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Divisibility properties of recurring sequences

by

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Let $v_0 = 0, v_1 = 0, \cdots, v_{n-2} = 0, v_{n-1} = 1, v_n \cdots$ be a sequence of rational integers, which satisfies the recursion

$$v_{i+n} = a_1 v_{i+n-1} + \cdots + a_n v_i \qquad i = 0, 1, 2, \cdots$$

where $a_1, a_2, \cdots, a_n$ are rational integers and $n \geqq 2$.

If in the sequence there exist $n-1$ consecutive elements with positive indices divisible by p, then let $j(p)$ be the smallest positive integer such that $v_{j(p)} \equiv v_{j(p)+1} \equiv \cdots \equiv v_{j(p)+n-2} \equiv 0 \mod p$.

H. J. A. Duparc proved in [1], that if the characteristic polynomial of the sequence

$$f(x) = x^n - a_1 x^{n-1} - a_2 x^{n-2} - \cdots - a_n$$

$\mod p$ is irreducible then $j(p)$ exists and

$$j(p) \,\Big|\, \frac{p^n-1}{p-1},$$

and he considered sequences with reducible characteristic polynomial, and he proved that every sequence which satisfies the recursion, is periodic mod p, in the following sense: there exists a rational integer c such that $u_{m+j(p)} \equiv c u_m \mod p$ $m = 0, 1, 2, \cdots$, where $u_0, u_1, u_2, \cdots$ is the sequence.

The assertions of theorem 2 and 3 are well-known results about the Fibonacci numbers to be found in [1].

Theorem 1. Let K be a finite field with p^n elements (where p is a prime number and n is a positive integer) whose prime field is P. Let $f(x)$ be an irreducible polynomial of $P[x]$ of degree n. If $x_0 \in K$ and $f(x_0) = 0$, then there exists a smallest positive integer j such that $x_0^j \in P$, and if

$$k \,\Big|\, \left(\frac{p^n-1}{p-1}, p-1\right), k > 0,$$

then

$$j \left| \frac{1}{k} \frac{p^n-1}{p-1} \right.$$

if and only if $(-1)^n f(0)$ is k-th power in P.

PROOF. Let $K^* = K-\{0\}$ and $P^* = P-\{0\}$. P^* is a normal subgroup of K^* since K^* is commutative. Let $q = (p^n-1)/(p-1)$. The order of $K_j^* \circ (K^*) = p^n-1$ and that of P_j; $\circ(P^*) = p-1$ then $o(K^*/P^*) = q$.

Let $\bar{a}$ be the coset modulo P^* containing "a" where $a \in K^*$. For every $a \in K^*$, $\bar{a}^q = P^*$, since $o(K^*/P^*) = q$.

Suppose $a \in K^*$ and let $N(a) = a^q$. Obviously, $N(a) \in P^*$ and $N(ab) = N(a)N(b)$ if $a, b \in K^*$.

Let for $a \in K^* N(a)$ be k-th power in P, where $k|(q, p-1)$.

$$\begin{aligned} q &= \frac{1}{p-1}[((p-1)+1)^n-1] \\ &= \frac{1}{p-1}\left[(p-1)^n + \binom{n}{1}(p-1)^{n-1} + \cdots + 1 - 1\right] \\ &= (p-1)\left[(p-1)^{n-2} + \cdots + \binom{n}{n-2}\right] + n. \end{aligned}$$

Therefore $(q, p-1) = (n, p-1)$, consequently $k|n$.

Let $b \in \bar{a}$. Since a and b are in the same coset, there exists an element c of P^* such that $b = ca$.

$$\begin{aligned} N(b) &= b^q = b \cdot b^p \cdot \cdots \cdot b^{p^{n-1}} \\ &= c \cdot a \cdot c^p \cdot a^p \cdot \cdots \cdot c^{p^{n-1}} a^{p^{n-1}} = c^n N(a), \end{aligned}$$

since $c^p = c$. $k|n$, hence c^n is k-th power in p, thus also $c^n N(a) = N(b)$ is also k-th power.

By this we proved the following:

(1) if $k|(q; p-1)$, then $N(b)$ is k-th power in p either for every b in a coset $\bar{a}$ of P^* or for none of the elements b of $\bar{a}$.

