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by

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1. Introduction to homotopy negligibility

We shall say that a subset A of a topological space X is *homotopy negligible* if the inclusion map $X-A \to X$ is a homotopy equivalence. The main result in this note is that under certain conditions this global property of A in X can be deduced from an analogous local property. In § 2, 3 examples are given where X is an infinite dimensional manifold. For a Palais-stable separable Hilbert manifold X the homotopy negligibility of A in X implies that X and $X-A$ are even diffeomorphic, by recent results of Kuiper, Burghelea [5] and Nicole Moulis [6]. These results are based on Bessaga [2] and Kuiper [4].

THEOREM 1. *Let X be an absolute neighborhood retract and A a closed subset. Assume that each point of A has a fundamental system of neighborhoods U in X for which $U \cap A$ is homotopy negligible in U. Then A is homotopy negligible in X.*

The theorem follows from the following lemma, for which we need another definition. A continuous map $f : Y \to Z$ of topological spaces is a *q-homotopy equivalence*, if f induces an isomorphism $f_i : \pi_i(Y) \to \pi_i(Z)$ of homotopy groups for all $i \leqq q$. If f is a q-homotopy equivalence for all q, then f is a *weak homotopy equivalence.*

Weak homotopy equivalence is again implied by compact homotopy equivalence: $X \overset{c}{\sim} Y \Leftrightarrow$ there are maps $f : X \to Y$ and $g : Y \to X$, such that for any compacts $K \subset X$ and $L \subset Y$, the restrictions $gf|K$ and $fg|L$ are homotopic to the inclusions respectively.

LEMMA. *Let X be a topological space, and A a closed subset with the following property P_q: for each $x \in A$ there is a fundamental*

[1] Much of this research was done while this author was a guest professor at the University of Amsterdam. The authors express their gratitude for some useful comments of the referee.

system of neighborhoods U of x in X for each of which the inclusion map $U-A = U-(U \cap A) \to U$ is a q-homotopy equivalence. Then for any neighborhood V of A in X the inclusion map $V-A \to V$ is a q-homotopy equivalence.

COROLLARY. *If A satisfies P_q for all q, then the inclusion map $X-A \to X$ is a weak homotopy equivalence.*

Assuming the Lemma we see that the conditions of Theorem 1 imply that the inclusion $X-A \to X$ is a weak homotopy equivalence. But a theorem of J. H. C. Whitehead [8] asserts that *if Y and Z are absolute neighborhood retracts and f a weak homotopy equivalence, then f is a homotopy equivalence* (i.e., there exists a map $g : Z \to Y$ which is a homotopy inverse of f).

If X is an absolute neighborhood retract and A a closed subset, then $X-A$ is also an absolute neighborhood retract. Theorem 1 now follows.

PROOF OF THE LEMMA. Under the assumptions of the lemma we first prove that $\pi_i(V-A) \to \pi_i(V)$ is *injective* for $i \leqq k$.

Let us call a neighborhood U of x with the property P_k a *preferred neighborhood* of x. We use the term for $x \in A$ as well as for $x \in V-A$. Let $f : (D^{i+1}, S^i) \to (V, V-A)$ be any map of the $i+1$-disc D^{i+1} into V carrying its boundary S^i into $V-A$. We want to move $f(D^{i+1})$ away from A, leaving f fixed on S^i. Now we cover the compact set $f(D^{i+1})$ by a finite number of preferred neighborhoods all in V. Choose a triangulation T_{i+1} of D^{i+1} so fine that every closed $i+1$-simplex $\sigma \in T_{i+1}$ has an image in some element, chosen once and for all, say $U_{i+1}(\sigma)$ of the covering, and such that simplices that meet S^i have their images in $V-A$.

