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# Entire methods of summation

by

H. I. Brown

## Introduction

In this paper we consider matrix transformations on the set of entire sequences into itself. We call such methods entire. By adopting M. S. Macphail's technique of applying a theorem of K. Knopp and G. G. Lorentz we obtain necessary and sufficient conditions on the elements of a matrix in order that it be an entire method. After some examples and preliminary Lemmas we then prove a consistency type theorem for entire methods of summation.

## 1. Entire methods of summation

Let  $s$  represent the set of all sequences of complex numbers. A member of  $s$ , say  $x = \{x_k\}$ ,  $k = 0, 1, 2, \dots$ , is called an *entire sequence* if  $\sum_{k=0}^{\infty} |x_k|p^k$  converges for every  $p > 0$ . Let  $\mathcal{E}$  designate the set of entire sequences and let  $A = (a_{nk})$  ( $n, k = 0, 1, 2, \dots$ ) be an infinite matrix of complex numbers. The set of equations

$$(1) \quad y_n = \sum_{k=0}^{\infty} a_{nk} x_k \quad (n = 0, 1, \dots)$$

defines an *entire method of summation* if each series in (1) converges and  $y = \{y_n\} \in \mathcal{E}$  whenever  $x \in \mathcal{E}$ . If, in addition,

$$\sum_{n=0}^{\infty} y_n = \sum_{k=0}^{\infty} x_k$$

then  $A$  is called a *regular entire method*.

For each positive integer  $p$ , let  $\mathcal{E}_p$  represent the set of sequences  $\{x_k\}$  such that

$$\sum_{k=0}^{\infty} |x_k|p^k < \infty.$$

In [3; p. 389], M. S. Macphail designates this set by  $l(p)$  and observes that the mapping

$$\{x_k\} \rightarrow \{x_k p^k\}$$

is a one-to-one correspondence between  $\mathcal{E}_p$  and  $l$  (the set of absolutely convergent series). It was shown by K. Knopp and G. G. Lorentz [2] that a necessary and sufficient condition for a matrix  $A = (a_{nk})$  to transform  $l$  into itself (that is, for  $A$  to be an  $l-l$  method) is that there exists a constant  $M$  such that

$$(2) \quad \sum_{n=0}^{\infty} |a_{nk}| < M \quad (k = 0, 1, 2, \dots),$$

and a necessary and sufficient condition for  $A$  to be absolutely regular (that is,  $\sum y_n = \sum x_k$  whenever  $x \in l$ ) is that in addition to (2) the equations

$$\sum_{n=0}^{\infty} a_{nk} = 1 \quad (k = 0, 1, 2, \dots)$$

hold. Thus, the matrix  $(a_{nk})$  maps  $\mathcal{E}_p$  into  $l$  if and only if the matrix  $(a_{nk} p^{-k})$  is an  $l-l$  method. That is,  $(a_{nk})$  maps  $\mathcal{E}_p$  into  $l$  if and only if there exists a constant  $M(p)$  such that

$$\sum_{n=0}^{\infty} |a_{nk}| p^{-k} < M(p) \quad (k = 0, 1, 2, \dots).$$

Similarly, for each positive integer  $q$ , a matrix  $(b_{nk})$  maps  $l$  into  $\mathcal{E}_q$  if and only if the matrix  $(b_{nk} q^n)$  is an  $l-l$  method, that is, if and only if there exists a constant  $M(q)$  such that

$$\sum_{n=0}^{\infty} |b_{nk}| q^n < M(q) \quad (k = 0, 1, 2, \dots).$$

Now  $\mathcal{E} = \cap \{\mathcal{E}_q : q = 1, 2, \dots\}$ ; hence, a matrix  $A = (a_{nk})$  is an entire method if and only if to each positive integer  $q$ , there corresponds a positive number  $p = p(q) \geq q$  such that  $A$  transforms  $\mathcal{E}_p$  into  $\mathcal{E}_q$ . In other words,  $A$  is an entire method if and only if to each  $q = 1, 2, \dots$ , there corresponds a  $p = p(q) \geq q$  such that the matrix  $(a_{nk} q^n p^{-k})$  is an  $l-l$  method. By taking  $q = 1$  we obtain necessary and sufficient conditions for  $A$  to be a regular entire method. We summarize these remarks in the following theorem.

