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New sets of postulates for intuitionistic topology

Dedicated to A. Heyting on the occasion of his 70^{th} birthday

by

A. S. Troelstra [1]

Introduction

In [1] and [4], sets of postulates were considered which used a decidable intersection relation between a countable collection of closed pointsets; the interiors of these pointsets constituted a basis for the topological space to be described. So the relation of strong inclusion (classically: V is strongly included in W iff the closure of V is contained in the interior of W) did not occur as a primitive notion, in contrast to some of the (classical) approaches as described in [3].

In order to get a manageable system with strong inclusion as a primitive notion, we should like to start with a basis (e.g. of open sets) with an enumerable relation of strong inclusion (i.e. the pairs for which the relation holds constitute the range of a function defined on the natural numbers.).

This is a very strong requirement, therefore not a fortunate choice if we want to prove without great effort for an important class of spaces that they satisfy our set of postulates.

But it turns out that we can find a basis and an enumerable binary relation (throughout this paper denoted by R) on the basis, which implies strong inclusion and which is sufficient to describe the topology.

With this starting point we are able to give a set of postulates which describes larger classes of spaces than was possible in [4]. In some respects the new sets of axioms are more easily managed, in other respects the systems of [4] possess technical advantages.

For the main theorems obtained for CIN-spaces in [4], analogous theorems can be proved for the S-spaces introduced in this paper. (Theorem 11 serves as an illustration).

[1] During part of the time of the preparation of this paper the author was supported by the Netherlands Organization for the advancement of pure research (Z.W.O.).

Our major aim however, is to show the possibility of an adequate non-metrical description of separable metric and complete separable metric spaces. The result is expressed in theorem **10**.

Conventions. The intuitionistic topological notions can be obtained by interpreting the well-known classical definitions intuitionistically.

In particular, we choose the following definitions: topological spaces are defined by their open pointspecies; a closure point of a species V is a point p such that every neighbourhood of p contains a point of V; a pointspecies is closed if it contains its closure points. A more detailed list of definitions can be found in [4].

$\mathfrak{A}$ is a denumerably infinite sequence of objects, R denotes a binary relation, $R \subset \mathfrak{A} \times \mathfrak{A}$. $\mathfrak{A}_0 = \{\emptyset\} \cup \mathfrak{A}$. Capitals A, B, C, D (indexed if necessary) are used to denote elements of $\mathfrak{A}$.

For sequences $x_1, x_2, \cdots$ we use the notation $\langle x_n \rangle_{n=1}^{\infty} = \langle x_n \rangle_n$. The set of natural numbers (zero not included) is denoted by N. i, j, k, m, n are variables used for natural numbers.

Postulates will be denoted by S1, S2 etc.

For R we postulate

S1. $$R(A, B) \; \& \; R(B, C) \rightarrow R(A, C).$$

Definitions of Σ, $\simeq$, Sec, Σ_0, $\#$.

1. $$\langle B_n \rangle_n \in \Sigma =_{\text{Def}} \bigwedge n R(B_{n+1}, B_n)$$

2. $$\langle B_n \rangle_n \simeq \langle C_n \rangle_n =_{\text{Def}} \bigwedge i \bigvee j (R(C_j, B_i) \; \& \; R(B_j, C_i))$$

3. $$\text{Sec}\,(A, B) =_{\text{Def}} \bigvee C (R(C, A) \; \& \; R(C, B))$$

4. $$\langle A_n \rangle_n \in \Sigma_0 =_{\text{Def}} \langle A_n \rangle_n \in \Sigma \; \& \bigwedge B, C (R(B, C) \rightarrow \rightarrow \bigvee m (\neg \, \text{Sec}\,(B, A_m) \vee R(A_m, C))$$

5. $$\langle A_n \rangle_n \# \langle B_n \rangle_n =_{\text{Def}} \bigvee n \neg \, \text{Sec}\,(A_n, B_n)$$

Theorem 1. $\#$ is an apartness relation on Σ_0 with $\simeq$ as the corresponding equivalence relation, i.e.

$$\neg \langle A_n \rangle_n \# \langle B_n \rangle_n \leftrightarrow \langle A_n \rangle_n \simeq \langle B_n \rangle_n,$$

$$\langle A_n \rangle_n \# \langle B_n \rangle_n \rightarrow \langle B_n \rangle_n \# \langle A_n \rangle_n,$$

and for all $\langle C_n \rangle_n \in \Sigma$,

if $\langle A_n \rangle_n \# \langle B_n \rangle_n$, then $\langle A_n \rangle_n \# \langle C_n \rangle$ or $\langle B_n \rangle_n \# \langle C_n \rangle_n$.