Next we consider the i-skeleton T^i_{i+1} of T_{i+1}. For the image point $f(x)$ of any point $x \in T^i_{i+1}$ there is a preferred neighborhood $U(f(x))$ which is contained in the intersection of all $U_{i+1}(\sigma)$ for which $x \in \sigma \in T_{i+1}$. We cover the compact set $f(T^i_{i+1})$ by a finite number of these preferred neighborhoods. Choose a subdivision T_i of T^i_{i+1} so fine that every closed i-simplex σ of T_i has an image in some element, chosen once and for all, say $U_i(\sigma)$ of the last mentioned covering.

Next we consider the $i-1$-skeleton T^{i-1}_i of T_i and we continue analogously. In this manner we obtain a sequence of skeletons T_j of dimension $j = 0, 1, 2, \cdots, i+1$, and for each j-simplex σ_j of T_j a preferred neighborhood $U_j(\sigma_j)$ covering $f(\sigma_j)$, such that moreover $U_j(\sigma_j) \subset U_{j+1}(\sigma_{j+1})$ whenever $\sigma_j \subset \sigma_{j+1}$.

Now we are ready to start our homotopic changes of f leaving its restriction to S^i invariant. We will define a homotopy f_t starting at time $t = 0$ with $f_0 = f$ and ending at time $t = i+2$ with f_{i+2}. f_{i+2} will have the required properties:

$$f_{i+2}|S^i = f|S^i, \qquad f_{i+2}(D^{i+1}) \subset V-A.$$

The homotopy will be such that

(a) $f_{j+1}(T_j) \subset V-A$ for $j = 0, 1, \cdots, i+1$

(b) $f_t|T_j = f_{j+1}|T_j$ for $t \geqq j+1$

(c) $f_t(\sigma_j) \subset U_j(\sigma_j)$ for $\sigma_j \in T_j$, all j, all t

(d) $f_t|S^i = f|S^i$.

Suppose f_t has been defined with these properties for $t \leqq j$ ($\leqq i+1$). We first describe what happens with a j-simplex σ of T_j in the time from $t = j$ to $t = j+1$. Because $f_j(\sigma) \subset U_j(\sigma)$ and $f_j(\partial\sigma) \subset V-A$ and because $j \leqq i+1 \leqq k+1$, the assumption P_k implies the existence of $g_t : \sigma \to U_j(\sigma)$ $j \leqq t \leqq j+1$ with the properties:

$$g_j = f_j|\sigma, \quad g_t|\partial\sigma = g_j|\partial\sigma,$$
$$g_{j+1}(\sigma) \subset U_j(\sigma) - A \subset V-A.$$

(As a matter of fact only the injectivity and not the isomorphisms are used here, as well as for the surjectivity consequence.)

We have to extend g_t to all of D^{i+1}. For that we consider a subdivision of T_{i+1} which contains T_j as a subcomplex, and such that the stars of the j-simplices σ of T_j have mutually disjoint interiors. Star σ is the union of all simplices that contain σ. It is also the join of σ and the link of σ: $\text{St}\,\sigma = \sigma * \text{Lk}\,\sigma$ and it can be obtained from $\sigma \times [0, 1] \times \text{Lk}\,\sigma$ by identification to one point of the sets $x \times 0 \times \text{Lk}\,\sigma$ and $\sigma \times 1 \times y$ for each $x \in \sigma$ and each $y \in \text{Lk}\,\sigma$. The second factor gives a coordinate s on star σ. The triples (x, s, y) are supernumary "coordinates" for $\text{St}\,\sigma$.

In these coordinates we define for $j \leqq t \leqq j+1$:

$$f_t(x, s, y) = \begin{cases} g_{t-2s}(x) & \text{for} \quad j \leqq t-2s \\ f_j(x) & \text{for} \quad t-1 \leqq t-2s \leqq j \\ f_j((x, \tfrac{1}{2}s, y)) & \text{for} \quad \tfrac{1}{2} \leqq s \leqq 1. \end{cases}$$

We apply this to the stars of all j-simplices of T_j which are not in S^i. For the remaining points z we take

$$f_t(z) = f_j(z).$$

We now easily check that during this inductive step the conditions (a), (b), (c) and (d) remain valid for $j \leqq t \leqq j+1$. After the step $j = i+1$ we obtain the required map f_{i+2}.