Since K^* is a cyclic group, there exists an element g of K^*, such that $\{g\} = K^*$, that is the elements $1, g, g^2, \cdots, g^{p^n-2}$ are different. Thus the elements

$$1, g^q = N(g), g^{2q} = (N(g))^2, \cdots, g^{(p-2)q} = (N(g))^{p-2}$$

are also different, consequently $\{N(g)\} = P^*$. Hence every $c \in P^*$ can be written in the form $c = (N(g))^m$, where m is uniquely determined mod $p-1$. $k|p-1$ implies that c is k-th power in P if and only if there exists an integer m_1 such that $m \equiv km_1$

(mod $p-1$). Obviously, $\{\bar{g}\} = (K^*/P^*)$ and it follows from (1) that for any $a \in \bar{g}^m$, $N(a)$ is k-th power in P if and only if $N(g^m) = (N(g))^m$ is also k-th power in P, that is $m \equiv km_1$ (mod $p-1$).

$k|(q, p-1)$, thus there are exactly q/k numbers in the sequence $1, 2, \cdots, q$ which can be m such that the above congruence with appropriate m_1 is satisfied. Thus P^* has exactly q/k cosets in which $N(a)$ is k-th power in P for every element "a", while the other cosets of P^* have no elements with this property.

Let H be the set of the former type cosets, then $P^* \in H$, since $1 \in P^*$ and $N(1) = 1$ is k-th power in P, thus H is non-vacuos.

If $m' \equiv m_1'k$ and $m'' \equiv m_1''k$ (mod $p-1$), then

$$m'+m'' \equiv (m_1'+m_1'')k$$

(mod $p-1$) and so H is closed relative to multiplication. These two properties imply that H is a subgroup of (K^*/P^*) and that $o(H) = q/k$.

$f(x)$ is irreducible in $P(x)$, $f(x_0) = 0$, thus $x_0, x_0^p, \cdots, x_0^{p^{n-1}}$ are different roots of $f(x)$ which has no other roots, hence $(-1)^n f(0) = x_0^q = N(x_0)$. Thus, if $(-1)^n f(0)$ is k-th power in P, then this holds also for $N(x_0)$ and consequently $\bar{x}_0 \in H$. Obviously $j = o(\bar{x}_0)|o(H) = (1/k)q$, and thus we proved the first part of the second assertion of the theorem.

Suppose $j|(1/k)q$, that is $o(x_0)|(1/k)q$. Since x_0 can be written in the form $\bar{x}_0 = \bar{g}^m$, then

$$\bar{1} = \bar{x}_0^{0(H)} = \bar{x}_0^{(1/k)/q} = \bar{g}^{m(1/k)q},$$

consequently $(m/k)q \equiv 0$ (mod q) and so m/k is an integer that is, $k|m$, hence $\bar{x}_0 \in H$, therefore $N(x_0) = (-1)^n f(0)$ is k-th power in P, thus we proved the theorem.

Let $u_0, u_1, u_2, \cdots$ be the Fibonacci sequence, that is $u_0 = 0$, $u_1 = 1$, and $u_{n+1} = u_n + u_{n-1}$ $(n = 1, 2, 3, \cdots)$. If there exists in the Fibonacci sequence any element different from u_0 and divisible by p, let $j(p)$ be the smallest positive integer such that $p|u_{j(p)}$.

THEOREM 2. Let p be a prime and $p \equiv 3$ or -3 (mod 5), then there exists in the Fibonacci sequence an element different from u_0 and divisible by p and

if $p \equiv 1$ (mod 4), then $j(p)|\frac{1}{2}(p+1)$
if $p \equiv -1$ (mod 4), then $j(p)|p+1$ but $j(p) \nmid \frac{1}{2}(p+1)$

PROOF. Let K_p be the field of the residue classes mod p, where p is an odd prime, and let R be the set of the matrices $\begin{pmatrix} a & b \\ b & a+b \end{pmatrix}$ where $a, b \in K_p$. R is a ring relative to the matrice operations, since if $a, b, c, d \in K_p$

$$\begin{pmatrix} a & b \\ b & a+b \end{pmatrix} + \begin{pmatrix} c & d \\ d & c+d \end{pmatrix} = \begin{pmatrix} a+c & b+d \\ b+d & (a+c)+(b+d) \end{pmatrix} \in R$$

$$\begin{pmatrix} -a & -b \\ -b & -a-b \end{pmatrix} \in R$$

$$\begin{pmatrix} a & b \\ b & a+b \end{pmatrix}\begin{pmatrix} c & d \\ d & c+d \end{pmatrix} = \begin{pmatrix} ac+bd & ad+bc+cd \\ ad+bc+bd & (ac+bd)+(ad+bc+bd) \end{pmatrix} \in R.$$

R is commutative, since its elements are symmetrical matrices and if the product of two symmetrical matrices is also symmetrical, then the two matrices are permutable.

$I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \in R$, consequently R is a commutative ring with a unit element. Let $A = \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}$. Obviously $A \in R$.

