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Almost direct products and saturation

Dedicated to A. Heyting on the occasion of his 70th birthday

by

Bjàrni Jónsson and Philip Olin [1]

1. Introduction

The purpose of this note is to prove that a certain reduced product of countably many relational structures of a countable similarity type always is an ω_1-saturated structure. In [1] (and later in [7]) this product was called an "almost everywhere direct product" but we shorten the name here to "almost direct product". A more special result of a related nature was obtained by Keisler in [5]; assuming the continuum hypothesis he showed that the almost direct product of countably many copies of a Boolean algebra of power at most ω_1 is a universal-homogeneous Boolean algebra of power ω_1. Our argument is to some extent similar to the one given by Keisler, and as we shall show, his result can be easily derived from ours.

The notion of an m-saturated structure, and the related notions of a universal-homogeneous structure and of a $\mathscr{K}$-universal-homogeneous structure ($\mathscr{K}$ a class of structures), have played a considerable role in recent model-theoretic investigations. In a few instances these structures occur in a natural way, e.g. the rational numbers with their ordering relation, but the known proofs of the general existence theorems in this domain employ highly non-constructive methods. Thus the original approach used by Jónsson in [2] and [3] and by Morley and Vaught in [6] is based on transfinite sequences of extensions and amalgamations of relational structures, while the later techniques developed by Keisler in [4] make use of special ultraproducts. Of course it

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should be observed that while our structures are obtained by a simple algebraic construction, the proof that they are ω_1-saturated is not an effective one. However, our results should serve to make these notions more concrete than they were before.

2. Preliminaries

We consider relational structures A of a fixed similarity type μ (briefly, μ-structures) consisting of a non-empty set $|A|$ and indexed families of operations and relations of finite rank over $|A|$. L_μ is the corresponding first order language with variables $v_0, v_1, v_2, \cdots, \Phi_\mu$ the set of all formulas of L_μ, and $\Phi_\mu(1)$ the set of all those formulas of L_μ in which no variable occurs free except possibly v_0. If t is a term in L_μ then t^A is the corresponding operation of rank ω over $|A|$; i.e. for each $x \in {}^\omega|A|$ (the set of all maps from ω into $|A|$), $t^A(x)$ is the value of the term t in A under the assignment of x_k for v_k $(k \in \omega)$. Similarly, if $\varphi \in \Phi_\mu$ then φ^A is the set of all $x \in {}^\omega|A|$ such that x satisfies φ in A. Of course, whether or not a given sequence x belongs to φ^A depends only on finitely many of the terms x_k; and φ^A may therefore be identified with a suitable relation of finite rank over A. In particular for $\varphi \in \Phi_\mu(1)$, φ^A may be identified with a subset of $|A|$.

A set $\Sigma \subseteqq \Phi_\mu$ is said to be satisfiable in A if some sequence of elements of $|A|$ satisfies every member of Σ in A. Σ is said to be finitely satisfiable in A if every finite subset of Σ is satisfiable in A. Given a cardinal m, the structure A is said to be m-saturated provided the following condition holds: for any set $X \subseteqq |A|$ with $\overline{\overline{X}} < m$, if $A' = (A, x)_{x \in X}$ is the structure obtained from A by adjoining the members of X as distinguished elements (operations of rank 0), and if μ' is the corresponding similarity type, then every subset of $\Phi_{\mu'}(1)$ that is finitely satisfiable in A' is satisfiable in A'. A is said to be saturated if it is m-saturated, where m is the cardinality of A.

By the almost direct product of μ-structures $A_i (i \in I)$ we mean the reduced product

$$B = \Pi_F(A_i, i \in I)$$

where F is the filter consisting of all cofinite subsets of I. We need to apply to this product the fundamental theorem of Feferman and Vaught on generalized products (Theorem 3.1 of [1]). Let C be the direct product of the structures A_i, and for $x \in |C|$ and $\psi \in {}^\omega\Phi_\mu(1)$ let $K(\psi, x)$ be the sequence of subsets of I such that, for all $k \in \omega$,

$$K(\psi, x)_k = \{i \in I : x(i) \in (\psi_k)^{A_i}\}.$$

Let D be the Boolean algebra of all subsets of I (with the usual set-theoretic operations), and let E be the quotient algebra of D mod F. Let σ be the similarity type of the Boolean algebras. Applied to this special situation, the Feferman-Vaught theorem yields the following result: for any $\varphi \in \Phi_\mu(1)$ there exist $\alpha \in \Phi_\sigma$ and $\psi \in {}^\omega\Phi_\mu(1)$ such that, for each $x \in |C|$,

