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Convergence of interpolatory polynomials on Tchebycheff abscissas

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by

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1

A systematic study of Lacumary interpolation [2] was first initiated by Suranyi, J. and Turán, P. [8] and Balázs and Turán, P. [1, 2, 3] in the special case when the values and second derivatives are prescribed on the zeros of $\pi_n(x) = (1-x^2)P'_{n-1}(x)$ where $P_{n-1}(x)$ is the Legendre polynomial of degree $\leq n-1$ while the existence and uniqueness have been shown for the abscissas as the zeros of ultraspherical polynomials $P_n^{(\lambda)}(x)$, $\lambda \geq -\frac{1}{2}$ the explicit representation and convergence theorems have been proved for π abscissas only. Later the convergence theorem of Balázs and Turán [3] is sharpened by Freud, G. [4] in the sense that the interpolatory polynomials of Balázs and Turán converges uniformly to given $f(x)$ in $[-1, +1]$ if $f(x)$ satisfies the Zygmund condition

$$|f(x+h) - 2f(x) + f(x-h)| = o(h) \quad \text{in} \quad [-1, +1].$$

Other interesting results are due to Saxena and Sharma [7, 8], Kis [5, 6], Varma and Sharma [11, 12] and Varma [15, 16].

The object of this paper is to consider the problem of existence, uniqueness, explicit representation and convergence of the sequence $R_n(x)$ of polynomials of degree $\leq 3n+3$ such that $R_n(x)$, $R'_n(x)$ are prescribed at the zeros of $(1-x^2)u_n(x)$ where

$$u_n(x) = \frac{\sin (n+1)\theta}{\sin \theta}, \qquad x = \cos \theta,$$

while $R'''_n(x)$ is prescribed at all the above abscissas except at -1 and $+1$. We shall call this "modified" (0, 1, 3) interpolation.

In § 2 we state the existence theorem and give the explicit rep-

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[2] They called it (0, 2) case.

resentation of these polynomials in a most suitable form and in § 3 and onwards we prove convergence theorem. It is interesting to remark that in modified $(0, 2)$ interpolation [15] we require for the uniform convergence of the sequence of polynomials $R_n(x)$ to $f(x)$ is that $f'(x) \in$ Lip α, $\alpha > \frac{1}{2}$, and this is best possible in a certain sense. So one would be inclined to think that in modified $(0, 1, 3)$ interpolation, we may require $f(x)$ to be twice differentiable or at least $f'(x) \in$ Lip α, $\alpha > \frac{1}{2}$ [compare corresponding theorem of Saxena and Sharma [7]]. But our theorem 3.1 asserts that this is not really the case. Here we need only $f'(x) \in$ Lip α, $\alpha > 0$. Although we could not prove that this is best possible, it seems quite plausible that this is really so in view of other known results in this direction [7, 8].

<div align="center">2</div>

Let us consider the set of numbers

(2.1) $-1 = x_{n+2} < x_{n+1} < \ldots < x_2 < x_1 = +1$

by which we shall denote the zeros of $(1-x^2)u_n(x)$, where

$$u_n(x) = \frac{\sin (n+1)\theta}{\sin \theta}, \qquad x = \cos \theta.$$

Then we have the following

THEOREM 2.1 *If* $n = 2k$, *then to prescribed values* $f(x_i)$, $f'(x_i)$ $(i = 1, 2, \ldots n+2)$ *and* δ_i $(i = 2, 3, \ldots, n+1)$ *there is a uniquely determined polynomial* $R_n(x)$ *of degree* $\leq 3n+3$ *such that*

(2.2) $R_n(x_i) = f(x_i), \quad R_n'(x_i) = f'(x_i), \quad i = 1, 2, \ldots, n+2$

(2.3) $R_n''(x_i) = \delta_i, \qquad\qquad\qquad i = 2, 3, \ldots, n+1.$

But if n *is odd,* $(n = 2k+1)$ *there is in general no polynomial* $R_n(x)$ *of degree* $\leq 3n+3$ *which satisfies* (2.2) *and* (2.3) *and if there exists such a polynomial then there is an infinity of them.*

From the uniqueness theorem it follows that $R_n(x)$ is given by

(2.4) $R_n(x) = \sum_{i=1}^{n+2} f(x_i) A_i(x) + \sum_{i=1}^{n+2} f'(x_i)B_i(x) + \sum_{i=2}^{n+1} \delta_i C_i(x)$

where the polynomials $A_i(x)$, $B_i(x)$ and $C_i(x)$ are the fundamental polynomials of degree $\leq 3n+3$. Their explicit forms are given by the following

