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The equivalence of two definitions of elementary formal system

Dedicated to A. Heyting on the occasion of his 70^{th} birthday

by

Haskell B. Curry

1. Introduction

In his book [TFS] [1] Smullyan introduced a notion of elementary formal system. This notion is evidently closely related to a notion which I called by the same name in [CFS] p. 267 and [FML] p. 68. It is clear that there is some sense in which these two definitions are equivalent; I found it interesting to work out explicitly the exact relationship between them. This note contains the result of this study.

In order to clarify matters I shall here use the term "*elementary formal system*" (efs) exclusively in the sense of [CFS], whereas I shall call Smullyan's notion a *smef*. Likewise, although I shall on the whole follow the terminology of the sources, I shall use modifications designed to avoid confusion between similar terms as applied to the different sorts of system.

Enough explanation is given here to enable the reader to follow the main argument without referring to the sources; but such reference may be useful for additional details, illustrative examples, discussion of terms which are here taken as self-explanatory, etc.

Some emendations of [FML] are considered, mostly in the footnotes.

2. Elementary formal systems

Let us begin by recapitulating the principal features of the notion of efs.

A formal theory is essentially a class $\mathfrak{T}$ of *elementary theorems*

[1] The letters in brackets refer to the Bibliography according to the system explained there.

constituting a subclass of a class $\mathfrak{E}$ of *elementary statements*. The class $\mathfrak{T}$ is generated from a subclass $\mathfrak{A}$ by rules $\mathfrak{R}$. The classes $\mathfrak{E}$, $\mathfrak{A}$, and $\mathfrak{R}$ are definite classes in the sense of [FML] § 2A5 [2]; and there must be an effective way of checking a proof of an elementary theorem, i.e. of seeing whether or not the rules have been correctly used; but the class $\mathfrak{T}$ is in general only semidefinite.

A *formal system* is a formal theory in which we postulate a class $\mathfrak{O}$ of formal objects and a class $\mathfrak{P}$ of elementary predicates of various degrees (where the *degree* of a predicate is the number of its arguments), such that each elementary statement asserts that a predicate P of degree n holds for an ordered n-tuple $u_1, \cdots, u_n$ of objects from $\mathfrak{O}$; this statement can be expressed thus:

$$Pu_1u_2 \cdots u_n.$$

In regard to the formal objects, [FML] describes two alternatives. In the first alternative, called a *syntactical system*, the formal objects are the *words* (i.e. finite strings), formed from the *letters* of a certain *alphabet* $\mathfrak{B}$. In the second alternative, called an *ob system*, the formal objects are generated from the certain *atoms* by certain *operations*.[3] The differences between these two alternatives are hardly relevant for our present purpose. If we assign to the atoms letters from an alphabet $\mathfrak{B}_0$, and to the operations of degree $k > 0$ (i.e. with k arguments) letters in an alphabet $\mathfrak{B}_k$, and if we assign to the obs words formed from all these letters by some standard notation such as the Łukasiewicz parenthesis-free notation, then the obs will indeed be words in the alphabet $\mathfrak{B}$ which is the union of all the $\mathfrak{B}_k$; only they will not constitute all

[2] One of my students, L. Fleischhacker, has made the following comment on the remarks made there. The notion of definiteness really has meaning, for an infinite class, only with reference to some wider class, which we may call a fundamental class or a universe. Thus, take the class $\mathfrak{W}$ of words of finite length in a finite alphabet $\mathfrak{B}$; this was cited in [FML] as a prime example of a definite class. But if there really were such things as words of infinite length it might not be possible to decide in finite terms when we had such a phenomenon. Thus when we say that $\mathfrak{E}$ is a definite class we are implicitly involving some universe of recognizably finite structure. Thus in [FML] p. 45 $\mathfrak{E}$ is taken as definite relative to the class of U-expressions; and in other places, where the situation is less exacting, it is said simply that, given a statement, it is a definite question whether it is in $\mathfrak{E}$. A similar remark applies to $\mathfrak{O}$ and some other classes below. But $\mathfrak{A}$ and $\mathfrak{T}$ are considered with reference to $\mathfrak{E}$ as universe.

[3] This is meant in the sense that to each ob there corresponds a unique construction which can be exhibited in the form of a labeled tree diagram. Thus the obs can be exhibited as symbolic constructs of treelike structure instead of strings.

the words in that alphabet, but only a certain well defined subset.[4] Thus if we admit the possibility that not all the words in $\mathfrak{B}$ are admissible as formal objects, but only certain "well formed" ones, we include both syntactical and ob systems.[5] I shall use the term "*ob*" for a formal object in either type of system; the term "*wef*", which I have sometimes suggested for the purpose, will be reserved for a related use in connection with a smef.