$$x/F \in \varphi^B \text{ if and only if } K(\psi, x)/F \in \alpha^E.$$

(Here x/F is the equivalence class of x mod F, and $K(\psi, x)/F$ is the sequence whose k-th term is the equivalence class of $K(\psi, x)_k$ mod F.) Of course the result as stated here is not in its strongest form; e.g. the infinite sequence ψ can obviously be replaced by a finite sequence, and both α and ψ can be constructed in an effective manner and are independent of the structures A_i.

We shall also need the fact (Skolem [8]) that, because E is atomless, every formula in Φ_σ is equivalent in E to a quantifier-free formula.

3. The fundamental theorem

THEOREM 1. If μ is a countable similarity type then the almost direct product of countably many μ-structures is ω_1-saturated.

PROOF. Let the given structures be A_ν ($\nu \in \omega$), and let B be their almost direct product and C their direct product. Also let D be the Boolean algebra of all subsets of ω, and E the quotient algebra of D modulo F, the filter of all cofinite subsets of ω. We shall prove that every subset Γ of $\Phi_\mu(1)$ which is finitely satisfiable in B is satisfiable in B. The theorem can then be obtained by applying this result to the augmented structures $(A_\nu, x(\nu))_{x \in X'}$ where X is a countable subset of $|C|$.

Since the similarity type μ is countable, so is Γ, and the members of Γ can be arranged in a sequence,

$$\Gamma = \{\varphi_0, \varphi_1, \varphi_2, \cdots, \varphi_n, \cdots\}.$$

By the theorem of Feferman and Vaught there exist, for each $n \in \omega$, a formula $\alpha_n \in \Phi_\sigma$ and a sequence of formulas $\psi_n \in {}^\omega\Phi_\mu$ such that for any $x \in |C|$,

$$x/F \in \varphi_n^B \text{ iff } K(\psi_n, x)/F \in \alpha_n^E.$$

As observed before, α_n can be taken to be quantifier-free and we may therefore assume that it is a disjunction

$$\bigvee_{k<r(n)} \beta_{n,k}$$

where each $\beta_{n,k}$ is a conjunction of equations and of negations of equations. Each equation can be written in the form $t = 0$, and the conjunction of two equations is equivalent (in a Boolean algebra) to a single equation. Consequently each $\beta_{n,k}$ may be assumed to be of the form

$$u_{n,k} = 0 \wedge \bigwedge_{j<s(n,k)} \sim(v_{n,k,j} = 0).$$

The set of formulas α_n has the property that for each finite subset J of ω there exists a member x of $|C|$ which leads to a solution of all the formulas α_n with $n \in J$, in the sense that

$$K(\psi_n, x)/F \in \alpha_n^E \text{ for all } n \in J.$$

It readily follows that for a suitable $k(0) < r(0)$ the set obtained by replacing α_0 by $\beta_{0,k(0)}$ has the same property. For otherwise there would exist for each $k < r(0)$ a finite set J_k of positive integers such that no member of $|C|$ leads to a solution of $\beta_{0,k}$ and of all the formulas α_n with $n \in J_k$. But then the union J of the sets J_k would be a finite set with the property that no member of $|C|$ leads to a solution of α_0 and of all the formulas α_n with $n \in J$. This contradiction proves our assertion.

By an iteration of this argument we obtain natural numbers $k(n) < r(n)$ for $n = 0, 1, 2, \cdots$ such that for each finite subset J of ω there exists a member x of $|C|$ that leads to a solution of $\beta_{n,k(n)}$ for each $n \in J$. Let $s(n, k(n)) = s(n)$, $\beta_{n,k(n)} = \beta_n$, $u_{n,k(n)} = u_n$, $v_{n,k(n),j} = v_{n,j}$, and for each $n \in \omega$ choose a member c_n of $|C|$ that leads to a solution of β_i for all $i < n$. This means that

$$u_i^D(K(\psi_i, c_n)) \text{ is finite,}$$
$$v_{i,j}^D(K(\psi_i, c_n)) \text{ is infinite}$$

whenever $i < n$ and $j < s(i)$. Therefore there exists an increasing sequence of natural numbers

$$0 = \lambda_0 < \lambda_1 < \lambda_2 < \cdots$$

such that if $i < n$ and $j < s(i)$ then

for all $\nu \geqq \lambda_n$, $\nu \notin u_i^D(K(\psi_i, c_n))$,

for some ν, $\lambda_n \leqq \nu < \lambda_{n+1}$ and $\nu \in v_{i,j}^D(K(\psi_i, c_n))$.

