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Laws of large numbers for functions of random walks with positive drift

by

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Summary. Let $X_1, X_2, \cdots$ be independent random variables, $X_2, X_3, \cdots$ having the same distribution with characteristic function φ and first moment $\mu > 0$, the absolute first moment being finite. Let $S_k = X_1 + \cdots + X_k$, $k = 1, 2, \cdots$ The paper gives conditions under which, for nonlattice and integer valued X_k, respectively,

$$\left\{\sum_{k=1}^{n} f(S_k) - \mu^{-1} \int_0^{n\mu} f(t)dt\right\} \Big/ b(n) \to 0,$$

or

$$\left\{\sum_{k=1}^{n} f(S_k) - \mu^{-1} \sum_{k=0}^{[n\mu]} f(k)\right\} \Big/ b(n) \to 0,$$

either in probability or a.s.

For bounded f and $b(n) = n$ these conditions take a simple form: For convergence in probability it is sufficient that, respectively, $\limsup_{|u|\to\infty} |\varphi(u)| < 1$ or the X_k are integer valued with span 1. If moreover $E|X_2|^\rho < \infty$ for some $\rho > 1$, there is convergence a.s.

As an application the condition on the renewal density in the Chung-Derman theorem on recurrent sets is eliminated.

Proofs are based on renewal theory.

1. Introduction

Throughout this paper we assume that $X_1, X_2, X_3, \cdots$ are independent nondegenerate random variables, that $X_2, X_3, \cdots$ have the same distribution function $F(\xi) \stackrel{\text{df}}{=} P\{X_k < \xi\}$, $k \geqq 2$, and that

$$\int |x| dF < \infty, \tag{1.1}$$

$$\mu \stackrel{\text{df}}{=} \int x dF > 0. \tag{1.2}$$

We denote the characteristic function of X_k, $k \geqq 2$, by φ and the distribution function and characteristic function of X_1 by F_1 and φ_1.

A random variable Y will be called here a lattice variable if

Y/c is a.s. integer valued for some $c > 0$. The span of a non-degenerate lattice variable Y is the largest c for which Y/c is integer valued.

Let

$$(1.3) \qquad S_k \overset{\text{df}}{=} X_1 + X_2 + \cdots + X_k, \qquad k = 1, 2, \cdots.$$

We intend to study convergence of

$$\sum_{k=1}^{n} f(S_k)/b(n),$$

where it will be assumed that

$$(1.4) \qquad \lim_{t\to\infty} b(t) = +\infty,$$

$$(1.5) \qquad b(s) \leqq b(t), \qquad 0 \leqq s \leqq t < \infty.$$

More precisely, we want to find conditions on the function b, the complex valued function f and the distribution of the X_k, such that for $n \to \infty$, either in probability or almost sure,

$$\left\{\sum_{k=1}^{n} f(S_k) - \mu^{-1} \int_0^{n\mu} f(t)dt\right\} \Big/ b(n) \to 0,$$

if the X_k are nonlattice, and

$$\left\{\sum_{k=1}^{n} f(S_k) - \mu^{-1} \sum_{k=0}^{[n\mu]} f(k)\right\} \Big/ b(n) \to 0,$$

if the X_k are integer valued. Here $[a] \overset{\text{df}}{=} \max\{n : n \leqq a, n \text{ integer}\}$.

Ergodic theory provides us with a number of results of this kind applying to processes of a type far more general than the random walk. On the other hand ergodic theory is mainly restricted to $b(n) = n$ and it imposes certain integrability conditions on f we do not require here. Ergodic theorems as given by Dunford and Miller [7] and Chacon and Ornstein [2] assume that the shift operator has norm not larger than 1, which does not hold for random walks.

We mention the work of Robbins [14], where f is assumed periodic or almost periodic and $\mu = 0$ or $E\{|x_k|\} = +\infty$ is not excluded, Kallianpur and Robbins [11], where f vanishes outside an interval, Harris and Robbins [10], where $f \in L_1$, Brunk [1], where $b(n) = n$ and the $E\{f(S_k)\}$ exist, and Shur [15], where $b(t) = t$ and $|f|^2$ is integrable with respect to a subinvariant measure of the transition matrix of the process. Kallianpur and

Robbins [11] and Skorokhod and Slobodenyuk [16] study the central limit problem for $\sum_{k=1}^{n} f(S_k)$.