THEOREM 2.2 *For n even, the fundamental polynomials have the following representation*

(a) *For $i = 2, 3, \ldots, n+1$ we have*

$$(2.5) \quad C_i(x) = \frac{(1-x^2)^{\frac{3}{2}} u_n^2(x)}{6(1-x_i^2) u_n'^2(x_i)} \left[a_i \int_{-1}^{x} \frac{u_n(t)}{(1-t^2)^{\frac{1}{2}}}\, dt + \int_{-1}^{x} \frac{l_i(t)}{(1-t^2)^{\frac{1}{2}}}\, dt \right]$$

where

$$(2.6) \quad \pi a_i = -\int_{-1}^{+1} \frac{l_i(t)}{(1-t^2)^{\frac{1}{2}}}\, dt, \qquad l_i(t) = \frac{u_n(t)}{(t-x_i) u_n'(x_i)}.$$

(b)

$$(2.7) \quad B_1(x) = \frac{(1-x^2) u_n^2(x)}{4(n+1)^3} \left[(1+x) u_n(x) + (1-x^2) u_n'(x) \right. $$
$$\left. + (1-x^2)^{\frac{1}{2}} \int_{-1}^{x} \frac{u_n'(t)}{(1-t^2)^{\frac{1}{2}}}\, dt \right]$$

$$(2.8) \quad B_{n+2}(x) = \frac{(1-x^2) u_n^2(x)}{4(n+1)^3} \left[(1-x) u_n(x) - (1-x^2) u_n'(x) \right.$$
$$\left. + (1-x^2)^{\frac{1}{2}} \int_{-1}^{x} \frac{u_n'(t)}{(1-t^2)^{\frac{1}{2}}}\, dt \right]$$

and for $i = 2, 3, \ldots, n+1$ we have

$$(2.9) \quad B_i(x) = \frac{(1-x^2)^2 l_i^2(x) u_n(x)}{(1-x_i^2)^2 u_n'(x_j)} + \frac{(1-x^2)^{\frac{3}{2}} u_n^2(x)}{(n+1)^2} \left[b_i \int_{-1}^{x} \frac{u_n(t)}{(1-t^2)^{\frac{1}{2}}}\, dt \right.$$
$$\left. + c_i \int_{-1}^{x} \frac{l_i(t)}{(1-t^2)^{\frac{1}{2}}}\, dt + \int_{-1}^{x} \frac{p_i(t)}{(1-t^2)^{\frac{1}{2}}}\, dt \right]$$

where

$$(2.10) \quad p_i(t) = \frac{3x_i l_i(t) - 2(1-t^2) l_i'(t)}{2(t-x_i)} = \frac{2(1-x_i^2)}{(n+1)} \sum_{r=0}^{n-1} \alpha_r u_r(x_i) u_r(t)$$
$$\alpha_r = n(n+2) - r(r+2) - 3$$

$$(2.11) \quad c_i = \frac{(n+3)(n-1)}{6} + \frac{x_i^2}{2(1-x_i^2)}$$

$$(2.12) \quad \pi b_i = -\int_{-1}^{+1} \frac{p_i(t)}{(1-t^2)^{\frac{1}{2}}}\, dt - c_i \int_{-1}^{+1} \frac{l_i(t)}{(1-t^2)^{\frac{1}{2}}}\, dt$$

(c)

$$(2.13) \quad A_1(x) = \frac{(1+x)^2 u_n^3(x)}{4(n+1)^3} - (n+1)^2 B_1(x) - \frac{(1-x^2)^2 u_n^2(x) u_n'(x)}{8(n+1)^3}$$
$$- \frac{(1-x^2)^{\frac{3}{2}} u_n^2(x)}{4(n+1)^3} \int_{-1}^{x} \frac{(1+t^2) u_n'(t)}{(1-t^2)^{\frac{1}{2}}}\, dt$$

$$(2.14) \quad A_{n+2}(x) = \frac{(1-x)^2 u_n^3(x)}{4(n+1)^3} + (n+1)^2 B_1(x) + \frac{(1-x^2)^2 u_n^2(x) u_n'(x)}{8(n+1)^3}$$
$$- \frac{(1-x^2)^{\frac{3}{2}} u_n^2(x)}{4(n+1)^3} \int_{-1}^{x} \frac{(1+t^2) u_n'(t)}{(1-t^2)^{\frac{1}{2}}} \, dt$$

