP is a free A-module) is faithfully projective.

In the first section we shall show that projective modules over clean orders in simple algebras have similar nice properties as projective modules over group rings of finite groups (cf. [13]), the prototypes of clean orders. In the second section we shall show that for lattices over maximal orders, a conjecture of Roiter [11] is true, if the skewfields of the simple components of A are not totally definite quaternion algebras. Finally, in the last section we shall extend a result of Swan [15]: If G' is an R-order in A, contained in the clean R-order G, and if $P_0(G)$ denotes the Grothendieck group of the special projective G-lattices, then we have an epimorphism

$$P_0(G') \to P_0(G).$$

2. Projective modules over clean orders in simple algebras

In this section we shall assume that A is a simple K-algebra, and that G is a clean R-order in A.

Moreover, in the first half of this section we assume that R is a discrete rank one valuation ring with quotient field K.

2.1 LEMMA. *Let M and N be projective G-lattices. Then*

$$M \cong N \Leftrightarrow KM \cong KN \qquad (KM = K \otimes_R M).$$

PROOF. Let L be the irreducible A-module. Assume

$$KM \cong KN \cong L^s.$$

Pick a positive integer t such that L^{st} is a free A-module. Put

$$M' = M \dotplus M^{(t-1)}$$
$$N' = N \dotplus M^{(t-1)}.$$

Then M' and N' are projective G-lattices, such that

$$KM' \cong KN' \cong L^{st};$$

hence M' and N' are special projective G-lattices, therefore they are G-free (cf. [12], 3.10) on the same number of elements, hence $M' \cong N'$. Since locally cancellation is allowed, we get $M \cong N$. The other direction of the lemma is trivial.   q.e.d. 2.1

2.2. LEMMA. *Up to isomorphism there is only one indecomposable projective G-lattice.*

PROOF. Let $\{e_i\}$ be a full set of indecomposable orthogonal idempotents in G. Let e be one of these idempotents. If P is an indecomposable projective G-lattice, we shall show: $P \cong Ge$, which will prove the lemma.

Again let L be the irreducible A-module. Then

(2.3)
$$KP \cong L^r$$
$$KGe \cong L^s$$

W.l.o.g. we can assume $r \neq s$, since otherwise the statement would follow from (2.1). Assume $r < s$. Pick two positive integers s', t' such that

(2.4) $$L^{rr'} = L^{ss'}$$

is a free A-module, then $r' > s'$. Let R^* be the completion of R. Decompose R^*P and R^*Ge into indecomposable R^*G-lattices:

$$R^* P \cong \sum_{i=1}^{n} \oplus P_i^*$$

(2.5)

$$R^* Ge \cong \sum_{i=1}^{n'} \oplus Q_i^*.$$

From (2.3) and (2.4) we conclude, using (2.1):

$$P^{r'} \cong Ge^{s'},$$

so by (2.5)

$$\sum_{i=1}^{n} \oplus P_i^{*r'} \cong \sum_{i=1}^{n'} \oplus Q_i^{*s'}.$$

Since the Krull-Schmidt-theorem is valid for $R^* G$-lattices, we get $nr' = n's'$, i.e. $n < n'$, and we conclude, that there is an $R^* G$-lattice X^*, such that

(2.6) $$X^* \oplus R^* P \cong R^* Ge.$$

Since $KP \oplus V \cong Ae$ $(r < s)$, for some A-module V,

$$X = V \cap X^*$$

is a G-lattice, such that $R^* X \cong X^*$. Hence we get from (2.6)

$$R^* X \oplus R^* P \cong R^* Ge,$$

i.e.

$$X \oplus P \cong Ge.$$

Since Ge was assumed to be indecomposable, $X = (0)$, thus

$$P \cong Ge.$$

If now $r > s$, we give a similar proof, using the fact that P was indecomposable.   q.e.d. 2.2

Now let us return to the global situation, where R is a Dedekind domain with quotient field K.

2.7 LEMMA. *Let $A = (D)_n$, D a field of finite dimension over K. If C is the maximal R-order in D, then G is maximal, if $C \subset G$, and G contains a primitive idempotent of A.*

PROOF. Let e be the primitive idempotent. Then e is in every $R_p \otimes_R G = G_p$, R_p being the localization of R at the prime p. Since G is maximal if and only if G_p is maximal for every prime p, it suffices to prove the lemma for G_p, and we shall omit the index p.