Proof. As an example, we only prove for

$\langle A_n\rangle_n, \langle B_n\rangle_n, \langle C_n\rangle_n \in \Sigma_0$

$$\langle A_n\rangle_n \# \langle B_n\rangle_n \to \langle A_n\rangle_n \# \langle C_n\rangle_n \vee \langle B_n\rangle_n \# \langle C_n\rangle_n.$$

We start with the remark (often used in the sequel) that

$$R(C, A) \,\&\, \mathrm{Sec}\,(C, D) \to \mathrm{Sec}\,(A, D)$$

hence also

$$R(C, A) \,\&\, \neg\, \mathrm{Sec}\,(A, D) \to \neg\, \mathrm{Sec}\,(C, D).$$

We can find a natural number ν such that $\neg\, \mathrm{Sec}\,(A_\nu, B_\nu)$. Since we have $R(A_{\nu+1}, A_\nu)$ and $R(B_{\nu+1}, B_\nu)$ we can also find a natural number μ such that $R(C_\mu, A_\nu) \vee \neg\, \mathrm{Sec}\,(C_\mu, A_{\nu+1})$, and $R(C_\mu, B_\nu) \vee \neg\, \mathrm{Sec}\,(C_\mu, B_{\nu+1})$. $R(C_\mu, A_\nu) \,\&\, R(C_\mu, B_\nu)$ would imply $\mathrm{Sec}\,(A_\nu, B_\nu)$, therefore $\neg\, \mathrm{Sec}\,(C_\mu, A_{\nu+1}) \vee \neg\, \mathrm{Sec}\,(C_\mu, B_{\nu+1})$.

Since $R(C, A) \,\&\, R(D, B) \,\&\, \mathrm{Sec}\,(C, D) \to \mathrm{Sec}\,(A, B)$, it follows that for $\lambda = \mu+\nu+1 : \neg\, \mathrm{Sec}\,(C_\lambda, A_\lambda) \vee \neg\, \mathrm{Sec}\,(C_\lambda, B_\lambda)$, hence $\langle A_n\rangle_n \# \langle C_n\rangle_n \vee \langle B_n\rangle_n \# \langle C_n\rangle_n$.

The remaining conditions can be proved by reasonings of the same kind.

Definitions of $\langle A_n\rangle_n^*$, $\langle A_n\rangle_n \in B$, $[B], \Pi^*$, $\langle A_n\rangle_n^* \in [B]$, open pointspecies, $\#$. ($\langle A_n\rangle_n, \langle B_n\rangle_n \in \Sigma_0$).

We introduce a primitive notion, the species of point generators $\Pi \subset \Sigma_0$. (For special classes of spaces a definable notion).

6. $\langle A_n\rangle_n^* =_{\mathrm{Def}} \{\langle B_n\rangle_n : \langle A_n\rangle_n \simeq \langle B_n\rangle_n \,\&\, \langle B_n\rangle_n \in \Pi\}$.

7. $\langle A_n\rangle_n \in B =_{\mathrm{Def}} \bigvee n\, R(A_n, B)$.

8. $[B] =_{\mathrm{Def}} \{\langle A_n\rangle_n^* : \langle A_n\rangle_n \in B \,\&\, \langle A_n\rangle_n \in \Pi\}$.

9. $\Pi^* = \{\langle A_n\rangle_n^* : \langle A_n\rangle_n \in \Pi\}$.

Π^* is called the species of points (of the space to be described). Arbitrary points are denoted by p, q, r; arbitrary pointspecies by U, V, W.

10. $\langle A_n\rangle_n^* \in B =_{\mathrm{Def}} \langle A_n\rangle_n^* \in [B]$.

11. V is called open, if $\bigwedge \langle B_n\rangle_n^* \in V \cap \Pi^* \bigvee m([B_m] \subset V)$.

12. $\langle A_n\rangle_n^* \# \langle B_n\rangle_n^* \leftrightarrow \langle A_n\rangle_n \# \langle B_n\rangle_n$.

Lemma 2. $R(A, B) \to [A] \subset [B]$.