and for $i = 2, 3, \ldots, n+1$ we have

$$(2.15) \quad A_i(x) = \frac{(1-x^2)^2 l_i^3(x)}{(1-x_i^2)^2} + \frac{x_i B_i(x)}{2(1-x_i^2)} + \frac{(1-x^2)^{\frac{3}{2}} u_n^2(x)}{(n+1)^2}$$
$$\cdot \left[d_i \int_{-1}^{x} \frac{u_n(t)}{(1-t^2)^{\frac{1}{2}}} \, dt + c_i' \int_{-1}^{x} \frac{l_i(t) dt}{(1-t^2)^{\frac{1}{2}}} + \int_{-1}^{x} \frac{q_i(t)}{(1-t^2)^{\frac{1}{2}}} \, dt \right]$$

where

$$(2.16) \quad c_i = 3x_i \left[\frac{(n+3)(n-1)-6}{4(1-x_i^2)} + \frac{37x_i^2+36}{12(1-x_i^2)^2} \right]$$

$$(2.17) \quad \pi d_i = - \int_{-1}^{+1} \frac{q_i(t)}{(1-t^2)^{\frac{1}{2}}} \, dt - c_i' \int_{-1}^{+1} \frac{l_i(t)}{(1-t^2)^{\frac{1}{2}}} \, dt$$

$$(2.18) \quad q_i(t) = \frac{(a_i't+b_i')l_i(t) - (1-t^2)l_i'(t)}{(t-x_i)^2}$$

$$(2.19) \quad a_i'x_i + b_i' = \frac{3x_i}{2}$$

$$(2.20) \quad a_i' = \left[\frac{-x_i^2}{4(1-x_i^2)} - \frac{(n+3)(n-1)}{3} \right].$$

In view of the uniqueness theorem 2.1, it remains to verify that the fundamental functions $A_i(x)$, $B_i(x)$ and $C_i(x)$ are polynomials of degree $\leq 3n+3$ and satisfy the following conditions.

$$(2.21) \quad A_i(x_j) = \begin{cases} 1 & i=j \\ 0 & i \neq j \end{cases} \quad A_i'(x_j) = 0 \quad i, j = 1, 2, \ldots, n+2$$

$$(2.22) \quad A_i'''(x_j) = 0 \qquad \begin{aligned} i &= 1, 2, \ldots, n+2 \\ j &= 2, 3, \ldots, n+1 \end{aligned}$$

$$(2.23) \quad B_i(x_j) = 0, \quad B_i'(x_j) = \begin{cases} 1 & i=j \\ 0 & i \neq j \end{cases} \quad i, j = 1, 2, \ldots, n+2$$

$$(2.24) \quad B_i'''(x_j) = 0 \qquad \begin{aligned} i &= 1, 2, \ldots, n+2 \\ j &= 2, 3, \ldots, n+1 \end{aligned}$$

$$(2.25) \quad C_i(x_j) = C'_j(x_j) = 0 \qquad \begin{aligned} & i = 2, 3, \ldots, n+1 \\ & j = 1, 2, \ldots, n+2 \end{aligned}$$

$$(2.26) \quad C'''_i(x_j) = \begin{cases} 1 & i = j \\ 0 & i \neq j \end{cases} \qquad j, i = 2, 3, \ldots, n+1.$$

To verify that the above conditions are satisfied by $A_i(x)$, $B_i(x)$, and $C_i(x)$ as given in the above theorem, we proceed on the lines of [15] and show that they are polynomials of degree $\leq 3n+3$.

<div align="center">3</div>

Let $f(x)$ be continuously differentiable in $[-1, +1]$ and consider the sequence of polynomials

$$(3.1) \quad R_n(x, f) = \sum_{i=1}^{n+2} f(x_{in}) A_{in}(x) + \sum_{i=1}^{n+2} f'(x_{in}) B_{in}(x) + \sum_{i=2}^{n+1} \delta_{in} C_{in}(x)$$

with arbitrary numbers δ_{in}. We shall prove the following

THEOREM 3.1 *Let $f(x)$ have a continuous first derivative in $[-1, +1]$ and let $f'(x) \in \text{Lip } \alpha$, $\alpha > 0$ and if*

$$(3.2) \qquad\qquad |\delta_{in}| = \frac{o(n^2)}{(1 - x_{in}^2)} \qquad i = 2, 3, \ldots, n+1$$

then the sequence $R_n(x, f)$ converges uniformly to $f(x)$ in $[-1, +1]$.