An elementary formal system (*efs*) is one in which the rules have a certain form. An elementary rule instance is a statement of the form

$$A_1, \cdots, A_m \Rightarrow A_0, \tag{1}$$

where $A_1, \cdots, A_m, A_0$ are elementary statements; (1) means that if $A_1, \cdots, A_m$ are in $\mathfrak{T}$ then A_0 is in $\mathfrak{T}$. An elementary rule scheme is a statement of the form (1) containing certain intuitive (or U-) variables such that when arbitrary formal objects are substituted for these variables the result is an elementary rule instance. An elementary formal system is then one in which the axioms are given by a finite number of axiom schemes and the rules are given by a finite number [6] of elementary rule schemes.

We can state this definition otherwise by introducing the notion of ob extension. Let $x_1, \cdots, x_q$ be all the variables appearing in the rule schemes of a system $\mathfrak{S}$. Let $\mathfrak{S}^*$ be the formal system obtained from $\mathfrak{S}$ by adjoining $x_1, \cdots, x_q$ to the formal objects as letters to be added to $\mathfrak{B}$ (or $\mathfrak{B}_0$) without making any other changes. Now if we define an elementary rule scheme of $\mathfrak{S}$ as an elementary rule instance of $\mathfrak{S}^*$, then it is clear that by substituting formal objects of $\mathfrak{S}$ for $x_1, \cdots, x_q$ in a rule scheme of $\mathfrak{S}$ we get a rule instance of $\mathfrak{S}$.

In an efs an elementary statement B is in $\mathfrak{T}$ just when there is a sequence $B_1, \cdots, B_p$ such that B_p is B and every B_k is either in $\mathfrak{A}$ or is a consequence of some of its predecessors by a rule R; this last means that there is a rule instance of form (1) such that A_0 is B_k and every A_i for $i > 0$ is some B_j for $j < k$. Further a

[4] The converse transformation, from a syntactical to an ob system, is also possible, but involves more complex ideas. If the operation is concatenation and the letters are the atoms, then the same word may have different constructions. Thus the syntactical system is a quotient structure of the ob system with respect to equality generated by the associative law. (Cf. [FML], § 2C3, Examples 3 and 4.)

[5] Of course still other types of system may be included.

[6] Conceivably we could generalize this by allowing, e.g., a recursive set of axiom and rule schemes. In that case we should need to make a corresponding change in the definition of a smef (see § 3).

statement (1) is a *derived rule* just when there is a sequence $B_1, \cdots, B_p$ such that B_p is A_0 and every B_k is either 1) in $\mathfrak{A}$, 2) one of the A_i for $i > 0$, or 3) the conclusion of a rule instance of form (1) (of course not with the same A_i) whose premises are all among the B_j for $j < k$.

It is perhaps worth remarking that in the foregoing an alphabet is only a finite set of objects which we call *letters*. Generally these letters have an order, called the alphabetical order, but that order plays no role here. What objects the letters (or atoms) are is completely arbitrary.

3. The notion of smef

We shall now consider the Smullyan definition. As explained in § 1, I shall change some of the terminology when there is conflict with that already introduced in § 2. In view of the explanation made in § 2 we can now be somewhat more brief.

A *smef* is a collection $\mathfrak{S}'$ consisting of the following items:

1) An alphabet $\mathfrak{K}$.

2) Another alphabet $\mathfrak{V}$ of variables, i.e. indeterminates.

3) An alphabet $\mathfrak{Q}$ of "predicates", each of which has a certain fixed degree. To avoid confusion with "predicate" as used in § 2, I shall call these *attributes*.

4) The implication sign "→" [7] and devices for indicating the application of attributes to their arguments (for this Smullyan uses a comma, I shall use the ordinary mathematical device).

5) A finite set [8] of axioms each of which is a wef in the sense below.

In terms of these notions Smullyan defines categories as follows:

Terms. These are certain words in the combined alphabets $\mathfrak{K}$ and $\mathfrak{V}$. Smullyan allows arbitrary words; but he could equally well allow any definite subset of these words, so they could be the obs of an ob system or its term extension. Terms without variables I shall call *constant* terms.

Elementary wefs. These are of the form $\Phi(t_1, \cdots, t_n)$ where $t_1, \cdots, t_n$ are terms and Φ is an attribute of degree n.

Wefs (well formed formulas). These constitute an inductive class generated as follows: (f1) every elementary wef is a wef,

[7] This use of "→" is not to be confused with the usage of [FML]; also "→" is not to be confused with "⇒".

[8] On this restriction cf. footnote 6.