**THEOREM 1.** *A necessary and sufficient condition for  $A$  to be an entire method is that for each positive integer  $q$  there exist  $p(q) \geq q$  and a constant  $M(p, q)$  such that*

$$(3) \quad \sum_{n=0}^{\infty} |a_{nk}| q^n p^{-k} < M(p, q) \quad (k = 0, 1, 2, \dots),$$

and a necessary and sufficient condition for  $A$  to be a regular entire method is that in addition to (3) the equations

$$\sum_{n=0}^{\infty} a_{nk} = 1 \quad (k = 0, 1, 2, \dots)$$

hold.

**REMARK.** In order that  $A$  be an entire method it is necessary that each column of  $A$  be an entire sequence. Also, by taking  $q = 1$  and  $p = p(1)$ , it is necessary that each row be analytic, that is, for each  $n = 0, 1, 2, \dots$ , the sequence

$$\{|a_{n0}|, |a_{nk}|^{1/k} : k = 1, 2, \dots\}$$

be bounded. However, one may easily show that these conditions are not sufficient. Indeed, the matrix defined by the set of equations

$$\begin{aligned} a_{nn} &= n!, & n &= 0, 1, \dots, \\ a_{nk} &= 0, & & \text{otherwise,} \end{aligned}$$

has both entire rows and entire columns. However, the entire sequence  $\{1/n!\}$  is transformed into the constant sequence  $\{1\}$ .

## 2. Examples

For each complex number  $t$ , the Euler-Knopp series-to-series method is defined by the set of equations

$$\begin{aligned} E_{nk}(t) &= \binom{n}{k} t^{k+1} (1-t)^{n-k}, & k &\leq n, \\ E_{nk}(t) &= 0, & k &> n. \end{aligned}$$

The transformations  $E(0)$  and  $E(1)$  are, respectively, the zero matrix and the identity, both of which are entire methods. However, if  $t$  is any other complex number, then the  $k^{\text{th}}$  column of  $(E_{nk})$  is not an entire sequence and so  $E(t)$  cannot be an entire method. (See the Remark.) Contrary to this, the Taylor matrix [1] is always entire. For each complex number  $t$ , the Taylor matrix  $T(t)$  is defined by the set of equations

$$\begin{aligned} T_{nk}(t) &= 0, & n &> k, \\ T_{nk}(t) &= \binom{k}{n} (1-t)^{n+1} t^{k-n}, & n &\leq k. \end{aligned}$$

The trivial cases  $T(0)$  (identity) and  $T(1)$  (zero) are certainly entire methods. If  $t$  is any complex number other than 0 or 1, then

$$\begin{aligned}
 (4) \quad \sum_{n=0}^k \binom{k}{n} |1-t|^{n+1} |t|^{k-n} q^n p^{-k} &= |1-t| p^{-k} (q|1-t| + |t|)^k \\
 &\leq |1-t| (q + (q+1)|t|)^k / p^k \\
 &\leq (1+R)(q + (q+1)R)^k / p^k,
 \end{aligned}$$

where  $R$  is chosen to be so large that  $|t| \leq R$ . We may now choose  $p = 2(q + (q+1)R)$ . Then (4) is dominated by  $(1+R)(1/2)^k$ , which shows that (3) is satisfied with  $M = 1+R$ . Thus,  $T(t)$  is entire.

Notice also that

$$\sum_{n=0}^k \binom{k}{n} (1-t)^{n+1} t^{k-n} = 1-t,$$

so that  $T(t)$  is regular if and only if  $t = 0$ , that is, if and only if  $T$  is the identity matrix.