4. Preliminary results

LEMMA 4.1 *For $x = \cos \theta$ we have*

$$(4.1) \qquad \int_{-1}^{x} \frac{u_{2r}(t)}{(1 - t^2)^{\frac{1}{2}}} \, dt = \pi - \theta - 2 \sum_{j=1}^{r} \frac{\sin(2j-1)\theta}{2j} \, ;$$

$$(4.2) \qquad \int_{-1}^{x} \frac{u_{2r-1}(t)}{(1 - t^2)^{\frac{1}{2}}} \, dt = -2 \sum_{j=1}^{r} \frac{\sin(2j-1)\theta}{2j-1} \, .$$

LEMMA 4.2 *For $-1 \leq x \leq +1$, $x = \cos \theta$ we have*

$$(4.3) \qquad \left| \int_{-1}^{x} \frac{u_p(t)}{(1 - t^2)^{\frac{1}{2}}} \, dt \right| \leq 12.$$

$$(4.4) \qquad \left| \int_{-1}^{x} \frac{l_{in}(t)}{(1 - t^2)^{\frac{1}{2}}} \, dt \right| \leq \frac{35}{n} \qquad i = 2, 3, \ldots, n+1.$$

$$(4.5) \qquad 0 \leqq \int_{-1}^{+1} \frac{l_{in}(t)}{(1-t^2)^{\frac{1}{2}}} \, dt = \frac{\pi}{(n+1)} \, (1-\cos n\theta_{in}) \leqq \frac{7}{n}$$

$$(4.6) \qquad \sum_{i=2}^{n+1} \frac{(1-x^2)\, l_{in}^2(x)}{(1-x_i^2)} \leqq 2.$$

$$(4.7) \qquad \left| \int_{-1}^{x} \frac{u_n'(t)}{(1-t^2)^{\frac{1}{2}}} \, dt \right| \leqq 6(n+1)^2$$

$$(4.8) \qquad \left| \int_{-1}^{x} \frac{p_{in}(t)}{(1-t^2)^{\frac{1}{2}}} \, dt \right| \leqq 8(n+1), \qquad i = 2, 3, \ldots, n+1$$

where $p_{in}(t)$ is defined by (2.10).

PROOF. Inequality (4.6) is due to P. Szasz [9]. (4.3) is immediate from (4.1) and (4.2). Since

$$(4.9) \qquad u_n'(x) = 4 \sum_{j=1}^{n/2} j u_{2j-1}(x) \qquad\qquad (n \text{ even})$$

we get (4.7) from (4.9) and (4.3). From a result of Fejer we have

$$(4.10) \qquad l_{in}(x) = \frac{2(1-x_{in}^2)}{(n+1)} \sum_{r=0}^{n-1} u_r(x_{in}) u_r(x), \qquad i = 2, 3, \ldots, n+1,$$

$$= \frac{1}{(n+1)} \Big[1+2 \sum_{r=1}^{n-1} \cos r\theta_{in} \cos r\theta - \cos n\theta_{in}\big(u_{n-2}(x) + \cos \theta_{in} u_{n-1}(x)\big) \Big]$$

Now using (4.3) we get (4.4) and (4.5). From (2.10) we have

$$(4.11) \qquad p_{in}(t) = \frac{1}{(n+1)} \Big[\sum_{r=1}^{n-1} \big(4r u_{r-2}(t) + 2\alpha_r T_r(t) \big) \cos r\theta_{in} - \alpha_0$$

$$- \cos n\theta_{in}\big(u_{n-2}(t) + \cos \theta_{in} u_{n-1}(t)\big) \Big]$$

where

$$T_r(t) = \cos r\theta \quad \text{and} \quad \int_{-1}^{x} \frac{T_r(t)dt}{(1-t^2)^{\frac{1}{2}}} = \frac{\sin r\theta}{r}, \quad \cos \theta = x.$$

Further using (4.3) we get (4.8).