We then define the member b of $|C|$ by letting

$$b(\nu) = c_n(\nu) \text{ whenever } \lambda_n \leqq \nu < \lambda_{n+1}.$$

If $\lambda_n \leqq \nu < \lambda_{n+1}$ then $b(\nu) = c_n(\nu)$, and therefore the conditions

$$\nu \in K(\psi_i, b)_k, \quad \nu \in K(\psi_i, c_n)_k$$

are equivalent. Hence for every term t in L_σ the conditions

$$\nu \in t^D(K(\psi_i, b)), \quad \nu \in t^D(K(\psi_i, c_n))$$

are equivalent. Consider a fixed $i \in \omega$. For each $\nu \geqq \lambda_{i+1}$ there exists $n \geqq i+1$ such that $\lambda_n \leqq \nu < \lambda_{n+1}$. Therefore ν does not belong to $u_i^D(K(\psi_i, c_n))$, and hence does not belong to $u_i^D(K(\psi_i, b))$. The set $u_i^D(K(\psi_i, b))$ is therefore finite. Also, given $j < s(i)$, for each $n \geqq i+1$ there exists ν such that $\lambda_n \leqq \nu < \lambda_{n+1}$ and $\nu \in v_{i,j}^D(K(\psi_i, c_n))$ and therefore $\nu \in v_{i,j}^D(K(\psi_i, b))$. Consequently each of the sets $v_{i,j}^D(K(\psi_i, b))$ is infinite. We infer that, for each $i \in \omega$, b leads to a solution of β_i in E, and a fortiori b therefore leads to a solution of α_i in E, so that b/F satisfies φ_i in B.

The proof is now complete.

Corollary 2. If the continuum hypothesis holds and if μ is a countable similarity type, then the almost direct product of countably many μ-structures of power at most ω_1 is saturated.

Proof. The cardinality of the almost direct product is in this case at most $\omega_1^\omega = \omega_1$.

4. Applications and open problems

Given a class $\mathscr{K}$ of μ-structures, a cardinal m, and a member A of $\mathscr{K}$, A is said to be $(\mathscr{K}, m)$-universal if every member of $\mathscr{K}$ of power at most m is isomorphic to a substructure of A. If, for any substructures B and B' of A that belong to $\mathscr{K}$ and whose power is less than m, every isomorphism of B onto B' can be extended to an automorphism of A, then A is said to be $(\mathscr{K}, m)$-homogeneous. For any A we let $A^* = (A, \varphi^A)_{\varphi \in \Phi_\mu}$, and we let $\mathscr{K}^* = \{A^* : A \in \mathscr{K}\}$. A μ-structure A of power m is said to be universal-homogeneous if and only if A^* is $(\mathscr{N}^*, m)$-universal-homogeneous, where $\mathscr{N}$ is the elementary type of A. Thus A is universal-homogeneous if and only if every μ-structure of power at most m that is elementarily equivalent to A is isomorphic to an elementary substructure of A, and every isomorphism between elementary substructures of A whose power is less than m can be extended to an automorphism of A. According to a result of Keisler (cf. Morley and Vaught [6], Theorem 3.4), A is universal-homogeneous if and only if it is saturated.

A μ-structure $A \in \mathscr{K}$ of power m may be universal-homogeneous without being $(\mathscr{K}, m)$-universal-homogeneous. Therefore Keisler's theorem mentioned in the introduction does not follow immediately from Theorem 1. In order to derive it we need the following result:

Lemma 3. Let m be an infinite cardinal and $\mathscr{B}$ the class of all Boolean algebras. A Boolean algebra B of power m is $(\mathscr{B}, m)$-universal-homogeneous if and only if B is atomless and saturated.

Proof. We make use of the fact that the class $\mathscr{N}$ of all atomless Boolean algebras is an elementary type, and that every infinite Boolean algebra can be embedded in an atomless Boolean algebra of equal power.