Proof immediate.

THEOREM 3. The open sets of Π^* define a topology with apartness relation on Π^*, i.e. Π^*, $\emptyset$ are open, arbitrary unions and finite intersections of open sets are open, and if V is open, $p \in V$, $q \notin V$, then $p \# q$.

PROOF. Straightforward, using lemma 2.

DEFINITION 13.

$V \Subset W =_{\mathrm{Def}} \bigwedge \langle B_n \rangle_n \in \Pi \bigvee m([B_m] \cap V = \emptyset \vee [B_m] \subset W)$.

THEOREM 4. (a) $\neg\, \mathrm{Sec}\, (A, B) \rightarrow [A] \cap [B] = \emptyset$

(b) $R(A, B) \rightarrow [A] \Subset [B]$

(c) $A \in \mathfrak{A} \rightarrow [A]$ open

(d) $V \Subset W \rightarrow V^- \subset \mathrm{Interior}\ W$, where V^- is the closure of V.

Proof presents no special difficulty.

We introduce a new postulate:

S2. $\bigwedge A \bigvee \langle B_n \rangle_n \in \Pi(\langle B_n \rangle_n \in A)$.

THEOREM 5. $\mathrm{Sec}\, (A, B) \leftrightarrow \bigvee \langle C_n \rangle_n \in \Pi(\langle C_n \rangle_n \in [A] \cap [B])$.

PROOF: immediate from S2.

Next we introduce

S3. There exists a mapping $f : N \rightarrow \mathfrak{A} \times \mathfrak{A}$ such that $R(A, B) \leftrightarrow \langle A, B \rangle \in f(N)$. ($R$ is enumerable).

S4. $R(A, B) \rightarrow \bigvee C(R(A, C)\ \&\ R(C, B))$.

DEFINITION 14. A topological space described by $\mathfrak{A}$, R, Π such that S1—S4 are fulfilled, is called an S_0-space.

LEMMA 6. If for a topological space (described by $\mathfrak{A}$, R, Π) S1, S2, S4 hold, and $R(A_1, A_0)$, then a function f, defined on Π^* can be constructed, such that $p \in A_1 \rightarrow f(p) = 1$, $p \notin A_0 \rightarrow f(p) = 0$ and $0 \not> f(p) \not> 1$ for every p.

PROOF. The proof closely parallels the proof of 3.2.27 in [4], which was inspired by [1], which was in turn an adaptation of the well-known Urysohn-construction. Therefore we do not present all straightforward details, which would be very tedious.

We construct A_α for every $\alpha = m2^{-n}$ ($n \in N$, $m \in N \cup \{0\}$, $m \leqq 2^n$) such that

$$\alpha < \beta \rightarrow R(A_\beta, A_\alpha).$$

This construction can be carried out inductively by inserting between every pair $A_{2m2^{-n}}$, $A_{2(m+1)2^{-n}}$ an $A_{(2m+1)2^{-n}}$ (using S4). Let $\langle B_n \rangle_n \in \Pi$. We introduce a partially defined function $\psi^B(n, k)$ (defined for all n in combination with some k) by:

$$\psi^B(n, k) = \psi(n, k) = \sup \{m2^{-n} : R(B_k, A_{m2^{-n}}) \vee m = 0\}.$$

We remark that, since $R(A_{(m+1)2^{-n}}, A_{m2^{-n}})$, for a certain number $t(n, m) : \neg \operatorname{Sec}(A_{(m+1)2^{-n}}, B_{t(n,m)}) \vee R(B_{t(n,m)}, A_{m2^{-n}})$.
It follows that if $t(n) = t^B(n) = \sup \{t(n, m) : 1 \leqq m \leqq 2^n\}$, then $\psi(n, k)$ is defined for $k \geqq t(n)$, while for all n, k, k':

$$(1) \qquad k, k' \geqq t(n) \rightarrow \psi(n, k) = \psi(n, k').$$

We may suppose $t(n)$ to increase monotonously, for an arbitrary t' which satisfies (1) can be replaced by a monotonous t:

$$t(n+1) = \sup \{t(n)+1, t'(n+1)\}.$$

Suppose $\psi(n, k)$ and $\psi(n', k)$ to be defined. Then it is easy to see that $n \leqq n' \rightarrow \psi(n, k) \leqq \psi(n', k)$.