In the estimation of $\sum_{i=1}^{n+2} |A_{in}(x)|$ we need the following results. Let

$$(4.12) \qquad k_1(t) \equiv k_1(x_{in}, t) = \sum_{r=1}^{n-1} u_r(x_{in}) u_r'(t)$$

$$(4.13) \qquad k_2(t) \equiv k_2(x_{in}, t) = \sum_{r=0}^{n-1} (e_r-1) u_r(x_{in}) u_r(t)$$

$$(4.14) \qquad k_3(t) \equiv k_3(x_{in}, t) = \sum_{r=1}^{n-1} e_r(1-x_{in}^2) u_r(x_{in}) u_r'(t)$$

where

$$(4.15) \qquad e_r = n(n+2) - r(r+2).$$

Now we prove the following

LEMMA 4.3 *For* $-1 \leqq x \leqq +1$, *we have*

$$(4.16) \qquad \left| \int_{-1}^{x} \frac{k_1(t)}{(1-t^2)^{\frac{1}{2}}} \, dt \right| \leqq \frac{36n^2}{(1-x_{in}^2)}, \qquad i = 2, 3, \ldots, n+1$$

$$(4.17) \qquad \left| \int_{-1}^{x} \frac{k_2(t)}{(1-t^2)^{\frac{1}{2}}} \, dt \right| \leqq \frac{36n^2}{(1-x_{in}^2)}, \qquad i = 2, 3, \ldots, n+1$$

and

$$(4.18) \qquad \left| \int_{-1}^{x} \frac{k_3(t)}{(1-t^2)^{\frac{1}{2}}} \, dt \right| \leqq \frac{70(n+1)^3}{\sqrt{-x_{in}^2}}, \qquad i = 2, 3, \ldots, n+1.$$

PROOF. Proof of (4.16) and (4.17) are similar to (4.8). A simple computation leads

$$4k_3(t) = \sum_{r=1}^{n-1} \beta_r(t) u_r(x_{in}) + k_4(t)$$

where

$$k_4(x_{in}, t) = k_4(t)$$
$$= -e_2 u_2'(t) - \cos(n+1)\theta_{in}(e_{n+1} u_{n+1}'(t) + e_{n-1} u_{n-1}'(t))$$

and

$$\beta_r(t) = 8u_r'(t) - 2(r+2)e_{r+2}T_{r+1}(t) + 4(5r(r+2) - n(n+2) + 8)u_{r-1}(t).$$

Using (4.15), (4.7) and (4.3) we have

$$(4.19) \qquad \left| \int_{-1}^{x} \frac{k_4(t)}{(1-t^2)^{\frac{1}{2}}} \, dt \right| \leqq 72(n+1)^3$$

and

$$\left| \sum_{r=1}^{n-1} u_r(x_{in}) \int_{-1}^{x} \frac{\beta_r(t)}{(1-t^2)^{\frac{1}{2}}} \, dt \right| \leqq \frac{16(n+1)^3}{\sqrt{1-x_{in}^2}} + \frac{2(n+1)^3}{\sqrt{1-x_{in}^2}} + \frac{192(n+1)^3}{\sqrt{1-x_{in}^2}}$$
$$= \frac{210(n+1)^3}{\sqrt{1-x_{in}^2}}.$$

Therefore

$$\left| \int_{-1}^{x} \frac{k_3(t)}{\sqrt{1-t^2}} \, dt \right| \leqq \frac{70(n+1)^3}{\sqrt{1-x_{in}^2}}.$$

LEMMA 4.4 *For* $-1 \leqq x \leqq +1$ *we have*

$$(4.20) \quad \left| \int_{-1}^{x} \frac{q_{in}(t)}{\sqrt{1-t^2}} \, dt \right| \leqq \frac{36(n+1)}{1-x_{in}^2} + \frac{23(n+1)^2}{(1-x_{in}^2)^{\frac{1}{2}}}$$

$$i = 2, 3, \ldots, n+1$$

where $q_{in}(t)$ *is defined by* (2.18).

PROOF. It can be shown that [15]

$$(4.21) \quad q_{in}(t) = \frac{x_{in}}{6(1-x_{in}^2)} \frac{u_n'(t)}{u_n'(x_{in})} + \frac{1}{3(n+1)}$$

$$\cdot \, [2x_{in}k_1(t) - x_{in}k_2(t) + k_3(t)].$$

Now using (4.7) and Lemma (4.3) we get (4.20).

5

Here we shall investigate the estimation of the fundamental polynomials.