Let $\bar{u}_0, \bar{u}_1, \bar{u}_2, \cdots$ be the residue classes mod p which contain the numbers $u_0, u_1, u_2, \cdots$. First we prove that

$$A^s = \begin{pmatrix} \bar{u}_{s-1} & \bar{u}_s \\ \bar{u}_s & \bar{u}_{s+1} \end{pmatrix} \qquad s = 1, 2, 3, \cdots.$$

For $s = 0$ the assertion is obvious. Suppose that

$$A^{s-1} = \begin{pmatrix} \bar{u}_{s-2} & \bar{u}_{s-1} \\ \bar{u}_{s-1} & \bar{u}_s \end{pmatrix}$$

then

$$A^s = AA^{s-1} = \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} \bar{u}_{s-2} & \bar{u}_{s-1} \\ \bar{u}_{s-1} & \bar{u}_s \end{pmatrix}$$

$$= \begin{pmatrix} \bar{u}_{s-1} & \bar{u}_s \\ \bar{u}_{s-2}+\bar{u}_{s-1} & \bar{u}_{s-1}+\bar{u}_s \end{pmatrix} = \begin{pmatrix} \bar{u}_{s-1} & \bar{u}_s \\ \bar{u}_s & \bar{u}_{s+1} \end{pmatrix}$$

thus the assertion is true.

If $p|u_s$, that is $\bar{u}_s = 0$, then $\bar{u}_{s+1} = \bar{u}_s+\bar{u}_{s-1} = \bar{u}_{s-1}$, hence $A^s = \bar{u}_{s-1}I$, and conversely, if there exists any $c \in K_p$ such that $A^s = cI$, then $\bar{u}_s = 0$, that is $p|u_s$.

(2) Thus $p|u_s$ if and only if there exists a $c \in K_p$ such that $A = cI$, hence if $j(p)$ exists it is the smallest positive integer satisfies the equation $A^{j(p)} = cI$ with appropriately chosen $c \in K_p$, and if there exists a positive integer t with $d \in K_p$ such that $A^t = dI$, then $j(p)$ exists.

Let $B = \begin{pmatrix} a & b \\ b & a+b \end{pmatrix} \in R$. If $|B| = d \neq 0$, then B^{-1} exists and

$$B^{-1} = \begin{pmatrix} (a+b)d^{-1} & -bd^{-1} \\ -bd^{-1} & ad^{-1} \end{pmatrix} \in R$$

Thus if $B \in R$, then $B^{-1} \in R$ exists if and only if $|B| \neq 0$. Now let $p \equiv \pm 3 \bmod 5$ and let $B \in R$, with $|B| = 0$.

$$(3) \qquad |B| = a^2+ab-b^2 = 0$$

if $b \neq 0$; $(ab^{-1})^2+ab^{-1}-1 = 0$, that is $(2ab^{-1})^2+4ab^{-1}+1 = 5$, hence $(2ab^{-1}+1)^2 = 5$ and it is in contradiction with $p \equiv \pm 3$ mod 5. Consequently, $b = 0$ and also $a = 0$. Thus $B = 0$ if and only if $|B| = 0$ and R is therefore a field. R has p^2 elements since the elements a and b of the matrice $\begin{pmatrix} a & b \\ b & a+b \end{pmatrix}$ can be chosen in p^2 different ways.

$f(x) = x^2-x-1$ is the characteristic polynomial of A, hence $f(A) = 0$. $f(x)$ is irreducible in $K[x]$, since its discriminant 5 and $(5/p) = -1$. Thus the theorem 1 can be applied to the cases $K = R$, $A = x_0$, $k = 1, 2$. The prime field of R is the set of matrices cI, $c \in K_p$, hence it follows that if j is the smallest positive integer such that $A^j = cI$ with appropriately chosen $c \in K_p$ then $j | \frac{1}{2}(p+1)$ if and only if

$$\left(\frac{f(0)}{p}\right) = \left(\frac{-1}{p}\right) = 1,$$

while $j | p+1$ in every case, which by (2) proves the theorem.

THEOREM 3. Let p be prime and $p \equiv 1$ or -1 mod 5. Then there exists in the Fibonacci sequence an element different from u_0 and divisible by p and

if $p \equiv 1 \bmod 4$, then $j(p) | \frac{1}{2}(p-1)$
if $p \equiv -1 \bmod 4$, then $j(p) \nmid \frac{1}{2}(p-1)$ but $j(p) | p-1$

PROOF. $(5/p) = 1$, hence there exists a $h \in K_p$ such that $h^2 = 5$. $g = (1+h)2^{-1}$ is a root of the polynomial x^2-x-1, therefore $\begin{pmatrix} 1 & g \\ g & g+1 \end{pmatrix} = 0$. $g^2-g-1 = 0$, thus $g^2+1 = g+2$. For $g+2 = 0$ it would follow that $g = -2$, that is $5 = 0$ which is impossible and therefore $g+2 = g^2+1 \neq 0$.