If $R(B_k, A_{\psi(n,k)})$, then $\neg R(B_k, A_{\psi(n,k)+2^{-n}})$; if $\psi(n, k) = 1$, then $\psi(n', k) = 1$. Hence

$$(2) \qquad n \leqq n' \rightarrow \psi(n, k) \leqq \psi(n', k) \leqq \psi(n, k)+2^{-n}.$$

Now we can prove that $\lim_{n\to\infty} \psi(n, t(n))$ exists, since for $n \leqq n'$

$$|\psi(n, t(n)) - \psi(n', t(n'))| \leqq |\psi(n, t(n)) - \psi(n, t(n'))| + |\psi(n, t(n')) - \psi(n', t(n'))| \leqq 2^{-n}.$$

Moreover, the value of this limit is independent of the particular function t which satisfies (1), and for which $\psi(n, t(n))$ is defined. This fact is readily verified.

Hence we can define a mapping F on Π by

$$F\langle B_n \rangle_n = \lim_{n\to\infty} \psi(n, t(n)).$$

A straightforward verification learns us that

$$\langle B_n \rangle_n \simeq \langle C_n \rangle_n \rightarrow F\langle B_n \rangle_n = F\langle C_n \rangle_n.$$

(By comparison of ψ^B, ψ^C, t^B and t^C.)

Therefore we can introduce f by

$$f\langle B_n \rangle_n^* = F\langle B_n \rangle_n.$$

Finally we have to verify the properties mentioned for f in the conclusion of the lemma. We verify easily that

$$(3) \qquad \begin{array}{l} \bigwedge p(p \in A_\alpha \rightarrow f(p) \nless \alpha) \\ \bigwedge p(p \notin A_\alpha \rightarrow f(p) \ngtr \alpha) \end{array}$$

(Hence $p \in A_1 \rightarrow f(p) = 1$, $p \notin A_0 \rightarrow f(p) = 0$.)

Finally f has to be proved to be continuous.

Let $\langle B_n \rangle_n, \langle C_n \rangle_n \in \Pi$, $\psi^B = \psi$, $t^B = t$. Suppose $0 < \psi(n, t(n)) < 1$. Then $\neg$ Sec $(B_{t(n)}, A_{\psi(n, t(n))+2^{-n}})$ & $R(B_{t(n)}, A_{\psi(n, t(n))})$.

$$\langle C_n \rangle_n \in B_{t(n)} \to \langle C_n \rangle_n^* \notin A_{\psi(n, t(n))+2^{-n}} \;\&\; \langle C_n \rangle_n^* \in A_{\psi(n, t(n))}$$
$$\to f\langle C_n \rangle_n^* \not> \psi(n, t(n)) + 2^{-n} \;\&\; f\langle C_n \rangle_n^* \not< \psi(n, t(n)).$$

Likewise with $\langle B_n \rangle_n$ substituted for $\langle C_n \rangle_n$, hence

$$|f\langle C_n \rangle_n^* - f\langle B_n \rangle_n^*| \not> 2^{-n}.$$

The special cases $\psi(n, t(n)) = 0$, $\psi(n, t(n)) = 1$, can be treated by adaptation of this reasoning, and produce the same result. Hence f is continuous.

Theorem 7. An S_0-space is metrizable.

Proof. (Adaptation of the proof of 3.2.28 in [4], which is again inspired by [1], just as the proof of lemma 6.)

Let Γ be an S_0-space, described by $\mathfrak{A}$, R, Π. Let $\langle\langle A_i, A_i' \rangle\rangle_i$ be an enumeration of all pairs $\langle A_i, A_i' \rangle$ such that $R(A_i, A_i')$. (This is made possible by S3).

With every pair $\langle A_i, A_i' \rangle$ we associate a continuous function f_i, such that $0 \not> f_i(p) \not> 1$, $p \in A_i \to f_i(p) = 1$, $p \notin A_i' \to f_i(p) = 0$ (according to lemma 6). We define

$$\rho(p, q) = \sum_{i=1}^{\infty} 2^{-i} |f_i(p) - f_i(q)|.$$

We must show that ρ is an adequate metric for Γ, i.e.