LEMMA 5.1 *For* $-1 \leqq x \leqq +1$ *we have*

$$(5.1) \quad |C_{in}(x)| \leqq \frac{10(1-x_{in}^2)}{n^3}, \qquad i = 2, 3, \ldots, n+1, \ n = 4, 6, \ldots$$

$$(5.2) \quad \sum_{i=2}^{n+1} (1-x_{in}^2)^{-1} |C_{in}(x)| \leqq 10n^{-2}.$$

LEMMA 5.2 *For* $n = 4, 6, \ldots$ *and for* $-1 \leqq x \leqq +1$ *we have*

$$(5.3) \quad |B_{1n}(x)| \leqq \frac{3}{n}, \qquad |B_{n+2,n}(x)| \leqq \frac{3}{n}$$

$$(5.4) \quad |B_{in}(x)| \leqq \frac{(1-x^2)\,l_{in}^2(x)}{n(1-x_{in}^2)} + \frac{80}{n} \qquad i + 2, 3, \ldots, n+1$$

and

$$(5.5) \quad \sum_{i=1}^{n+2} |B_{in}(x)| \leqq 88.$$

The proof of these two lemmas follows very easily from Lemma 4.2 and (2.5)—(2.12).

LEMMA 5.3 *For* $-1 \leqq x \leqq +1$ *we have*

$$(5.6) \quad |A_{1n}(x)| \leqq 7n, \quad |A_{n+2n}(x)| \leqq 7n$$

$$(5.7) \quad |A_{in}(x)| \leqq \frac{6n(1-x^2)\,l_{in}^2(x)}{1-x_{in}^2} + \frac{1400}{(n+1)(1-x_{in}^2)} + \frac{119}{\sqrt{1-x_{in}^2}}$$

and

$$(5.8) \qquad \sum_{i=1}^{n+2} |A_{in}(x)| \leqq 1626\, n \log n.$$

PROOF. (5.6) follows easily from (2.13), (2.14) and (4.7). From (2.16)—(2.18) and (4.20) we have

$$(5.9) \qquad |c_i'| \leqq \frac{20(n+1)^2}{(1-x_{in}^2)} \qquad i = 2, 3, \ldots, n+1$$

$$(5.10) \qquad |d_i'| \leqq \frac{52(n+1)}{1-x_{in}^2} + \frac{8(n+1)^2}{\sqrt{1-x_{in}^2}}$$

$$(5.11) \qquad \left| \frac{(1-x^2)l_{in}(x)}{(1-x_{in}^2)} \right| \leqq 5 \qquad -1 \leqq x \leqq +1.$$

Therefore using (5.4), lemma 4.2

$$|A_{in}(x)| \leqq 6n\, \frac{(1-x^2)l_{in}^2(x)}{1-x_{in}^2} + \frac{1400}{(n+1)(1-x_{in}^2)} + \frac{119}{\sqrt{1-x_{in}^2}}$$

$$\therefore \sum_{i=2}^{n+1} |A_{in}(x)| \leqq 12n + 1400(n+1) + 119n \log n \leqq 1612n \log n.$$

LEMMA 5.4 *Let* $f'(x) \in \text{Lip } \alpha, \, 0 < \alpha < 1$ *in* $[-1, +1]$. *Then there exists a sequence of polynomial* $\{\varphi_n(x)\}$ *of degree at most* n *with the following properties for* $-1 \leqq x \leqq +1$

$$(5.12) \qquad |f(x) - \varphi_n(x)| \leqq \frac{c}{n^{1+\alpha}} \left[(\sqrt{1-x^2})^{1+\alpha} + \frac{1}{n^{1+\alpha}} \right]$$

$$(5.13) \qquad |f'(x) - \varphi_n'(x)| \leqq \frac{c_1}{n^\alpha} \left[(\sqrt{1-x^2})^\alpha + \frac{1}{n^\alpha} \right]$$

$$(5.14) \qquad |\varphi_n'''(x)| \leqq \frac{n^{2-\alpha}}{(1-x^2)} \quad \text{in} \quad -1 < x < +1.$$

PROOF. The existence of $\varphi_n(x)$ satisfying (5.12) and (5.13) are well known [see Timan [14]]. (5.14) follows closely on the lines of Freud G. [4].