Ge is irreducible, hence $S(Ge) = \text{End}_G(Ge)$ is an R-order in D. Moreover from (2.2) it follows that Ge is faithfully projective, i.e. (cf. [12], 1.5; [1], A.6)

$$G = \mathrm{Hom}_{S(Ge)}(Ge, Ge) \qquad (\text{cf. } [2]).$$

Since $S(Ge) \subset C \subset G$, we have $C = S(Ge)$ (cf. [9], lemma 14). But Ge is a $S(Ge)$-lattice, hence G is a maximal R-order in A (cf. [1], 3.6).   q.e.d. 2.7

2.8 LEMMA. *There exists only one genus of indecomposable projective G-lattices.*

PROOF. Because of (2.1) we are finished, if we can show: P, Q are indecomposable projective G-lattices $\Rightarrow KP \cong KQ$. Assume the contrary. Let L be the irreducible A-module,

$$KP \cong L^s, \qquad KQ \cong L^t.$$

W.l.o.g. we can assume $s > t$.

Then for every prime p we have

$$P_p \cong (G_p e_p)^{(s_p)}, \qquad Q_p \cong (G_p e_p)^{(t_p)}, \qquad \text{by (2.2)},$$

where e_p is an indecomposable idempotent in G_p. Since $KP_p \cong KP$ and $KQ_p \cong KQ$, we have $s_p > t_p$ for every p. But that would imply that Q_p is a direct summand of P_p for every p, thence P decomposes (cf. [3]). So we have arrived at a contradiction.

q.e.d. 2.8

Using these results one can prove most of the statements in [3], § 78, (cf. [13]).

2.9 LEMMA. *Let M be a projective G-lattice, e an indecomposable idempotent in G. If I is any non zero ideal in R, then there exists a natural number n, such that*

$$(Ge)^n \supset M, \quad and \quad I + ann((Ge)/^n M) = R.$$

PROOF. Let M be a projective G-lattice, then there is an idempotent e in G, such that

$$M \vee Ge^{(n)} \text{ for some natural } n.$$

This follows from Lemma 2.8 by looking at the indecomposable components of M. Now we apply [11], Lemma 1, to embed M in Ge^n such that $I + ann((Ge)^n/M) = R$.

2.10 LEMMA. *Let M be a projective G-lattice, I a non zero ideal in R, and e an indecomposable idempotent in G. Then there exists a finite number of indecomposable projective lattices M_i in G, such that*

$$I + ann(Ge/M_i) = R \quad and \quad M \cong \sum \oplus M_i.$$

PROOF. In the proof we need the following statement, which is of interest for itself:

2.11 CLAIM. *Let $i(G)$ denote the Higman-ideal of G (cf. [3]). Then a G-lattice M is projective if and only if we can embed M into Ge^n for some n, such that*

$$i(G) + ann(Ge^n/M) = R.$$

PROOF OF CLAIM. If M is projective, the statement follows from (2.9). Conversely, if $i(G) + ann(Ge^n/M) = R$, then $M \vee Ge^n$, (cf. [7]), hence M is projective.   q.e.d. Claim

Now let M be projective, then the proof of (2.10) is the same as in [3], 78.8, using Ge^n instead of F'.   q.e.d. 2.10

2.12 THEOREM. *If M is a projective G-lattice, I a non zero ideal in R, and e an indecomposable idempotent in G, then there exists a projective indecomposable lattice M' in G, and a natural number n, such that*

$$M \cong Ge^n \oplus M' \quad and \quad I + ann(Ge/M') = R.$$

PROOF. We can use the proof of [3], 78.9, if we can show: Let M_1, M_2 be indecomposable projective lattices in G, such that

$$I + ann(Ge/M_i) = R, \qquad\qquad i = 1, 2;$$

then there exists an indecomposable projective G-lattice M_3, such that

$$I + ann(Ge/M_3) = R, \qquad M_1 \oplus M_2 \cong Ge \oplus M_3.$$

By [11], Prop. 5, since $M_1 \vee Ge$, and since M_2 is a faithful G-lattice, we can find M_3, such that

$$M_1 \oplus M_2 \cong Ge \oplus M_3, \qquad M_3 \vee Ge.$$

It is easily checked that M_3 is projective, and by (2.9) we can embed M_3 in Ge in the desired form.   q.e.d. 2.12

If the Krull-Schmidt-theorem is valid for G-lattices, one can characterize clean orders in terms of idempotents.

2.13 LEMMA. *Let G be an R-order in the semisimple finite dimensional K-algebra. Assume, that the Krull-Schmidt-theorem is valid for G-lattices. If there exists an idempotent e in G, such that Ge is an indecomposable faithfully projective G-lattice, then G is a clean R-order in A.*

PROOF. We shall show: If P is an indecomposable projective G-lattice, then $P \cong Ge$. This will prove the lemma. Since P

is projective, there exists a projective G-lattice X such that $P \oplus X \cong G^s$. But Ge is faithfully projective, hence

$$G \oplus Y \cong (Ge)^t, \text{ for some } G\text{-lattice } Y.$$

Since the Krull-Schmidt-theorem is valid, and since Ge is indecomposable, we get $G \cong (Ge)^r$, hence

$$P \oplus X \cong (Ge)^{sr}.$$

But P was indecomposable, and the Krull-Schmidt-theorem is valid, so $P \cong Ge$.   q.e.d. 2.13

3. On the genera of lattices over maximal orders

Let A be a semisimple finite dimensional K-algebra and B a maximal R-order in A. By $\# V(M)$, $M \in C(B)$ we denote the number of non-isomorphic B-lattices in the genus of M. Roiter [11] stated the following conjecture:

(3.1) $\# V(M) \leqq \# V(B)$, for every $M \in C(B)$.