To prove that $\pi_i(V-A) \to \pi_i(V)$ is surjective $(i \leqq k)$ it suffices to show that any map $f : S^i \to V$ can be deformed in V to a map into $V-A$; the same argument yields that (even for $i \leqq k+1$, but we do not need that). The lemma is proved, and so is Theorem 1.

2. Applications

EXAMPLE 1. *Submanifolds.*

Let E be a metrizable locally convex topological vector space. Suppose that X is a manifold modeled on E; more precisely, X is a paracompact Hausdorff space such that with every point $x \in X$ we can associate a chart (κ, U) i.e., a neighborhood U and a homeomorphism κ mapping U onto an open subset of E. *Every such manifold X is an absolute neighborhood retract* [3, 7].

Let A be a closed submanifold of X; thus A is a closed subset of X with the following property: There is a closed linear subspace F of E and for each $x \in A$ a chart (κ, U) containing x such that $\kappa(U \cap A) = \kappa(U) \cap F$. We will call $\dim E/F$ the *codimension* codim (X, A) *of A in X*. Since $E-F \to E$ is a weak k-homotopy equivalence for all $k \leqq \text{codim}\,(E, F)-2$, we conclude that A satisfies P_k for all $k \leqq \text{codim}\,(X, A)-2$. Thus we obtain for $k = \infty$:

THEOREM 2. *Any infinite codimensional closed submanifold A in X is homotopy negligible.*

EXAMPLE 2. *Fibre bundles and zero sections.*

Let $\xi : X \to B$ be a vector bundle over the base space B with total space the ANR X, whose fibres are infinite dimensional metrizable locally convex topological vector spaces. If A denotes the image of the zero section, then $X-A \to X$ is a homotopy equivalence; since ξ is also a homotopy equivalence, we see that $X-A$ and A have the same homotopy type.

EXAMPLE 3. *A mapping space.*

Let M be a separable smooth (i.e., C^∞) manifold modeled on a Hilbert space (possibly finite dimensional!), and S a compact topological space. Then the mapping space $C(S, M)$ of all continuous maps $S \to M$ with the topology of uniform convergence is a smooth manifold modeled on a Banach space (infinite dimensional except in trivial cases, which we exclude). The identification of each point $m \in M$ with the point mapping $S \to m$ defines an imbedding of M into $C(S, M)$ of infinite codimension. The theorem implies that $C(S, M) - M \to C(S, M)$ is a homotopy equivalence.

3. Approximately infinite codimension

The next lemma prepares the way to a more general class of homotopy negligible subsets. Let ρ be a metric for the model E.

Lemma. *Let A be a closed subset of the open subset U of E. Suppose that for every $\varepsilon > 0$ there is a closed linear subspace $F_{A,\varepsilon}$ of E with* codim $(E, F_{A,\varepsilon}) = \infty$ *and such that $\rho(x, F_{A,\varepsilon}) < \varepsilon$ for all $x \in A$. Let D be a compact subset of U, and S a closed subset of D disjoint from A. Then there is a homotopy $f_I : (D, S) \to (U, U-A)$ such that $f_0 =$ identity, $f_t(x) = x$ for all $x \in S$, $t \in I$, and $f_1(D) \subset U-A$.*

We will say that such subsets A are *approximately infinite codimensional in* E. For example, any *compact* set in E has this property.