6

PROOF OF THEOREM 3.1 From the uniqueness theorem we have

$$R_n(x,f) - f(x) = \sum_{i=1}^{n+2} [f(x_{in}) - \varphi_n(x_{in})]A_{in}(x)$$

$$+ \sum_{i=1}^{n+2} [f'(x_{in}) - \varphi_n'(x_{in})]B_{in}(x) + \sum_{i=2}^{n+1} [\delta_{in} - \varphi_n'''(x_{in})]C_{in}(x)$$

$$+ \varphi_n(x) - f(x) = I_1 + I_2 + I_3 + I_4 \qquad \text{say.}$$

From (5.8) and (5.12) $I_1 = c/n^{1+\alpha}$ 1616 $n \log n = o(1)$.
From (5.13) and (5.5) we have $I_2 = o(1)$.
Using (5.2) and (3.2), and (5.14) we have immediately

$$I_3 = \sum_{i=2}^{n+1} \left[\frac{o(n^2)}{(1-x_{in}^2)} + \frac{n^{2-\alpha}}{(1-x_{in}^2)} \right] |C_{in}(x)|$$
$$= o(1) \quad \text{for} \quad 0 < \alpha < 1.$$

and lastly from (5.12) $I_4 = o(1)$.
   Thus $R_n(f, x) - f(x) = o(1)$ which proves the theorem.

7

   Here we shall consider existence and uniqueness theorem for modified $(0, 1, 3)$ interpolation when nodes are taken as zeros of ultraspherical polynomials. Since λ and n are fixed, we shall denote $P_n^{(\lambda)}(x) = \varphi_n(x)$. It is well known [11] that the differential equation for $\varphi_n(x)$ for all non-negative integers n's is given by

(7.1) $(1-x^2)\varphi_n''(x) - (2\lambda+1)x\varphi_n'(x) + n(n+2\lambda)\varphi_n(x) = 0.$

It is also known [11] that all the zeros of $\varphi_n(x)$ are real, simple and lying in $-1 < x < 1$.
   We shall prove the following theorem (a special case of which is theorem 2.1 which corresponds to $\lambda = 1$).

   THEOREM 7.1 *Let* $n = 2k$, *and* $\lambda \neq \pm\frac{1}{4}$,

$$\lambda \neq \frac{m-1}{2}, \qquad m = 1, 2, \ldots, n+2,$$

then to prescribed values $a_i, b_i, (i = 1, 2, \ldots, n+2)$, c_i $(i = 2, 3, \ldots, n+1)$ *there is a uniquely determined polynomial* $f(x)$ *of degree* $\leq 3n+3$ *such that*

(7.2) $f(x_i) = a_i, f'(x_i) = b_i, \quad i = 1, 2, \ldots, n+2$

(7.3) $f'''(x_i) = c_i \qquad\qquad i = 2, 3, \ldots, n+1.$

Here x_i's are zeros of $(1-x^2)\varphi_n(x)$

(7.4) $+1 = x_1 > x_2 > \ldots \qquad > x_{n+1} > x_{n+2} = -1$

   PROOF. Proof of theorem 7.1 is on the lines of the paper [9] by J. Suranyi and P. Turán. We shall show that in the case

(7.5) $\begin{aligned} f(x_i) &= f'(x_i) = 0, \quad i = 1, 2, \ldots, n+2, \\ f'''(x_i) &= 0, \qquad\qquad i = 2, 3, \ldots, n+1 \end{aligned}$

the only polynomial of degree $\leqq 3n+3$ is $f(x) \equiv 0$. Thus, using first part of (7.5) we have

$$(7.6) \qquad f(x) = (1-x^2)^2 \varphi_n^2(x) r_{n-1}(x)$$

where $r_{n-1}(x)$ is a polynomial in x of degree $\leqq n-1$. Also, $f'''(x_i) = 0$, $i = 2, 3, \ldots, n+1$ and since the zeros of $\varphi_n(x)$ are simple, we obtain

$$(7.7) \quad (1-x^2) r'_{n-1}(x_i) + (2\lambda-3) x_i r_{n-1}(x_i) = 0 \quad i = 2, 3, \ldots, n+1.$$

Since the polynomial

$$(1-x^2) r'_{n-1}(x) + (2\lambda-3) x r_{n-1}(x)$$

is of degree $\leqq n$, and by (7.7) all its zeros are the same as those of $\varphi_n(x)$, we obtain

$$(7.8) \qquad (1-x^2) r'_{n-1}(x) + (2\lambda-3) x r_{n-1}(x) = c \varphi_n(x)$$

with numerical c.