Our proofs are based on renewal theory. Let

$$N[a, b) \stackrel{\text{df}}{=} \text{number of } k \text{ with } S_k \in [a, b), \tag{1.6}$$

$$U[a, b) \stackrel{\text{df}}{=} E\{N[a, b)\}. \tag{1.7}$$

For any distribution function G, bounded or not, we write

$$G[a, b) \stackrel{\text{df}}{=} G(b)-G(a). \tag{1.8}$$

Accordingly the process $N(t)$ and the function $U(t)$ are defined up to an additive random variable or constant, respectively, by (1.6) or (1.7) and

$$N(b)-N(a) = N[a, b), \qquad -\infty < a < b < \infty, \tag{1.6$^{\text{a}}$}$$

$$U(b)-U(a) = U[a, b), \qquad -\infty < a < b < \infty. \tag{1.7$^{\text{a}}$}$$

If the X_k are integer valued, we put

$$z_n \stackrel{\text{df}}{=} \text{number of } k \text{ with } S_k = n, \quad n = \cdots -1, 0, 1, 2, \cdots, \tag{1.9}$$

$$u(n) \stackrel{\text{df}}{=} E\{z_n\}, \qquad n = \cdots -1, 0, 1, 2, \cdots. \tag{1.10}$$

It is possible to choose F_1 in such a way that the process $\{N(t), t \geqq 0\}$ has stationary increments, in particular

$$U(t) = \mu^{-1} t, \qquad t \geqq 0. \tag{1.11}$$

This is accomplished by taking

$$\begin{aligned} F_1(\xi) &= 0, & \xi &\leqq 0, \\ F_1(\xi) &= \mu_L^{-1} \int_0^\xi \{1-L(x)\} dx, & \xi &> 0, \end{aligned} \tag{1.12}$$

where L is the distribution function of the strict ascending ladder heights in the random walk $\{X_2, X_2+X_3, \cdots\}$ and

$$\mu_L \stackrel{\text{df}}{=} \int x dL, \tag{1.13}$$

which is finite (Feller [8], Ch. XII). That $N[a_j, b_j)$, $j = 1, \cdots, m$, under (1.12) have the same joint distribution as $N[a_j+h, b_j+h)$, $j = 1, \cdots, m$, if

$$0 \leqq a_1 \leqq b_1 \leqq a_2 \leqq \cdots \leqq a_m \leqq b_m; \qquad h > 0,$$

is seen by considering the first entrances in $[a_1, \infty)$ and $[a_1+h, \infty)$. These must be strict ascending ladder points of the random walk $\{S_k\}$. Our assertion follows from the well known stationarity property of the initial distribution (1.12) for the residual lifetime in a renewal process with distribution function L of times between renewals. The relation (1.11) follows with $\lim_{t\to\infty} t^{-1}U[0, t) = \mu^{-1}$.

If the X_k, $k \geqq 2$, are integer valued, it is possible to take X_1 integer valued with a distribution such that the process $\{z_n, n = 0, 1, 2, \cdots\}$ is stationary, in particular

$$(1.14) \qquad u(n) = \mu^{-1}, \qquad n = 0, 1, 2, \cdots.$$

To this end we take

$$(1.15) \qquad P\{X_1 = n\} = \mu_L^{-1} \sum_{k>n} p_L(k), \qquad n = 0, 1, 2, \cdots,$$

where $p_L(k)$ is the probability that a strict ascending ladder height in the random walk $\{X_2, X_2+X_3, \cdots\}$ is equal to k and $\mu_L = \sum k p_L(k)$.

Following Takács [19], Appendix 3, we call the processes $\{N(t)\}$ and $\{z_n\}$ *homogeneous renewal processes* if the distribution of X_1 is given by (1.12) and (1.15), respectively. For easy reference we will call the corresponding process $\{S_k\}$ the *homogeneous random walk* and we will use the terms *basic renewal process* and *basic random walk* if $F_1 = F$. The expected numbers of renewals in the basic process will be denoted by U_0 and u_0, so that

$$(1.16) \qquad U_0(b)-U_0(a) = U_0[a, b) = \sum_{m=1}^{\infty} F^{(m)}[a, b),$$

where $F^{(m)}$ is the m-fold convolution of F, and, for integer valued X_k,

$$(1.17) \qquad u_0(k) = \sum_{m=1}^{\infty} p^{(m)}(k), \qquad k = \cdots, -1, 0, 1, 2, \cdots,$$

where $p^{(m)}(k) = P\{X_2+ \cdots +X_{m+1} = k\}$.