(1) $$\rho(p, q) = \rho(q, p),$$

(2) $$\rho(p, q) \not< 0,$$

(3) $$p \# q \leftrightarrow \rho(p, q) > 0,$$

(4) $$\rho(p, q) \not> \rho(p, r) + \rho(r, q),$$

(5) $$\bigwedge p \bigwedge \varepsilon \bigvee B(p \in [B] \;\&\; [B] \subset U(\varepsilon, p)),$$

(6) $\bigwedge B \bigwedge p \in B \bigvee \varepsilon(U(\varepsilon, p) \subset [B])$, where

$$U(\varepsilon, p) = \{q : \rho(p, q) < \varepsilon\}.$$

(1) and (2) are trivial. Let $p = \langle B_n \rangle_n^*$, $q = \langle C_n \rangle_n^*$. Proof of (3). Let $\rho(p, q) > 0$.

$\rho(p, q) > 0 \to |f_\nu(p) - f_\nu(q)| > 2^{-\mu}$ for certain $\nu, \mu \in N$. Suppose $f_\nu(p) - f_\nu(q) > 2^{-\mu}$. f_ν is continuous (lemma 6), hence there are D_1, D_2 such that $p \in [D_1]$, $q \in [D_2]$ and

$$\bigwedge r \in [D_1](|f_\nu(p)-f_\nu(r)| < 2^{-\mu-1}) \\ \& \bigwedge r \in [D_2](|f_\nu(q)-f_\nu(r)| < 2^{-\mu-1}).$$

Hence $[D_1] \cap [D_2] = \emptyset$, so (by theorem 5) $\neg \operatorname{Sec}(D_1, D_2)$. Therefore, for a λ such that $R(B_\lambda, D_1)$ & $R(C_\lambda, D_2) : \neg \operatorname{Sec}(B_\lambda, C_\lambda)$, hence $p \# q$. Conversely, if $p \# q$, then $\neg \operatorname{Sec}(B_\lambda, C_\lambda)$ for a certain $\lambda \in N$. Let $\langle B_{\lambda+1}, B_\lambda \rangle = \langle A_\nu, A'_\nu \rangle$. Then $f_\nu(p) = 1$, $f_\nu(q) = 0$, hence $\rho(p, q) > 0$.

Proof of (4). (4) follows from the fact that for every i

$$|f_i(p)-f_i(q)| \ngtr |f_i(p)-f_i(r)|+|f_i(r)-f_i(q)|.$$

Proof of (5). Let $p \in \Gamma$, and let ν be chosen such that $\sum_{i=\nu+1}^{\infty} 2^{-i} = 2^{-\nu} < 2^{-1}\varepsilon$. $f_1, f_2, \cdots, f_\nu$ are continuous functions, hence there are $D_1, D_2, \cdots, D_\nu$ such that

$$\bigwedge q \in [D_i](|f_i(p)-f_i(q)| < 2^{-i-1}\varepsilon\nu^{-1}),\ p \in [D_i] \text{ for } 1 \leqq i \leqq \nu.$$

Therefore we can find a B_μ such that $R(B_\mu, D_i)$ for $1 \leqq i \leqq \nu$. Let $q \in [B_\mu]$. Then

$$\rho(p, q) = \sum_{i=1}^{\nu} |f_i(p)-f_i(q)|2^{-i} + \sum_{i=\nu+1}^{\infty} 2^{-i}|f_i(p)-f_i(q)| \\ < \sum_{i=1}^{\nu} 2^{i-1}\varepsilon\nu^{-1}2^{-i} + 2^{-1}\varepsilon = \varepsilon$$

Proof of (6). Let $p \in [D]$. For a certain natural number λ, $R(B_\lambda, D)$. $R(B_{\lambda+1}, B_\lambda)$; let $\langle B_{\lambda+1}, B_\lambda \rangle = \langle A_\nu, A'_\nu \rangle$.

$$\rho(p, q) < 2^{-\nu} \rightarrow |f_\nu(p)-f_\nu(q)|2^{-\nu} < 2^{-\nu} \\ \rightarrow |f_\nu(p)-f_\nu(q)| < 1 \\ \rightarrow f_\nu(q) > 0 \\ \rightarrow \neg\neg\, q \in [B_\lambda].$$

Since $R(B_\lambda, D)$ implies $\bigvee m(\neg \operatorname{Sec}(B_\lambda, C_m) \vee R(C_m, D))$, and the first possibility would imply $q \notin [B_\lambda]$, we conclude to $R(C_m, D)$, hence $q \in [D]$. So $U(2^{-\nu}, p) \subset [D]$. This proves (6).