Let

$$C = \begin{pmatrix} (g+2)^{-1} & g(g+2)^{-1} \\ g(g+2)^{-1} & (g+1)(g+2)^{-1} \end{pmatrix} \neq 0$$

$$D = \begin{pmatrix} (g+1)(g+2)^{-1} & -g(g+2)^{-1} \\ -g(g+2)^{-1} & (g+2)^{-1} \end{pmatrix} \neq 0$$

Obviously $C, D \in R$ and since $g^2-g-1 = 0$, $|C| = 0$ and $|D| = 0$ and

$$CD = \begin{pmatrix} g+1-g^2 & -g+2 \\ g^2+g-g^2-g & -g^2+g+1 \end{pmatrix} = 0$$

$C+D = I$, that is $C^2+CD = C$, $C^2 = C$ and similarly $D^2 = D$. Suppose $B \in R$ and

$$B = c_1C+d_1D = c_2C+d_2D, \text{ where } c_1, c_2, d_1, d_2 \in K_p.$$

Then $(c_1-c_2)C^2 = (c_1-c_2)C = 0$, $C \neq 0$ so $c_1-c_2 = 0$, hence $c_1 = c_2$ and $d_1 = d_2$. Thus the elements $cC+dD$ are different if c and d run over the elements of K_p independently of each other. Hence we get p^2 different elements and since R has p^2 elements, each element of R is uniquely written in the form $cC+dD$, where $c, d \in K_p$.

If $B_1 = c_1C+d_1D$ and $B_2 = c_2C+d_2D$, then it follows from $DC = 0$, $C^2 = C$, $D^2 = D$ that

(4) $B_1B_2 = c_1c_2C+d_1d_2D$ and $B_1+B_2 = (c_1+c_2)C+(d_1+d_2)D$

(that is R is the direct sum of the ideals generated by C and D). Let $\begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix} = A = c'C+d'D$. "$A$" is a root of the polynomial $f(x) = x^2-x-1$, hence because of (4) c' and d' are also roots of $f(x)$. $c' \neq d'$, since $A \neq bI$ if $b \in K_p$, thus, c' and d' are two different roots of $f(x)$ and therefore $c'd' = -1$.

(5) Let s be the smallest positive integer such that there exists a $v \in K_p$ which satisfies the equation $A^s = vI$. Such s is sure to exist, since $A^{p-1} = c^{p-1}C^{p-1}+d^{p-1}D^{p-1} = C+D = I$. Obviously, if $A^t = vI$ with $v \in K_p$, then $s|t$.

Suppose $(-1/p) = 1$. Since $c'd' = -1$, $(c'/p) = (d'/p)$, so

$$A^{(p-1)/2} = c'^{(p-1)/2}C+d'^{(p-1)/2}D = C+D = I$$

or

$$A^{(p-1)/2} = c^{(p-1)/2}C+d^{(p-1)/2}D = -C-D = -I$$

thus by (5) $s|\frac{1}{2}(p-1)$ and this is by (2) the first assertion of the theorem.

Suppose $(-1/p) = -1$. $A^{p-1} = c^{p-1}C+d^{p-1}D = C+D = I$, thus by (5) and (2) $j(p)|p-1$.

$cd = -1$, thus $(c'/p) = (d'/p)$ and because of uniqueness

$$A^{(p-1)/2} = c^{(p-1)/2}C+d^{(p-1)/2}D = \pm C \mp D \neq vC+vD = vI$$

for any $v \in K_p$, thus by (5) $s \nmid \frac{1}{2}(p-1)$ and by (2) $j(p) \nmid \frac{1}{2}(p-1)$ which is the second assertion of the theorem.