We start studying the convergence to zero of the expressions

$$(1.18) \qquad \Lambda(T) \overset{\text{df}}{=} \left\{\int_{[0,T)} f(t)dN(t)-\mu^{-1}\int_0^T f(t)dt\right\}\Big/ b(T),$$

and, for integer valued X_k,

$$(1.19) \qquad \lambda(n) \overset{\text{df}}{=} \left\{\sum_{k=0}^{n-1} f(k)z_k-\mu^{-1}\sum_{k=0}^{n-1} f(k)\right\}\Big/ b(n).$$

For the homogeneous renewal processes

$$(1.18^{\text{a}}) \qquad \Lambda(T) = \{b(T)\}^{-1}\int_{[0,T)} f(t)d\tilde{N}(t),$$

$$(1.19^{\text{a}}) \qquad \lambda(n) = \{b(n)\}^{-1}\sum_{k=0}^{n-1} f(k)\tilde{z}_k,$$

where $\sim$ denotes centering at expectation. The right-hand sides of (1.18) and (1.19) contain a random number of summands $f(S_j)$ and afterwards they will have to be compared with sums of a deterministic number of terms.

Our study of (1.18) and (1.19) is based on consideration of second moments, i.e. only wide sense stationarity is used. This means that our results on a.s. convergence will not be the best possible. Moreover, if $\limsup_{|u|\to\infty} |\varphi(u)| = 1$ in the nonlattice case, our methods fail to a large extent (cf. the last part of section 4). Therefore it would seem important to look for ergodic theorems ensuring a.s. existence of $\lim_{n\to\infty} \sum_k a_{nk}\zeta_k$ and $\lim_{T\to\infty} \int a(T, \tau)dZ(\tau)$ for stationary processes $\{\zeta_k\}$ and processes $\{Z(t)\}$ with stationary increments. The only general result of this type (Cohen [4]) known to the author did not seem to fit the problem at hand.

One of our results on (1.18) made it possible to eliminate the condition on the renewal density in the Chung-Derman recurrence theorem (Chung and Derman [3], Derman [5]). We refer to section 6 below.

Assumptions. Throughout the paper the assumptions on the random walk, stated on the first page, will apply, and also (1.4) and (1.5). The sets of further assumptions in our theorems will be suitable subsets of the following list of conditions:

$$\limsup_{t\to\infty} b(at)/b(t) < \infty, \qquad a > 1, \tag{1.20}$$

$$\limsup_{t\to\infty} b(t+c)/b(t) < \infty, \qquad c > 0, \tag{1.20a}$$

$$X_k \text{ is integer valued with span } 1, \qquad k \geqq 2, \tag{1.21}$$

$$\limsup_{|u|\to\infty} |\varphi(u)| < 1, \tag{1.22}$$

$$\int x^2 dF(x) < \infty, \tag{1.23}$$

$$\varphi(u)-1-i\mu u = O(|u|^{1+\theta}), \qquad u \to 0, \tag{1.24}$$

for some $\theta \in (0, 1)$, which is implied by

$$\int |x|^{1+\theta} dF(x) < \infty, \tag{1.24*}$$

$$\{b(n)\}^{-1} \sum_{j=0}^{n-1} |f(j)| \leqq c < \infty, \qquad n \geqq 1, \tag{1.25}$$

$$\lim_{n\to\infty} \{b(n)\}^{-2} \sum_{j=0}^{n-1} |f(j)|^2 = 0, \tag{1.26}$$

$$\sum_{k=1}^{\infty} |f(k)/b(k)|^2 ({}^2\log k)^2 < \infty. \tag{1.27}$$

There exist $\varepsilon_1(x) \downarrow 0$ and $\varepsilon_2(x) \downarrow 0$ such that

$$\sum_{r=1}^{\infty} \varepsilon_1^{\theta}(r) < \infty, \tag{1.28^a}$$

$$\sum_{k=1}^{\infty} \varepsilon_1^{\theta-1}({}^2\log k)\{b(k)\}^{-2}k^{-1}\sum_{j=0}^{k-1} |f(j)|^2 < \infty, \tag{1.28^b}$$