Now we introduce the following postulate:

S5. There exists a sequence $\langle \mathfrak{A}_i \rangle_i$ such that

(a) $\mathfrak{A}_i = \langle B_{i,n} \rangle_n$; $\mathfrak{A} = \mathfrak{A}_1$, $\bigwedge i(\mathfrak{A}_{i+1} \subset \mathfrak{A}_i)$

(b) $\bigwedge i(\cup \{B : B \in \mathfrak{A}_i\} \supset \Pi^*)$

(c) $\bigwedge A \in \mathfrak{A}_i \bigvee B \in \mathfrak{A}_{i+1}(R(B, A))$

(d) $\bigwedge A, B(A \in \mathfrak{A}_i \ \&\ R(B, A) \rightarrow B \in \mathfrak{A}_i)$

(e) $\bigwedge n(A_n \in \mathfrak{A}_n \ \&\ R(A_{n+1}, A_n)) \rightarrow \langle A_n \rangle_n \in \Pi$.

DEFINITION 15. A space defined by $\mathfrak{A}$, R, Π such that S1—5 are satisfied will be called an S^*-space.

DEFINITION 16. A space Γ which is described by $\mathfrak{A}$, R, Π, such that S1—2 are satisfied, is said to possess a perfect representation by a species $\Pi_1 \subset \Pi$, if Π_1 can be described as the species of elements of a spread $\langle \Theta, \vartheta \rangle$ (Θ spread law, ϑ complementary law) such that

(a) $\bigwedge \langle A_n \rangle_n \in \Pi \bigvee \langle B_n \rangle_n \in \Pi_1 (\langle A_n \rangle_n \simeq \langle B_n \rangle_n)$

(b) There is a mapping ϑ^* such that

$\vartheta \langle k_1, k_2, \cdots, k_n \rangle = \langle \vartheta^* \langle k_1 \rangle, \vartheta^* \langle k_1, k_2 \rangle, \cdots, \vartheta^* \langle k_1, \cdots, k_n \rangle \rangle.$

(c) Let α denote a sequence of natural numbers,

$\alpha = \langle \alpha(1), \alpha(2), \cdots \rangle$, $\bar{\alpha}(n) = \langle \alpha(1), \cdots, \alpha(n) \rangle$. Then

$$\sigma \in \Theta \;\&\; p \in \vartheta^* \sigma \to \bigvee \alpha \bigvee n (\bigwedge m (\bar{\alpha}(m) \in \Theta) \;\&\; \bar{\alpha}(n) = \sigma \;\&\; \langle \vartheta^* \bar{\alpha}(m) \rangle_m^* = p).$$

THEOREM 8. Every S^*-space possesses a perfect representation.

PROOF. For Π_1 we can take:

$$\langle A_n \rangle_n \in \Pi_1 \leftrightarrow \langle A_n \rangle_n \in \Pi \;\&\; \bigwedge n (A_n \in \mathfrak{A}_n).$$

The spread corresponding to Π_1 then can be defined according to the standard construction in [5]. Take for the relation R occurring in the description of the standard construction the relation R' defined by

$$R'(\langle A_1, \cdots, A_n \rangle, A_{n+1}) \leftrightarrow R(A_{n+1}, A_n) \;\&\; A_{n+1} \in \mathfrak{A}_{n+1}.$$

Requirement (a) of definition 16 is fulfilled since, if $\langle C_n \rangle_n \in \Pi$, we can find a number $n(k)$ for every k, such that $R(C_{n(k)}, B)$ for some $B \in \mathfrak{A}_k$ (a consequence of S5(b)).

Hence by S5(d) $C_{n(k)} \in \mathfrak{A}_k$. Therefore

$$\langle C_{n(k)} \rangle_k \in \Pi_1, \langle C_{n(k)} \rangle_k \simeq \langle C_n \rangle_n.$$

The perfectness follows from the fact that if

$$\langle A_n \rangle_n \in \Pi_1, p \in A_\mu, p = \langle C_n \rangle_n^*, \langle C_n \rangle_n \in \Pi_1,$$

then there exists a $C_\nu (\nu \geqq \mu + 1)$ such that $R(C_\nu, A_\mu) \;\&\; C_\nu \in \mathfrak{A}_{\mu+1}$. Hence if we take $\langle D_n \rangle_n$ to be given by

$$\langle D_1, \cdots, D_\mu \rangle = \langle A_1, \cdots, A_\mu \rangle, D_{\mu+n} = C_{\nu+n}$$

for $n \in N$, then $\langle D_n \rangle_n \in \Pi_1$, $\langle D_n \rangle_n \simeq \langle C_n \rangle_n$. From the description of the standard construction it is seen that this implies the validity of (c) in definition 16. (b) also follows from a consideration of the standard construction.