As we showed in [10], this conjecture is not true in general. If S is any R-order in some semisimple finite dimensional K-algebra, we denote by $h(S)$ the number of left ideal classes in S (an ideal in S is an S-lattice I, such that $KI \cong KS$). $h(S)$ is always finite by the Jordan-Zassenhaus-theorem (cf. [16]). If $M \in C(B)$, we write

$$S(M) = \mathrm{End}_B(M).$$

The following lemma is easily proved, using the Morita-theorems (cf. [2]):

3.2 LEMMA. *Let $B = \sum \oplus B_i$ the decomposition of B into maximal R-orders in simple algebras. If E_i is an irreducible B_i-lattice, then*

$$\# V(M) \leqq \# V\left(\sum \oplus E_i\right) \text{ for every } M \in C(B).$$

We have equality, if KM contains every irreducible A-module exactly once. Moreover $\# V\left(\sum \oplus E_i\right) = \prod h(S(E_i))$.

(Q) We shall say that A *satisfies* (Q), if none of the skewfields in the simple components of A is a totally definite quaternion algebra.

3.3 LEMMA. *If A satisfies* (Q), *then Roiter's conjecture is true for maximal R-orders in A.*

PROOF. Let $\{e_i\}$ be the set of mutually orthogonal central primitive idempotents of B. If A satisfies (Q), then (cf. [4])

$$(3.4) \qquad h(S(M)) = \prod_{e_i M \neq 0} h(Be_i).$$

As in [7], (cf. [6], Satz 1) one shows easily

$$\# V(M) = h(S(M)), \qquad M \in C(M).$$

Now the statement follows from (3.4).   q.e.d. 3.3

3.5 COROLLARY. *If A satisfies (Q), $M, N_1, N_2 \in C(B)$, such that*

$$M \oplus N_1 \cong M \oplus N_2,$$

then

$$N_1 \cong N_2.$$

PROOF. The hypotheses imply, that for each e_i, we have

$$e_i M \oplus e_i N_1 \cong e_i M \oplus e_i N_2.$$

If we can show that for every e_i, $e_i N_1 \cong e_i N_2$, we have proved the corollary, since B is maximal. Hence we can assume that A is simple. Since locally we can cancel, $N_1 \vee N_2$. Assume the statement were not true, and let

$$N_1, N_2, \cdots, N_k \text{ be the non isomorphic } B\text{-lattices,}$$

in the same genus as N_1.

CLAIM. $V(N_1 \oplus M) = \{M \oplus N_i\}, \qquad i = 1, \cdots, k.$

Obviously all these B-lattices lie in the same genus as $N_1 \oplus M$. Now let $X \in V(N_1 \oplus M)$, then $X \cong N' \oplus M', N' \vee N_1, M' \vee M$. Since A was assumed to be simple, all the B-lattices are faithful, hence by [11], Prop. 5, we can find $N_i \in V(N_1)$, such that

$$N' \oplus M' \cong N_i \oplus M.$$

This proves the claim.

Since A satisfies (Q), we conclude from (3.4)

$$\# V(N_1 \oplus M) = \# V(N_1),$$

hence no two of the $N_i \oplus M$ can be isomorphic.   q.e.d. 3.5

The following lemma was proved by Jacobinski [6] under weaker conditions:

3.6 LEMMA. *Assume that A satisfies (Q); let G be an R-order in A and B a maximal R-order in A containing G. If M, N are G-lattices such that $M \vee N$, then*

$$BM \cong BN \Leftrightarrow M \oplus B \cong N \oplus B,$$

B considered as G-lattice.

PROOF. (i) If $M \oplus B \cong N \oplus B$, then $BM \oplus B \cong BN \oplus B$; hence by (3.5) $BM \cong BN$.

(ii) Conversely: Assume $BM \cong BN$. Since B is a faithful G-lattice, there is a G-lattice $E \vee B$ such that

(3.7) $M \oplus E \cong N \oplus B$ (by [11], prop. 5, since $M \vee N$).

Since $B \vee E$, E is a B-lattice, hence $BE = E$.