$$\sum_{r=1}^{\infty} r^2\varepsilon_2^{\theta}(r) < \infty, \tag{1.28^c}$$

$$\sum_{k=1}^{\infty} ({}^2\log k)^2\varepsilon_1^{\theta-1}({}^2\log k)|f(k)/b(k)|^2 < \infty, \tag{1.28^d}$$

with θ the same as in (1.24). Sufficient for (1.28) is

$$\sum_{k=1}^{\infty} ({}^2\log k)^{\alpha_1} \{b(k)\}^{-2}k^{-1}\sum_{j=0}^{k-1} |f(j)|^2 < \infty, \tag{1.28^a)*}$$

$$\sum_{k=1}^{\infty} ({}^2\log k)^{\alpha_2} |f(k)/b(k)|^2 < \infty, \tag{1.28^b)*}$$

for some $\alpha_1 > (1-\theta)/\theta$ and $\alpha_2 > (3-\theta)/\theta$.

$$\lim_{\delta\downarrow 0}\limsup_{n\to\infty} \{b(n)\}^{-1}\sum_{j=[n-n\delta]}^{[n+n\delta]} |f(j)| = 0, \tag{1.29}$$

$$\{b(T)\}^{-1}\int_0^T |f(t)|dt \leqq c < \infty, \qquad T \geqq T_1, \tag{1.30}$$

$$\lim_{T\to\infty} \{b(T)\}^{-2}\int_0^T |f(t)|^2 dt = 0, \tag{1.31}$$

$$\int_2^{\infty} |f(t)/b(t)|^2({}^2\log t)^2 dt < \infty. \tag{1.32}$$

There exist $\varepsilon_1(x) \downarrow 0$ and $\varepsilon_2(x) \downarrow 0$, such that

$$\sum_{r=1}^{\infty} \varepsilon_1^{\theta}(r) < \infty, \tag{1.33^a}$$

$$\int_2^{\infty} \varepsilon_1^{\theta-1}({}^2\log t)\{b(t)\}^{-2}t^{-1}\int_0^t |f(\tau)|^2 d\tau\, dt < \infty, \tag{1.33^b}$$

$$\sum_{r=1}^{\infty} r^2\varepsilon_2^{\theta}(r) < \infty, \tag{1.33^c}$$

$$\int_2^{\infty} ({}^2\log t)^2\varepsilon_2^{\theta-1}({}^2\log t)|f(t)/b(t)|^2 dt < \infty, \tag{1.33^d}$$

with θ the same as in (1.24). Sufficient for (1.33) is

$$(1.33^{a})^* \qquad \int_2^\infty ({}^2\log t)^{\alpha_1}\{b(t)\}^{-2}t^{-1}\int_0^t |f(\tau)|^2 d\tau\, dt < \infty,$$

$$(1.33^{b})^* \qquad \int_2^\infty ({}^2\log t)^{\alpha_2}|f(t)/b(t)|^2 dt < \infty,$$

for some $\alpha_1 > (1-\theta)/\theta$ and $\alpha_2 > (3-\theta)/\theta$.

$$(1.34) \qquad \lim_{\delta\downarrow 0}\limsup_{T\to\infty}\{b(T)\}^{-1}\int_{T-T\delta}^{T+T\delta}|f(\tau)|d\tau = 0.$$

The principal conditions for convergence to zero in probability of (1.18) are (1.22), (1.31) and (1.23) or (1.30), and for (1.19) they are (1.26) and (1.23) or (1.25). We use (1.34) and (1.29) for extending results on (1.18) and (1.19) to sums $\sum_1^n f(S_k)$. The other conditions on f, φ and F are applied to obtain a.s. convergence. For bounded f and $b(n) = n$, simple results appear.

Sections 2 and 3 deal with convergence in probability and a.s. for integer valued X_k. In sections 4 and 5 we consider convergence in probability and a.s. for nonlattice X_k, mainly under (1.22). Section 6 is on recurrent sets.