($f2$) if X_2 is a wef and X_1 is an elementary wef, then $X_1 \to X_2$ is a wef. Thus the wefs are of the form

$$X_1 \to X_2 \to \cdot \;\cdots\; \cdot \to X_n \to X_0 \tag{2}$$

where $X_0, X_1, \cdots, X_n$ are elementary wefs.[9]

The theorems in $\mathfrak{S}'$ are an inductive class generated from the axioms by two rules, substitution and modus ponens. The elementary theorems are those theorems which are also elementary wefs. By the usual process of "Rückverlegung der Einsetzungen" ([FML] § 3D3, p. 115)[10] we can suppose that the substitutions are made in the axioms, so that the axioms are given by axiom schemes; then the sole rule is modus ponens.

4. The equivalence theorem

The result of this note is the following:

THEOREM 1. *Let $\mathfrak{S}$ be an efs and $\mathfrak{S}'$ be a smef such that the following conditions are satisfied:*

(i) *The formal objects of $\mathfrak{S}$ are precisely the constant terms of $\mathfrak{S}'$.*

(ii) *There is a one-one correspondence preserving degree between the elementary predicates of $\mathfrak{S}$ and the attributes of $\mathfrak{S}'$. This establishes a correspondence which associates to each elementary statement X of $\mathfrak{S}$ (or some term extension of it) a unique elementary wef X' of $\mathfrak{S}'$ and vice versa.*

(iii) *The rules and (for $n = 0$) the axioms of $\mathfrak{S}$ are certain schemes of the form*

$$X_1, X_2, \cdots, X_n \Rightarrow X_0, \tag{3}$$

where $X_0, X_1, \cdots, X_n$ are obs in a term extension of $\mathfrak{S}$, the indeterminates functioning as U-variables for formal objects; the axioms of $\mathfrak{S}'$ are precisely the corresponding schemes

$$X'_1 \to X'_2 \to \cdot \;\cdots\; \cdot \to X'_n \to X'_0. \tag{4}$$

Further, let $A_0, A_1, \cdots, A_m$ be elementary statements of $\mathfrak{S}$, and $A'_0, A'_1, \cdots, A'_m$ be the corresponding elementary wefs of $\mathfrak{S}'$.

Then a necessary and sufficient condition that (1) *be a derived*

[9] Note that if "$\to$" is regarded as an operational sign it is associated to the right; i.e. (2) is formed from X_1 and $X_2 \to \cdots \to X_n \to X_0$, but not, e.g., from $X_1 \to X_2$ and $X_3 \to \cdots \to X_n \to X_0$.

[10] This is frequently ascribed to von Neumann; but in principle it appears in Post [IGT], Lemma 2, p. 178.

rule in $\mathfrak{S}$, is that A'_0 be an elementary theorem in that smef $\mathfrak{S}''$ which is formed by adjoining $A'_1, \cdots, A'_m$ to the axioms of $\mathfrak{S}'$.

Proof of necessity. Let (1) hold. Then there exists a sequence $B_1, B_2, \cdots, B_p$ constituting a derivation Δ of A_0 from $A_1, \cdots, A_n$. I shall show by deductive induction that every B'_k is an elementary theorem of $\mathfrak{S}''$. If B_k is one of the A_i, then B'_k is A'_i and hence an axiom of $\mathfrak{S}''$. Likewise, if B_k is an axiom of $\mathfrak{S}$, B'_k is an axiom of $\mathfrak{S}'$, and so of $\mathfrak{S}''$. Finally, if B_k is obtained by an inference, let the rule instance be

$$C_1, C_2, \cdots, C_r \to B_k,$$

where $C_1, \cdots, C_r$ precede B_k in Δ. Then

$$C'_1 \to C'_2 \to \cdots \to C'_r \to B'_k$$

is an axiom of $\mathfrak{S}'$ and hence a theorem of $\mathfrak{S}''$. By the inductive hypothesis each C'_j is a theorem of $\mathfrak{S}''$. Hence, by repeated applications of modus ponens, B'_k is also.

Proof of sufficiency.[11] Let $B'_1, \cdots, B'_p$ be a derivation Δ' in $\mathfrak{S}''$ of A'_0. Each B'_k is then of the form

$$C'_1 \to C'_2 \to \cdots C'_r \to C'_0,$$

where the C'_j are elementary. Then I shall show by induction on k that

$$A_1, A_2, \cdots, A_m, C_1, \cdots, C_r \Rightarrow C_0 \qquad (5)$$

is a derived rule in $\mathfrak{S}$. If B'_k is A'_i, then $r = 0$ and $C_0 \equiv A_i$; (5) is obvious. If B'_k is an axiom of $\mathfrak{S}'$, then (5) holds without the $A_1, \cdots, A_m$, and hence a fortiori with them. If B'_k is obtained by an inference, then there is an elementary wef D' such that D' and $D' \to B'_k$ are both theorems of $\mathfrak{S}''$. By the inductive hypothesis we have

$$A_1, A_2, \cdots, A_m \Rightarrow D,$$
$$A_1, A_2, \cdots, A_m, D, C_1, \cdots, C_r \Rightarrow C_0.$$

Let the second derivation be exhibited as a tree proof; over each occurrence of D as a top node put the first derivation of D; the result will be a tree proof of (5).