REMARK 1. Without restriction we may always suppose Π for an S^*-space to be defined by

$$\langle A_n \rangle_n \in \Pi \to \bigwedge n(A_n \in \mathfrak{A}_n \ \& \ R(A_{n+1}, A_n))$$

as follows from the proof of theorem 8.

REMARK 2. The definition of an S^*-space by means of coverings $\mathfrak{A}_i$ reminds one of uniform spaces, and hence constitutes a "non-topological" feature in the definition of S^*-spaces.

Now we introduce the class of the S-spaces, (which turn out to be special cases of S^*-spaces) in which this trait is eliminated. We introduce a postulate:

S6. There exists a binary predicate $S \subset \mathfrak{A}_0 \times \mathfrak{A}_0$, such that

(a) $S(\emptyset, \emptyset)$, $S(\emptyset, A)$, $S(A, \emptyset)$ for every A,

(b) $S(A, B) \to S(B, A)$,

(c) $S(A, B) \ \& \ R(C, B) \to S(A, C)$,

(d) There exists a mapping $g : N \to \mathfrak{A}_0 \times \mathfrak{A}_0$, such that $S(A, B) \leftrightarrow \langle A, B \rangle \in g(N)$; $\langle \emptyset, \emptyset \rangle$, $\langle \emptyset, A \rangle$, $\langle A, \emptyset \rangle \in g(N)$ (S is enumerable),

(e) $S(A, B) \to \neg \operatorname{Sec}(A, B)$,

(f) $\langle C_n \rangle_n \in \Pi \leftrightarrow \bigwedge A, B\big(R(A, B) \to \bigvee n(S(C_n, A) \vee R(C_n, B))\big) \ \& \ \bigwedge n\big(R(C_{n+1}, C_n)\big)$.

DEFINITION 17. A space described by means of $\mathfrak{A}$, R, S, Π such that S1—4, S6 are satisfied, is called an S-space.

THEOREM 9. Every S-space is an S^*-space.

PROOF. Let Γ be an S-space defined by $\mathfrak{A}$, R, S, Π, i.e. S1—4, S6 are satisfied.

We suppose $f(i) = \langle A_i, A'_i \rangle$ (see S3) and define

$$\mathfrak{A}_1 = \mathfrak{A}, \ \mathfrak{A}_{k+1} = \{B : (\bigwedge i \leqq k)\big(S(A_i, B) \vee R(B, A'_i)\big)\}.$$

$\mathfrak{A}_k$ is enumerable for every k, and clearly $\bigwedge k(\mathfrak{A}_{k+1} \subset \mathfrak{A}_k)$.

S5(e) is satisfied as a consequence of S6(f).

S5 (d): Let $A \in \mathfrak{A}_{k+1} \ \& \ R(B, A)$. Hence

$$(\bigwedge i \leqq k)\big(S(A_i, A) \vee R(A, A'_i)\big);$$

this implies

$$(\bigwedge i \leqq k)\big(S(A_i, B) \vee R(B, A'_i)\big) \ \ (\text{S6 (c)}).$$

S5 (c): Let $A \in \mathfrak{A}_{k+1}$, $\langle B_n \rangle_n \in A$, $\langle B_n \rangle_n \in \Pi$.

Then for a certain ν: $R(B_\nu, A)$. Since $\mathfrak{A}_{k+1}$ is a covering, there is a $C \in \mathfrak{A}_{k+2}$, such that $\langle B_n \rangle_n \in C$, hence $R(B_\mu, C)$, so (S5 (d)) $B_\mu \in \mathfrak{A}_{k+2}$. Therefore $B_{\mu+\nu} \in \mathfrak{A}_{k+2}$ & $R(B_{\mu+\nu}, A)$.

S5 (b) is an immediate consequence of S6 (f) and the definition of the $\mathfrak{A}_i$.

REMARK 3. For an S-space, S6 (f) can be taken as a definition of Π.