### 3. Preliminary lemmas

It is well known that  $\mathcal{E}$  is a locally convex  $FK$  space with its  $FK$  topology being given by the family of seminorms  $\{h_n : n = 1, 2, \dots\}$ , where for each  $x \in \mathcal{E}$ ,

$$h_n(x) = \max_{|z|=n} \left| \sum_{i=0}^{\infty} x_i z^i \right|.$$

Also, if we define an analytic sequence  $x$  to mean that the sequence  $\{|x_0|, |x_k|^{1/k} : k = 1, 2, \dots\}$  is a bounded sequence, then every continuous linear functional  $f$  on  $\mathcal{E}$  has the representation

$$f(x) = \sum_{n=0}^{\infty} t_n x_n,$$

for some analytic sequence  $t$ . (For a discussion of  $\mathcal{E}$ , see, for example, C. Goffman and G. Pedrick, *First Course in Functional Analysis*, pp. 220–222, 224, Prentice-Hall, New Jersey.)

Now let  $A$  be an entire method of summation and let  $\mathcal{E}_A$  represent its summability field, that is,

$$\mathcal{E}_A = \{x \in s : Ax \in \mathcal{E}\}.$$

An application of [4, Theorem 1, p. 226 and Theorem 5, p. 230] shows how  $\mathcal{E}_A$  may inherit a locally convex  $FK$  topology given by the following family of seminorms:

$$p_n(x) = |x_n|, \quad (n = 0, 1, 2, \dots),$$

$$q_n(x) = \sup_m \left| \sum_{k=0}^m a_{nk} x_k \right|, \quad (n = 0, 1, 2, \dots),$$

$$h_n(x) = \max_{|z|=n} \left| \sum_{i=0}^{\infty} \left( \sum_{k=0}^{\infty} a_{nk} x_k \right) z^i \right|, \quad (n = 1, 2, 3, \dots).$$

Also, every  $f \in \mathcal{E}'_A$  (the dual space of  $\mathcal{E}_A$ ) may be evaluated as

$$(5) \quad f(x) = \sum_{n=0}^{\infty} t_n \sum_{k=0}^{\infty} a_{nk} x_k + \sum_{k=0}^{\infty} \alpha_k x_k$$

for some analytic sequences  $t$  and  $\alpha$ , and all  $x \in \mathcal{E}_A$ . ( $\alpha$  is analytic because  $\mathcal{E}_A \supseteq \mathcal{E}$  and so  $\sum \alpha_k x_k$  converges for every  $x \in \mathcal{E}$ .)

To each entire method  $A$  there corresponds the functional  $S_A$  given by

$$S_A(x) = \sum_{n=0}^{\infty} \sum_{k=0}^{\infty} a_{nk} x_k.$$

Since every matrix map between  $FK$  spaces is continuous [4; p. 204], it follows that  $S_A \in \mathcal{E}'_A$ .

**LEMMA 1.** *If  $f \in \mathcal{E}'_A$ , then there exists an entire method  $B$  such that  $\mathcal{E}_B \supseteq \mathcal{E}_A$  and  $S_B(x) = f(x)$  for every  $x \in \mathcal{E}_A$ .*

**PROOF.** Given an entire method  $A$  define the matrix  $B = (b_{nk})$  by the set of equations

$$\begin{aligned} b_{0k} &= \alpha_k + t_0 a_{0k} & (k = 0, 1, 2, \dots) \\ b_{nk} &= t_n a_{nk} & (n = 1, 2, \dots; k = 0, 1, 2, \dots), \end{aligned}$$

where  $t$  and  $\alpha$  are the analytic sequences given by equation (5) in the representation of  $f$ .

Let  $N$  be the smallest integer greater than or equal to the number  $\max(M(\alpha), M(t))$ , where

$$M(\alpha) = \max \left( \sup_k (|\alpha_0|, |\alpha_k|^{1/k}), 1 \right)$$

and

$$M(t) = \max \left( \sup_n (|t_0|, |t_n|^{1/n}), 1 \right).$$

$N$  depends only on  $f$ .