For $k = p$ we have $r = 0$ and $C_0 = A_0$. Then (5) gives us (1), q.e.d. This completes the proof of the theorem.

[11] Note that in this proof the B'_k are not necessarily elementary wefs.

REMARK 1. The restriction that the A'_i be constant can be set aside by adjoining any variables they contain to $\mathfrak{S}$, i.e. using a suitable term extension of $\mathfrak{S}$. There is no difficulty with substitutions because we have excluded the substitution rule.

COROLLARY 1.1. *The elementary theorems of $\mathfrak{S}$ correspond to the elementary theorems of $\mathfrak{S}'$ and vice versa.*

PROOF. This is the special case $n = 0$.

COROLLARY 1.2. *Any theorem of $\mathfrak{S}'$ corresponds to a derived rule of $\mathfrak{S}$.*

PROOF. If

$$A'_1 \to A'_2 \cdots \to A'_m \to A'_0$$

is a theorem of $\mathfrak{S}'$, then A'_0 is a theorem of $\mathfrak{S}''$.

REMARK 2. The converse of Corollary 1.2 does not hold because there is no deduction theorem in $\mathfrak{S}'$. A counterexample is the case where $n = 0$ in all axioms (4); then one cannot establish $A'_0 \to A'_0$. Moreover not every derived rule of $\mathfrak{S}$ is a theorem of $\mathfrak{S}'$. In fact every theorem of $\mathfrak{S}'$ is an end-segment of an axiom; i.e., it is obtained from an axiom of the form (4) by cutting off some of the initial X'_i. Thus if the axioms and (primitive) rules of $\mathfrak{S}$ are

$$A, \quad A \Rightarrow B, \quad B \Rightarrow C,$$

then

$$A \Rightarrow C$$

is a derived rule, but

$$A' \to C'$$

is not a theorem of $\mathfrak{S}'$.

REMARK 3. If the elementary wefs are statements, then $\to$ is a conjunctor, and a smef is not an efs. This can be got around by regarding the Φ_j and $\to$ as operators, much as in the reduction of a formal system to assertional form (of [FML] § 2D1). The restriction in modus ponens to the effect that the premise of an implication must be elementary can also be got rid of, provided that the axioms have that character.[12] By this maneuver a smef becomes a special kind of assertional efs.

[12] This, of course, depends on the nonassociativity mentioned in footnote 9. The conclusion follows by an induction on the number of steps in the proof.

5. Concluding remarks

Theorem 1 shows that a smef and an efs are essentially equivalent notions. Note that either $\mathfrak{S}$ or $\mathfrak{S}'$ could be given first and the other determined from it. Thus every efs can be exhibited as a smef and vice versa; and if one identifies the attributes of $\mathfrak{S}'$ with the elementary predicates of $\mathfrak{S}$ the two systems will have the same elementary theorems (Corollary 1.1.) and the same derived elementary rules.[13] A smef has the advantage that all smefs have the same rules; on the other hand it has certain disadvantages when it comes to applying techniques of formal deducibility.[14]

BIBLIOGRAPHY

Works are cited by three letter abbreviations which have been used, except where editors insisted otherwise, in all my publications since [CLg.I.]. Where no author's name is expressed or implied the author is Curry or Curry and Feys.

H. B. Curry

[CFS]. Calculuses and formal systems. Dialectica 12: 249—273 (1958).

[FML]. Foundations of Mathematical Logic. New York, McGraw-Hill Book Co., 1963.

H. B. Curry and Robert Feys

[CLg.I] Combinatory Logic, vol. I. Amsterdam, North-Holland Publishing Co., 1958.

Emil L. Post

[IGT]. Introduction to a general theory of elementary propositions. Amer. J. Math. 43: 163—185 (1921).

Raymond M. Smullyan

[TFS]. Theory of Formal Systems. Princeton, N.J., Princeton University Press, 1961.

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[13] Note that in the example at the end of Remark 2,

$$A', B' \rightarrow C'$$

is indeed an elementary derived rule (in a self-explanatory sense) of $\mathfrak{S}'$, although its analogue is not a theorem of $\mathfrak{S}'$.

[14] Such as those which are considered in the second half of [FML]. The situation noted in Remark 2 is also a disadvantage.