Now we have to investigate whether or not the equation (7.8) has a polynomial solution of degree $\leqq n-1$ (n even). We try to solve the equation by

$$(7.9) \qquad r_{n-1}(x) = \sum_{i=0}^{n-1} c_i \varphi_i(x).$$

We shall be using the identities for $i \geqq 1$ [see Szegö [11]].

$$(7.10) \quad (1-x^2) \varphi_i'(x) = \frac{1}{2(i+\lambda)} \{ (i+2\lambda-1)(i+2\lambda) \varphi_{i-1}(x) \\ -i(i+1) \varphi_{i+1}(x) \}$$

and

$$(7.11) \quad x \varphi_i(x) = \frac{1}{2(i+\lambda)} \{ (i+1) \varphi_{i+1}(x) + (i+2\lambda-1) \varphi_{i-1}(x) \}.$$

Substituting (7.9) in (7.8) we obtain

$$c \varphi_n(x) = \sum_{i=1}^{n-1} c_i (1-x^2) \varphi_i'(x) + \sum_{i=0}^{n-1} c_i (2\lambda-3) x \varphi_i(x)$$

and using (7.10) (7.11)

$$c\varphi_n(x) = \sum_{i=1}^{n-1} \frac{c_i}{2(i+\lambda)} \{(i+2\lambda-1)(i+2\lambda)\varphi_{i-1}(x) - i(i+1)\varphi_{i+1}(x)\}$$

$$+ \frac{c_0(2\lambda-3)\varphi_1(x)}{2\lambda} + \sum_{i=1}^{n-1} \frac{c_i(2\lambda-3)}{2(i+\lambda)}$$

$$\cdot \{(i+1)\varphi_{i+1}(x) + (i+2\lambda-1)\varphi_{i-1}(x)\}$$

$$= \frac{c_0(2\lambda-3)}{2\lambda}\varphi_1(x) + \sum_{i=1}^{n-1} \frac{c_i(i+1)(2\lambda-3-i)}{2(i+\lambda)}\varphi_{i+1}(x)$$

$$+ \sum_{i=1}^{n-1} \frac{c_i(i+2\lambda-1)(i+4\lambda-3)}{2(i+\lambda)}\varphi_{i-1}(x)$$

$$= \frac{c_0(2\lambda-3)\varphi_1(x)}{2\lambda} + \sum_{i=2}^{n} \frac{ic_{i-1}(2\lambda-i-2)\varphi_i(x)}{2(i-1+\lambda)}$$

$$+ \sum_{i=0}^{n-2} c_{i+1} \frac{(i+2\lambda)(i+4\lambda-2)\varphi_i(x)}{2(i+1+\lambda)} .$$

We have to compare the coefficients of $\varphi_i(x)$ in (7.12). Comparing the coefficient of $\varphi_{n-1}(x)$ we find for $n \geqq 4$

(7.13) $$0 = \frac{c_{n-2}(n-1)(2\lambda-n-1)}{2} .$$

If $n \geqq 4$ and $n \neq 2\lambda-1$ then (7.13) implies

(7.14) $$c_{n-2} = 0.$$

Comparing the coefficients of $\varphi_0(x)$ in (7.12) we obtain

$$0 = \frac{c_1(2\lambda)(2\lambda-1)}{\lambda+1} .$$

From the conditions imposed in λ in theorem 7.1

$$(\lambda \neq 0, \lambda \neq \tfrac{1}{2} \lambda > -\tfrac{1}{2})$$

we have

(7.15) $$c_1 = 0.$$

If $n \geqq 4$ and $i = 1, 2, \ldots, n-2$, the comparison of the coefficients of $\varphi_i(x)$ in (7.12) gives

(7.16) $$0 = \frac{ic_{i-1}(2\lambda-i-2)}{2(i-1+\lambda)} + \frac{(i+2\lambda)(i+4\lambda-2)}{2(i+1+\lambda)} c_{i+1}.$$

Evidently c_{i+1} can be expressed always by c_{i-1} from (7.16).
    Starting from $c_1 = 0$ we have

(7.18) $$c_1 = c_3 = \ldots = c_{n-1} = 0 \qquad (n \text{ even}).$$

Similarly starting from (7.14) (i.e. $c_{n-2} = 0$) and using (7.16) we have

(7.19) $c_{n-2} = c_{n-4} = \ldots = c_2 = c_0 = 0$ (n even).

Here we remark that (7.19) was possible owing to the condition $\lambda \neq (m-1)/2 \; m = 1, 2, n+2$. Therefore we conclude from (7.10) and (7.19) that $r_{n-1}(x) \equiv 0$. This implies that $f(x) \equiv 0$. Therefore in general equations (7.2) and (7.3) determines a unique polynomial $f(x)$ degree $\leqq 3n+3$. This proves the theorem.

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