2. Convergence in probability, X_k integer valued with span 1

LEMMA 2.1. Under (1.21) the centered homogeneous renewal process $\tilde{z}_n = z_n - \mu^{-1}$, $n \geqq 0$, is wide sense stationary with covariance function

$$(2.1) \qquad \begin{aligned} R(h) \overset{\mathrm{df}}{=} E\{\tilde{z}_n\tilde{z}_{n+h}\} &= \mu^{-1}\delta_{0h}+\mu^{-1}u_0(h)+\mu^{-1}u_0(-h)-\mu^{-2}, \\ & n \geqq 0,\ n+h \geqq 0,\ h = \cdots -2, -1, 0, 1, \cdots, \end{aligned}$$

where $u_0(h)$ is defined by (1.17). The process has spectral density

$$(2.2) \qquad \rho(u) = \frac{1}{2\pi\mu}\frac{1-|\varphi(u)|^2}{|1-\varphi(u)|^2}, \qquad |u| \leqq \pi.$$

PROOF. We have

$$\begin{aligned} E\{z_n z_{n+h}\} &= \sum_{j=1}^\infty\sum_{k=1}^\infty P\{S_j = n,\ S_k = n+h\} \\ &= \delta_{0h}\sum_{j=1}^\infty P\{S_j = n\} + \sum_{j=1}^\infty\sum_{k=j+1}^\infty P\{S_j = n\}p^{(k-j)}(h) \\ &\quad + \sum_{k=1}^\infty\sum_{j=k+1}^\infty P\{S_k = n+h\}p^{(j-k)}(-h), \end{aligned}$$

where $p^{(m)}(r) = P\{X_2+\cdots+X_{m+1} = r\}$, and (2.1) follows with (1.17) and (1.14).

In Spitzer [**17**], section 9, it is shown that

$$2\delta_{0k}+u_0(k)+u_0(-k) = \mu^{-1}+(2\pi)^{-1}\int_{-\pi}^{\pi}\psi(u)e^{iuk}du,$$

where

$$\psi(u) = 2Re\{1-\varphi(u)\}^{-1} = 1+\{1-|\varphi(u)|^2\}|1-\varphi(u)|^{-2}.$$

It follows that

$$R(h) = \int_{-\pi}^{\pi}\rho(u)e^{iuh}du.$$

Lemma 2.2. Let $\rho(u)$ be given by (2.2).

(a) If (1.21) holds, $\rho(u)$ is bounded for $0 < \varepsilon \leqq |u| \leqq \pi$.

(b) If (1.21) and (1.23) hold, $\rho(u)$ is bounded on $[-\pi, \pi]$.

(c) If (1.21) holds and there is no a such that X_2-a has span $d > 1$, then $\rho(u)$ is bounded away from zero on $[-\pi, \pi]$.

Proof. The lemma follows from well-known properties of characteristic functions (Loeve [12], § 12, 13; Lukacs [13], Section 2). With respect to (c) it is noted that either

$$\lim_{u\to 0} u^{-2}\{1-|\varphi(u)|^2\} = +\infty$$

or, if not, $\int x^2dF < \infty$ by Fatou's lemma.

Theorem 2.1. Let $\lambda(n)$ be defined by (1.19). Under (1.21) we have for the homogeneous and the basic renewal process: If either

(a) (1.23) and (1.26) hold,

or

(b) (1.25) and (1.26) hold,

$$\lim_{n\to\infty} E|\lambda(n)|^2 = 0. \tag{2.3}$$

Proof. First consider the homogeneous renewal process. From (1.19^a) and lemma 2.1

$$E|\lambda(n)|^2 = \{b(n)\}^{-2}\int_{-\pi}^{\pi}\rho(u)|\sum_{k=0}^{n-1}f(k)e^{iuk}|^2du. \tag{2.4}$$

Under (1.23) we have by lemma 2.2^b

$$E|\lambda(n)|^2 \leqq C\{b(n)\}^{-2}\sum_{k=0}^{n-1}|f(k)|^2,$$

and (a) follows.

To prove (b) we put

$$
\begin{aligned}
E|\lambda(n)|^2 &= \{b(n)\}^{-2}\sum_{j=0}^{n-1}\sum_{k=0}^{n-1} f(j)\bar{f}(k)\int_{-\pi}^{\pi}\rho(u)e^{iu(j-k)}du \\
&\leqq c_1\{b(n)\}^{-2}\sum_{|k-j|\leqq M}|f(j)\bar{f}(k)| \\
&\quad +\{b(n)\}^{-2}\sum_{|k-j|>M}|f(j)\bar{f}(k)\int_{-\pi}^{\pi}\rho(u)e^{iu(j-k)}du|.
\end{aligned}
$$

With the Riemann-Lebesgue lemma and (1.25) we may take M so large that the second term here is smaller than $\frac{1}{2}\varepsilon$. The first term then is smaller than $\frac{1}{2}\varepsilon$ for $n \geqq n(\varepsilon)$ by (1.26), as is seen by making use of the Schwartz inequality.