THEOREM 10. (a) The class of spaces homeomorphic to an S_0-space coincides with the class of separable metrizable spaces.

(b) Every complete separable metrizable space is homeomorphic to an S-space.

PROOF. Let Γ be a separable, metrizable space; we suppose ρ to be a metric, adequate with respect to Γ.

Let $\langle p_n \rangle_n$ denote a sequence of points, dense in Γ. There exists a mapping $r : N^3 \to$ rational numbers, such that

$$\bigwedge n, m, k(|\rho(p_n, p_m) - r(n, m, k)| < 2^{-k}).$$

We put $\mathfrak{A} = \{U(r_i, p_j) : i \in N \ \& \ j \in N\}$, where $\langle r_n \rangle_n$ is an enumeration of the rational numbers of $(0, 1)$. We write $U_{i,j}$ for $U(r_i, p_j)$, and define

$$R(U_{i,j}, U_{s,t}) \leftrightarrow \bigvee k(r_i + r(j, t, k) + 2^{-k} < r_s).$$

Clearly R is enumerable; we see that

$$R(U_{i,j}, U_{s,t}) \leftrightarrow r_i + \rho(p_j, p_t) < r_s.$$

S1 is satisfied for R.

We proceed to show that for $\langle B_n \rangle_n \in \Sigma_0$, $\bigcap_{n=1}^{\infty} B_n$ contains at most one point. Let $B_n = U(s_n, q_n)$; $s_n \in \langle r_n \rangle_n$, $q_n \in \langle p_n \rangle_n$. We want to prove that $\lim_{n\to\infty} s_n = 0$.

$R(U(4^{-1}s_\nu, q_\nu), U(2^{-1}s_\nu, q_\nu))$ holds; therefore a $\mu > \nu$ can be found such that

$$U(4^{-1}s_\nu, q_\nu) \cap U(s_\mu, q_\mu) = \emptyset \vee R(U(s_\mu, q_\mu), U(2^{-1}s_\nu, q_\nu))$$

(since $\neg$ Sec (A, B) always implies $A \cap B = \emptyset$).

In the first case $\rho(q_\mu, q_\nu) + s_\mu < s_\nu$ and $\rho(q_\mu, q_\nu) \not< s_\mu$ (since $q_\nu \notin U(s_\mu, q_\mu)$). So $s_\mu < s_\nu - \rho(q_\mu, q_\nu)$, hence $2s_\mu < s_\nu$, therefore $s_\mu < 2^{-1}s_\nu$. In the second case immediately $s_\mu < 2^{-1}s_\nu$. Hence $\lim_{n\to\infty} s_n = 0$.

Now we can choose $\Pi \subset \Sigma_0$ such that S2 holds. Let Γ be complete. We define

$$S(U_{i,j}, U_{s,t}) \leftrightarrow \rho(p_j, p_t) > r_i + r_s.$$

S is then easily proved to be enumerable; all requirements of S6 are satisfied. If our only interest is to prove Γ to be an S^*-space we may take $\mathfrak{A}_k = \{U(r_i, p_j) : r_i < k^{-1}\}$.

The verification of S4 is easy.

Since $\langle B_n \rangle_n \simeq \langle C_n \rangle_n$ is equivalent to $\lim_{n\to\infty} q_n = \lim_{n\to\infty} x_n$, if $B_n = U(s_n, q_n)$, $C_n = U(t_n, x_n)$, $\langle B_n \rangle_n, \langle C_n \rangle_n \in \Pi$, there exists a bi-unique correspondence between the equivalence classes with respect to $\simeq$ and the points of Γ. By a comparison of the definitions of open species it follows directly that the same topology is obtained in both cases.

Conversely, since an S_0-space is metrizable, and hence as a consequence of S2 separable, (a) and (b) of our theorem are proved.

REMARK 4. It is not difficult to verify (notation of [4], 3.3.1) that, by taking $\mathfrak{A} = \langle \text{Interior } A_{i,j} \rangle_{i,j}$, an S^*-space can be constructed from every CIN-space.

THEOREM 11. Let Γ be an S-space, and let $I \subset N$.
$\cup \{W_i : i \in I\} \supset \Gamma \to \cup \{\text{Interior } W_i : i \in I\} \supset \Gamma.$

PROOF. Analogous to the proof of theorem 2 in [5], using theorems 8 and 9 of this papeı.

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