THEOREM 4. Let

$$v_0 = 0,\ v_1 = 0, \cdots, v_{n-2} = 0,\ v_{n-1} = 1,\ v_n,\ v_{n+1}, \cdots$$

be a sequence of integers which satisfies the recursion

$$v_{i+n} = a_i v_{i+n-1} + a_2 v_{i+n-2} + \cdots + a_n v_i \quad i = 0, 1, 2, \cdots$$

where $a_1, a_2, \cdots, a_n$ are integers and $n \geqq 2$. If the characteristic polynomial of the sequence

$$f(x) = x^n - a_1 x^{n-1} - a_2 x^{n-2} - \cdots - a_n \bmod p$$

irreducible where p is prime, then in the sequence there exist $n-1$ consecutive elements with positive indices which are divisible by p, and if $j(p)$ is the smallest positive integer such that

$$v_{j(p)} \equiv v_{j(p)+1} \equiv \cdots \equiv v_{j(p)+n-2} \equiv 0 \bmod p,$$

then for

$$k \left| \left(\frac{p^n - 1}{p-1}, p-1 \right) k > 0; \quad j(p) \right| \frac{1}{k} \frac{p^n-1}{p-1}$$

if and only if $(-1)^{n+1} a_n$ is k-th power mod p.

PROOF. Let

$$A = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 1 & & 0 & 0 \\ \vdots & & & \ddots & \vdots & \vdots \\ \vdots & & & & 1 & 0 \\ \vdots & & & & 0 & 1 \\ \bar{a}_n & \cdots & \cdots & \cdots & \bar{a}_2 & \bar{a}_1 \end{bmatrix}$$

where $\bar{a}_1, \bar{a}_2, \cdots, \bar{a}_n$ are the residue classes mod p which contain the numbers $a_1, a_2, \cdots, a_n$.

Let K be the set of matrices $g(A)$, where $g \in K_p[x]$. The characteristic polynomial of A is $x^n - a_1 x^{n-1} - \cdots - a_n = f(x)$. $f(x)$ is irreducible in $K_p[x]$ and since $f(A) = 0$, $g_1(A) = g_2(A)$ if and only if $g_1(x) \equiv g_2(x) \pmod{f(x)}$, hence K is a finite field with p^n elements.

Let

$$\underline{a} = \begin{bmatrix} \bar{v}_0 \\ \bar{v}_1 \\ \vdots \\ \bar{v}_{n-1} \end{bmatrix}$$

where $\bar{v}_0, \bar{v}_1, \bar{v}_2, \cdots$ are the residue classes mod p which contain the numbers $\bar{v}_0, \bar{v}_1, \bar{v}_2, \cdots$ and prove that:

(6) if $B, C \in K$, then $B = C$ if and only if $Ba = Ca$.

With immediate calculation we have

$$A \begin{bmatrix} \bar{v}_s \\ \bar{v}_{s+1} \\ \vdots \\ \bar{v}_{s+n-1} \end{bmatrix} = \begin{bmatrix} \bar{v}_{s+1} \\ \bar{v}_{s+2} \\ \vdots \\ \bar{v}_{s+n} \end{bmatrix} \qquad s = 0, 1, 2, \cdots. \tag{7}$$

The vectors

$$\underline{a} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ \vdots \\ 1 \end{bmatrix}, \; A\underline{a} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 1 \\ \cdot \end{bmatrix}, \cdots, A^{n-1} = \begin{bmatrix} 1 \\ \cdot \\ \vdots \\ \cdot \\ \cdot \end{bmatrix}$$

are obviously linearly independent over K, since the determinant constructed these vectors is $-1 \neq 0$. Thus every n dimensional vectors over K_p can be written in the form

$$\sum_{j=0}^{n-1} c_j A^j \underline{a} = \left(\sum_{j=0}^{n-1} c_j A^j \right) \underline{a},$$

where $c_j \in K_p$.

There exist over K_p exactly p^n n-dimensional vectors, K has p^n elements and $\sum_{j=0}^{n-1} c_j A^j \in K$, thus if $B, C \in K$, then $B \neq C$ implies $Ba \neq Ca$, and (6) is true.

Since (7)

$$A^s \underline{a} = \begin{bmatrix} \bar{v}_s \\ \bar{v}_{s+1} \\ \vdots \\ \bar{v}_{s+n-1} \end{bmatrix}.$$

The prime field P of K is the set of matrices cI, $c \in K_p$. If $A^s \in P$ obviously

$$\bar{v}_s = \bar{v}_{s+1} = \cdots = \bar{v}_{s+n-2} = 0$$

and conversely, if,

$$\bar{v}_s = \bar{v}_{s+1} = \cdots = \bar{v}_{s+n-2} = 0$$

then by (6) $A^s = \bar{v}_{s+n-1} I \in P$. Consequently if j is the smallest positive integer such that $A^j \in P$, then $j = j(p)$. (Such j is sure to exist since $A^{p^n-1} = I \in P$.)

By applying the first theorem to the case of $x_0 = A$ we get the assertion of theorem 4.

REFERENCE

H. J. A. Duparc

[1] Divisibility properties of recurring sequences, Dissertation Amsterdam 1963. p. 1—96.

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