Proof. Choose $\varepsilon > 0$ so that $\rho(A, S) > 5\varepsilon$ and $\rho(E-U, D) > 6\varepsilon$. The compact set D can be covered by a finite number of discs of radius ε; their centers span a finite dimensional subspace F_D, and $\rho(x, F_D) < \varepsilon$ for all $x \in D$. Let F be the closed linear subspace spanned by $F_{A,\varepsilon}$ and F_D; then codim $(E, F) = \infty$, and $\rho(x, F) < \varepsilon$ for all $x \in A \cup D$. Now take any point x_2 with $\rho(x_2, F) = 3\varepsilon$, and let $x_1 \in F$ be a point such that $\rho(x_1, x_2) < 4\varepsilon$. Set $v = x_2 - x_1$, and define the map $f_1 : D \to E$ by

$$f_1(x) = x + \varphi(\rho(x, S)/\varepsilon)v,$$

where $\varphi : \boldsymbol{R} \to \boldsymbol{R}$ is given by

$$\varphi(t) = 0 \ (t \leqq 0), \ \varphi(t) = t(0 < t < 1), \ \varphi(t) = 1(t \geqq 1).$$

Then $f_t = (1-t)f_0 + tf_1$, where $f_0 =$ identity map, is a homotopy of the desired sort.

Theorem 3. *Let A be a closed subset of a manifold X modeled on E. Assume that every point $x \in A$ is contained in a chart (κ, U) of X such that $\kappa(U \cap A)$ is approximately infinite codimensional. Then A is homotopy negligible in X. For example, any locally compact set in X has this property.*

Proof. Since $\kappa(U \cap A)$ is closed in the open subset $\kappa(U)$ of the model E, it follows from the lemma that all relative homotopy groups $\pi_i(U, U-A) = 0$ $(i \in \boldsymbol{Z})$, whence that the inclusion map $U-A \to U$ is a weak homotopy equivalence. Since for any chart (κ, U) containing x we know that U and $U-A$ are absolute neighborhood retracts, it follows that $U-A \to U$ is a homotopy equivalence. Therefore A is homotopy negligible in X.

Application. (Nicole Moulis [6]). *If X is a separable Hilbert manifold and X is diffeomorphic with $X \times H$* (H is Hilbert space; X is called *Palais-stable* in this case) *and A is locally compact in X, then X and $X-A$ are diffeomorphic.*

Proof. By the above theorems X and $X-A$ are homotopy equivalent. As X is Palais-stable it is diffeomorphic with an open set in H. Then so is $X-A$. By Moulis [6], $X-A$ has a non-degenerate Morse function with minimum fulfilling Condition C of Palais-Smale. By Kuiper-Burghelea [5], $X-A$ is also Palais-stable, and $X-A$ and X are diffeomorphic.

Example. If A is an approximately infinite codimensional closed subset of a contractible manifold X, then $X-A$ is contractible.

As another instance, let H be an infinite dimensional Hilbert space and $X = GL(H)$ the group of bounded linear automorphisms of H; then X is an open nonseparable subset of the Banach space of bounded endomorphisms of H, and $GL(H)$ is a contractible Lie group [4]. Let $GL_c(H)$ denote the closed subgroup of automorphisms of the form $I+K$ where $I =$ identity map, and K is a compact endomorphism. Then $GL(H)-GL_c(H)$ is contractible.

Added in proof.

1) If X is a separable metrizable C^0-manifold modeled on an infinite dimensional Fréchet space, and A is a closed subset as in Theorem 1, then there is a homeomorphism of $X-A$ onto X; see [1].

2) Very general conditions to insure that $X-A$ is homeomorphic to X (with good control on the homeomorphism) have recently been given by W. H. Cutler, *Negligible subsets of non-separable Hilbert manifolds* (to appear). For instance, if X is a C^0-manifold modeled on a non-separable Hilbert space and A is a countable union of locally compact subsets, then $X-A$ is homeomorphic to X, by a homeomorphism near the inclusion map $X-A \to X$.

3) Every separable metrizable C^∞-manifold modeled on infinite dimensional Hilbert space is Palais-stable. This follows from combined work of [5], [6], and J. Eells–K. D. Elworthy, *On the differential topology of Hilbertian manifolds*, Proc. Summer Institute, Berkeley, 1968.

4) The application of § 3 (without the unnecessary hypothesis of Palais-stability) is also due to J. E. West, *The diffeomorphic excision of closed local compacta from infinite-dimensional Hilbert manifolds* (to appear). Comp. Math.

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