For the basic process the theorem is proved by writing for the homogeneous process

$$E|\lambda(n)|^2 = \sum_{r=0}^{\infty} P\{X_1 = r\}E\{|\lambda(n)|^2|X_1 = r\},$$

and considering the term with $r = 0$, where $P\{X_1 = 0\} > 0$ by (1.15).

If (1.21) holds and there is no a such that X_2-a has span $d > 1$, the condition (1.26) is necessary in order that we have (2.3) for the homogeneous renewal process. This is seen from (2.4) and lemma 2.2c.

For the random walk (1.3), lattice or not, we define the random variables

$$M_1(t) \overset{\text{df}}{=} \min\{k : S_k \geqq t\}, \qquad t > 0, \tag{2.5}$$

$$M(t) \overset{\text{df}}{=} M_1(t)-1, \qquad t > 0. \tag{2.6}$$

To convert $\lambda(n)$ into a nonrandom sum we need the following lemmas on $M(t)$.

LEMMA 2.3. For any random walk satisfying (1.1) and (1.2), irrespective of the distribution of X_1,

$$\lim_{t\to\infty} t^{-1}M(t) = \mu^{-1}, \text{ a.s.}$$

PROOF We have

$$M_1(t) = \sum_{k=1}^{\gamma(t)} \lambda_k, \tag{2.7}$$

where λ_k denotes the k^{th} strict ascending ladder epoch of the random walk (Feller [8], Ch. XII. 1) and $\gamma(t)$ the number of steps in which the strict ascending ladder process reaches $[t, \infty)$.

The λ_k are independent and for $k \geqq 2$ have the same distribution with finite expectation. A similar assertion holds for the ladder heights and we have

$$\mu_L = \mu E\{\lambda_2\}, \tag{2.8}$$

where μ_L is the expectation of the m^{th} ladder height, $m \geqq 2$. We refer to Feller [8], Ch. XII. 2, theorem 2. From the strong law of large numbers of renewal theory, applied to the ladder process,

$$\lim_{t\to\infty} t^{-1}\gamma(t) = \mu_L^{-1}, \text{ a.s.,} \tag{2.9}$$

and from the strong law of large numbers for the λ_k

$$\lim_{t\to\infty} \{\gamma(t)\}^{-1} \sum_{k=1}^{\gamma(t)} \lambda_k = \lim_{n\to\infty} n^{-1} \sum_{k=1}^{n} \lambda_k = E\{\lambda_2\}, \text{ a.s.} \tag{2.10}$$

The lemma follows from (2.7), (2.10), (2.9) and (2.8).

Lemma 2.4. Under (1.21) we have for the homogeneous random walk: If either

(a) (1.23) and (1.26) hold,

or

(b) (1.25) and (1.26) hold,

then

$$\left\{ \sum_{k=1}^{M(T)} f(S_k) - \mu^{-1} \sum_{k=0}^{T^*-1} f(k) \right\} \Big/ b(T^*) \xrightarrow{P} 0.$$

Here

$$x^* \overset{\text{df}}{=} \min\{n : n \text{ integer}, n \geqq x\}. \tag{2.11}$$

Proof. Let $\lambda(n)$ be defined by (1.19). Then

$$\lambda(T^*) = \left\{ \sum_{k=1}^{M(T)} f(S_k) - \mu^{-1} \sum_{k=0}^{T^*-1} f(k) \right\} \Big/ b(T^*) - A(T) + B(T), \tag{2.12}$$

with

$$A(T) \overset{\text{df}}{=} \{b(T^*)\}^{-1} \sum_{\substack{k=1 \\ S_k<0}}^{M(T)} f(S_k),$$

$$B(T) \overset{\text{df}}{=} \{b(T^*)\}^{-1} \sum_{\substack{k>M(T) \\ 0\leqq S_k<T}} f(S_k).$$