To show that  $B$  is an entire method we apply Theorem 1. Let  $q$  be any positive integer whatsoever. Choose  $p \geq N \cdot q$  so that

$$\sup_k \sum_{n=0}^{\infty} |a_{nk}| (N \cdot q)^n p^{-k} = M(p, q) < \infty.$$

(This is possible because  $A$  is an entire method.) For this choice

of  $p$  observe that for each  $k = 0, 1, 2, \dots$ ,

$$|\alpha_k|^{1/k}/p \leq 1.$$

Thus, for each  $k = 0, 1, 2, \dots$ ,

$$\begin{aligned} \sum_{n=0}^{\infty} |b_{nk}| q^n p^{-k} &\leq \frac{|\alpha_k|}{p^k} + \sum_{n=0}^{\infty} |t_n a_{nk}| q^n p^{-k} \\ &= \left( \frac{|\alpha_k|^{1/k}}{p} \right)^k + \sum_{n=0}^{\infty} |a_{nk}| (|t_n|^{1/n} \cdot q)^n p^{-k} \\ &\leq \left( \frac{|\alpha_k|^{1/k}}{p} \right)^k + \sum_{n=0}^{\infty} |a_{nk}| (N \cdot q)^n p^{-k}. \end{aligned}$$

It follows that

$$\sup_k \sum_{n=0}^{\infty} |b_{nk}| q^n p^{-k} < \infty;$$

hence,  $B$  is an entire method.

That  $\mathcal{E}_B \supseteq \mathcal{E}_A$  follows immediately from the construction of  $B$ .

Finally, if  $x \in \mathcal{E}_A$ , then

$$\begin{aligned} S_B(x) &= \sum_{n=0}^{\infty} \sum_{k=0}^{\infty} b_{nk} x_k \\ &= \sum_{k=0}^{\infty} (\alpha_k + t_0 a_{0k}) x_k + \sum_{n=1}^{\infty} \sum_{k=0}^{\infty} t_n a_{nk} x_k \\ &= f(x), \end{aligned}$$

which proves the Lemma.

Let now  $A$  and  $B$  be two entire methods. Define  $C = (C_{nk})$  by the set of equations

$$\begin{aligned} C_{2n,k} &= a_{nk} & (n, k = 0, 1, 2, \dots), \\ C_{2n+1,k} &= -b_{nk} & (n, k = 0, 1, 2, \dots). \end{aligned}$$

**LEMMA 2.**  $C$  is an entire method such that  $\mathcal{E}_C = \mathcal{E}_A \cap \mathcal{E}_B$  and  $S_C(x) = S_A(x) - S_B(x)$  for every  $x \in \mathcal{E}_C$ .

**PROOF.** Since  $A$  and  $B$  are entire methods, given any positive integer  $q$  we may choose  $p \geq q^2$  so that

$$\sup_k \sum_{n=0}^{\infty} |a_{nk}| (q^2)^n p^{-k} < \infty$$

and

$$q \cdot \sup_k \sum_{n=0}^{\infty} |b_{nk}| (q^2)^n p^{-k} < \infty.$$

Thus,

$$\sup_k \sum_{n=0}^{\infty} |C_{nk}| q^n p^{-k} \leq \sup_k \sum_{n=0}^{\infty} |a_{nk}| q^{2n} p^{-k} + \sup_k \sum_{n=0}^{\infty} |b_{nk}| q^{2n+1} p^{-k} < \infty,$$

so that  $C$  is an entire method.

Next,  $x \in \mathcal{E}_C$  if and only if for every  $p > 0$ ,

$$(6) \quad \left| \sum_{k=0}^{\infty} a_{0k} x_k \right| p^0 + \left| \sum_{k=0}^{\infty} b_{0k} x_k \right| p^1 + \left| \sum_{k=0}^{\infty} a_{1k} x_k \right| p^2 + \left| \sum_{k=0}^{\infty} b_{1k} x_k \right| p^3 + \cdots < \infty.$$

Since this is a series of non-negative terms, it is satisfied for every  $p > 0$  if and only if

$$\sum_{n=0}^{\infty} \left| \sum_{k=0}^{\infty} a_{nk} x_k \right| (p^2)^n + p \cdot \sum_{n=0}^{\infty} \left| \sum_{k=0}^{\infty} b_{nk} x_k \right| (p^2)^n < \infty$$

for every  $p > 0$ , that is, if and only if  $x \in \mathcal{E}_A \cap \mathcal{E}_B$ .