From (3.7) we now get

$$BM \oplus BE \cong BN \oplus B,$$

since $BN \cong BM$, this implies (3.5)

$$BE \cong B,$$

hence $E \cong B$.   q.e.d. 3.6

4. Grothendieck groups of special projective modules

Let A be a semisimple finite dimensional K-algebra, and G a clean R-order in A. By $P_0(G)$ we denote the Grothendieck group of the special projective G-lattices. Let G' be an R-order in A, contained in G; then we have a map

$$P' \to G \otimes_{G'} P',$$

from the category of special projective G'-lattices into the category of special projective G-lattices, which induces a homomorphism of abelian groups:

$$f : P_0(G') \to P_0(G)$$
$$f : [P'] \to [G \otimes_{G'} P'],$$

where $[P']$ denotes the class of P' in $P_0(G')$.

4.1 THEOREM. *f is an epimorphism.*

PROOF. Let P be a special projective G-lattice; we have to show $[P] \in \text{im } f$. Since G is clean, P is locally free, i.e. $P \vee F$, $F = G^n$ for some natural n. Let $F' = (G')^n$, then $G \otimes_{G'} F' \cong F$, hence

$$G \otimes_{G'} F' \vee P.$$

This means that $G \otimes_{G'} F'$ and P lie also in the same genus as G'-lattices, hence (by [11], lemma 1) we can embed P into $G \otimes_{G'} F'$ (as G'-lattice), such that

$$ann\ ((G \otimes_{G'} F')/P) + i(G)I_0 = R,$$

and such that the hypotheses of [11], lemma 2 are satisfied; $I_0 = \{r \in R : rG \subset G'\}$. We write U for $(G \otimes_{G'} F')/P$. Since F' is a free G'-lattice, it is a faithful G'-lattice, hence by [11], lemma 2, there exists a G'-lattice P', $P' \vee F'$, such that $F'/P' \cong U$. Since $P' \vee F'$, P' is a special projective G'-lattice. Therefore we get an exact sequence of G'-lattices:

$$0 \to P' \to F' \to U \to 0,$$

where $ann\ (U) + I_0 = R$. Now we can proceed as in [15], proof of Proposition (5.2), to conclude

$$G \otimes_{G'} P' \oplus G \otimes_{G'} F' \cong P \oplus G \otimes_{G'} F',$$

hence

$$f[P'] = [P].$$

q.e.d. 4.1

REMARK. This extends a result of Swan [15].

4.2 REMARK. With the same argument we can prove the following:

Let $G' \subset G$ be any two R-orders in A, then we have an epimorphism

$$P_f(G') \to P_f(G),$$

where $P_f(G)$ denotes the Grothendieck group of the locally free G-lattices.

REFERENCES

M. AUSLANDER and O. GOLDMAN
  [1] Maximal orders. Trans. Amer. Math. Soc. 97 (1960), 1−24.

H. BASS
  [2] The Morita theorems. Mimeo. notes, Univ. of Oregon, 1962.

C. W. CURTIS and I. REINER
  [3] Representation theory of finite groups and associative algebras. Interscience, N.Y. 1962.

M. EICHLER
  [4] Über die Idealklassenzahl hyperkomplexer Systeme, Math. Z. 43 (1937), 481−494.

D. G. HIGMAN
  [5] On orders in separable algebras. Can. J. Math. 7 (1955), 509−515.

H. JACOBINSKI
  [6] Über die Geschlechter von Gittern über Ordnungen. J. für reine und angew. Math., to appear.

J. M. Maranda

[7] On the equivalence of representations of finite groups by groups of automorphisms of modules over Dedekind rings. Can. J. Math. 7 (1955), 516—526.

I. Reiner

[8] The Krull-Schmidt-theorem for integral group representations. Bull. Amer. Math. Soc. 67 (1961), 365—367.

K. W. Roggenkamp

[9] On the irreducible representations of orders. MS, U. of Illinois, 1967.

K. W. Roggenkamp

[10] A counterexample to a conjecture of A. V. Roiter. Notices Amer. Math. Soc. 14 (1967), 530 (67T—372).

A. V. Roiter

[11] On the integral representations, belonging to one genus. Izv. Akad. Nauk, SSSR 30 (1966), 1315—1324.

J. R. Strooker

[12] Faithfully projective modules and clean algebras. Ph.D. thesis, Reichsuniversität Utrecht, 1965.

R. Swan

[13] Induced representations and projective modules. Ann. Math. 71 (1960), 552—578.

R. Swan

[14] Projective modules over group rings and maximal orders. Ann. Math. 76 (1962), 55—61.

R. Swan

[15] The Grothendieck ring of a finite group. Topology, 2 (1963), 85—110.

H. Zassenhaus

[16] Neuer Beweis der Endlichkeit der Klassen- bei unimodularer Äquivalenz endlicher ganzzähliger Substitutionsgruppen. Hamb. Abh. 12 (1938), 276—288.