Suppose B is $(\mathscr{B}, m)$-universal-homogeneous. Given $a \in |B|$ with $0 < a < 1$, the four-element Boolean algebra A generated by a has an automorphism f that takes a into $\bar{a}$, and f can be extended to an automorphism of B. Since a and $\bar{a}$ cannot both be atoms of B, it follows that a is not an atom of B. Thus B is atomless. Because of the theorem on elimination of quantifiers in atomless Boolean algebras, every atomless subalgebra of B is an elementary subalgebra of B. Therefore every atomless Boolean algebra of power at most m is isomorphic to an elementary subalgebra of B. Consequently B^* is $(\mathscr{N}^*, m)$-universal. Furthermore, the assumption that B is $(\mathscr{B}, m)$-homogeneous obviously implies that B^* is $(\mathscr{N}^*, m)$-homogeneous. Thus B is universal-homogeneous and is therefore saturated.

Next suppose B is atomless and saturated. Then B^* is $(\mathscr{N}^*, m)$-universal and hence B is $(\mathscr{N}, m)$-universal. Since every infinite Boolean algebra can be embedded in an atomless Boolean algebra of equal power, it follows that B is $(\mathscr{B}, m)$-universal. To obtain the homogeneity of B we observe that every Boolean algebra has up to isomorphism a unique minimal embedding in an atomless Boolean algebra. Given two subalgebras C and D of B of power less than m, and an isomorphism f of C onto D, we can embed them minimally in atomless subalgebras C' and D' of B and extend f to an isomorphism f' of C' onto D'. If $m = \omega$ then C' and D' can both be taken to be equal to B; but if $m > \omega$, then C' and D' are of power less than m. Since C' and D' are elementary subalgebras of B, f' can be extended to an automorphism of B. We therefore conclude that B is $(\mathscr{B}, m)$-universal-homogeneous, and the proof is complete.

THEOREM 4. (Keisler) Let $\mathscr{B}$ be the class of all Boolean algebras. Assuming the continuum hypothesis, the almost direct product of countably many Boolean algebras of power at most ω_1 is $(\mathscr{B}, \omega_1)$-universal-homogeneous.

PROOF. The given almost direct product is obviously atomless. Hence the conclusion follows by Corollary 2. and Lemma 3.

It would be interesting to have further examples where our method yields $(\mathscr{K}, \omega_1)$-universal-homogeneous structures B, for familiar classes $\mathscr{K}$. There are of course trivial examples, e.g. the class $\mathscr{K}$ of all Abelian groups of some fixed exponent n, but there B can be obtained by even more elementary methods. One can easily find examples of classes $\mathscr{K}$ for which this method does not work. For example, let $\mathscr{K}$ be the class of all partially ordered sets. We may assume that all the factors A_ν contain two elements p and q with $p < q$. In the direct product C, consider the elements

$$a_1 = \langle p, p, q, p, p, q, p, p, q, p, p, q, \cdots \rangle$$
$$a_2 = \langle p, q, p, p, q, p, p, q, p, p, q, p, \cdots \rangle$$
$$a_3 = \langle p, p, q, p, q, p, p, p, q, p, q, p, \cdots \rangle$$
$$a_3' = \langle q, p, p, q, p, p, q, p, p, q, p, p, \cdots \rangle$$

In the almost direct product B, the set $\{a_1/F, a_2/F, a_3/F\}$ is unordered, and so is $\{a_1/F, a_2/F, a_3'/F\}$. Hence the first set can be mapped isomorphically onto the second. However, every common upper bound for a_1/F and a_2/F is also an upper bound for a_3/F, while the corresponding statement for the second set is false. Hence the given isomorphism cannot be extended to an automorphism of B.

A similar example can be found in section 3 of [7].

If the factors A_ν are Boolean algebras, then B is an atomless Boolean algebra and this determines its elementary type. In general, however, even if all the structures A_ν are equal to the same finite structure A, we do not know how to describe the elementary type of B. This is true even in the very simple case when A is a three-element linearly ordered set. The case when A is the ring of integers Z is discussed in [7].

Our principal result also suggests the problem of determining which filters F have the property that every F-reduced product is ω_1-saturated, as well as the corresponding problem with ω_1

replaced by a larger cardinal. In this connection we mention an unpublished result of Fred Galvin which states that if F is the union of a chain of countably complete filters, and if some countable subfamily of F has an empty intersection, then every F-reduced product is ω_1-saturated. However, we have no reason to believe that these sufficient conditions are also necessary.

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