Finally, let  $x \in \mathcal{E}_C$ . Then

$$S_C(x) = \sum_{k=0}^{\infty} a_{0k} x_k - \sum_{k=0}^{\infty} b_{0k} x_k + \sum_{k=0}^{\infty} a_{1k} x_k - \sum_{k=0}^{\infty} b_{1k} x_k \pm \cdots.$$

Since this is an absolutely convergent series [take  $p = 1$  in equation (6)], we may rearrange its terms to obtain

$$\begin{aligned} S_C(x) &= \sum_n \sum_k a_{nk} x_k - \sum_n \sum_k b_{nk} x_k \\ &= S_A(x) - S_B(x). \end{aligned}$$

#### 4. Consistency of entire methods of summation

Two entire methods  $A$  and  $B$  will be called *consistent* (relative to the functionals  $S_A$  and  $S_B$ ) if  $S_A(x) = S_B(x)$  for every  $x \in \mathcal{E}_A \cap \mathcal{E}_B$ .

**THEOREM 2.** *In order that an entire method  $A$  be consistent with every entire method  $B$  whenever  $S_A(x) = S_B(x)$  for  $x \in \mathcal{E}$ , it is necessary and sufficient that  $\mathcal{E}$  be dense in  $\mathcal{E}_A \cap \mathcal{E}_B$  whenever  $S_B(x) = S_A(x)$  for  $x \in \mathcal{E}$  (where the closure is taken in the FK topology of  $\mathcal{E}_A \cap \mathcal{E}_B$ ).*

**PROOF.** Assume  $\mathcal{E}$  is dense in  $\mathcal{E}_A \cap \mathcal{E}_B$  and that  $S_A(x) = S_B(x)$  for every  $x \in \mathcal{E}$ . Then  $F(x) = S_A(x) - S_B(x)$  defines a continuous linear functional on  $\mathcal{E}_A \cap \mathcal{E}_B$  which vanishes on  $\mathcal{E}$ ; hence, it must vanish on  $\mathcal{E}_A \cap \mathcal{E}_B$ . Thus,  $A$  and  $B$  are consistent.

Conversely, assume that  $A$  is an entire method which is con-



sistent with every entire method that agrees with  $A$  on  $\mathcal{E}$ . Suppose there exists an entire method  $B$  such that  $S_B(x) = S_A(x)$  for  $x \in \mathcal{E}$  and yet  $\mathcal{E}$  is not dense in  $\mathcal{E}_A \cap \mathcal{E}_B$ . Then there exists  $f \in \mathcal{E}'_C$  such that  $f$  vanishes on  $\mathcal{E}$  and  $f(y) \neq 0$  for some  $y \in \mathcal{E}_C$ , where  $C$  is the entire method constructed from  $A$  and  $B$  as in Lemma 2.

By Lemma 1, there exists an entire method  $D$  such that  $\mathcal{E}_D \supseteq \mathcal{E}_C$  and  $S_D(x) = f(x)$  for  $x \in \mathcal{E}_C$ .

Define  $E = (e_{nk})$  by the set of equations

$$e_{nk} = d_{nk} + a_{nk} \quad (n, k = 0, 1, 2, \dots).$$

Then  $E$  is an entire method because for every  $k$ ,

$$\sum_{n=0}^{\infty} |e_{nk}| q^n p^{-k} \leq \sum_{n=0}^{\infty} |d_{nk}| q^n p^{-k} + \sum_{n=0}^{\infty} |a_{nk}| q^n p^{-k}.$$

Since  $\mathcal{E}_D \supseteq \mathcal{E}_C$  we have  $\mathcal{E}_E \supseteq \mathcal{E}_C$ . Moreover, for

$$x \in \mathcal{E}, S_E(x) = S_D(x) + S_A(x) = S_A(x)$$

since  $f$  vanishes on  $\mathcal{E}$ . However,  $E$  is not consistent with  $A$  since  $y \in \mathcal{E}_E \cap \mathcal{E}_A$ ,  $f(y) \neq 0$ , and  $S_E(y) = f(y) + S_A(y)$ . This contradicts our assumption, and the Theorem is proved.

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