Since $N(-\infty, 0) < \infty$, a.s.,

$$\lim_{T\to\infty} A(T) = 0, \text{ a.s.} \tag{2.13}$$

Moreover

$$|B(T)| \leqq \{b(T^*)\}^{-1} G(T^*) \max_{0\leqq h<T^*} |f(h)|,$$

where $G(T^*)$ is the number of k with $k > M(T)$ and $S_k \in [0, T^*)$. Now $G(T^*)$ has the same distribution as $N[-T^*, 0)$ and $N[-T^*, 0) \leqq N(-\infty, 0)$. From (1.26) it follows that

$$\lim_{T\to\infty} \{b(T^*)\}^{-1} \max_{0\leqq h< T^*} |f(h)| = 0.$$

So

$$B(T) \xrightarrow{P} 0. \tag{2.14}$$

The lemma follows from (2.12), (2.13), (2.14) and theorem 2.1.

For the final result of this section we also need the following lemma on convergence in probability.

Lemma 2.5. Let $\{Y(t), t \geqq 0\}$ be any family of complex valued random variables. If to every sequence $t_k \to \infty$ and every $\varepsilon > 0$ there is a subsequence $\{\tau_j\}$ of $\{t_k\}$, depending on ε, with

$$\limsup_{j\to\infty} |Y(\tau_j)| < \varepsilon, \text{ a.s.,}$$

then $Y(t) \xrightarrow{P} 0$ as $t \to \infty$.

Proof. The assumption of the lemma with Fatou's lemma implies that to every sequence $t_k \to \infty$ and every $\varepsilon > 0$ there is a subsequence $\{\tau_j\}$ with

$$\limsup_{j\to\infty} E\,\theta(|Y(\tau_j)|) < \varepsilon,$$

where $\theta(x) \stackrel{\mathrm{df}}{=} x(1+x)^{-1}$, $x \geqq 0$. So

$$\lim_{t\to\infty} E\,\theta(|Y(t)|) = 0,$$

and the lemma follows.

Theorem 2.2. Under (1.21) we have for the homogeneous and the basic random walk: If either

(a)(1.20), (1.23), (1.26) and (1.29) hold,

or

(b) (1.20), (1.25), (1.26) and (1.29) hold,

then for $n \to \infty$

$$\left\{\sum_{k=1}^{n} f(S_k) - \mu^{-1} \sum_{k=0}^{[n\mu]} f(k)\right\} \Big/ b(n) \xrightarrow{P} 0. \tag{2.15}$$

Corollary 1. Under the conditions of the theorem

$$\sum_{k\ 1}^{n} f(S_k)/b(n) \xrightarrow{P} \lambda,$$

with λ necessarily degenerate, if and only if

$$\sum_{k=0}^{[n\mu]} f(k)/b(n) \to \lambda\mu.$$

Corollary 2. Under (1.21), if f is bounded,

$$n^{-1}\left\{\sum_{k=1}^{n} f(S_k) - \mu^{-1}\sum_{k=0}^{[n\mu]} f(k)\right\} \xrightarrow{P} 0.$$

Proof. First we consider the homogeneous process and show that for $T \to \infty$

$$w(T) \overset{\text{df}}{=} \left\{\sum_{k=1}^{M(T)} f(S_k) - \sum_{k=1}^{[T/\mu]} f(S_k)\right\}\Big/ b(T) \xrightarrow{P} 0, \tag{2.16}$$

with $M(T)$ defined by (2.5) and (2.6). We make use of lemma 2.5. Take $\varepsilon > 0$ fixed. By (1.29) there is $\delta = \delta(\varepsilon) > 0$ with

$$\limsup_{T\to\infty} \{\mu b(T)\}^{-1} \sum_{k=(T-T\delta)^*}^{(T+T\delta)^*} |f(k)| < \varepsilon. \tag{2.17}$$

Here x^* is defined by (2.11). By lemma 2.3 there is $\tau_1(\delta)$ a.s. finite such that for $T \geqq \tau_1(\delta)$

$$\begin{aligned} |w(T)| &\leqq \{b(T)\}^{-1} \sum_{k=1+M(T-T\delta)}^{M(T+T\delta)} |f(S_k)| \\ &= \left\{\sum_{k=1}^{M(T+T\delta)} |f(S_k)| - \mu^{-1} \sum_{k=0}^{(T+T\delta)^*-1} |f(k)|\right\}\Big/ b(T) \\ &\quad - \left\{\sum_{k=1}^{M(T-T\delta)} |f(S_k)| - \mu^{-1} \sum_{k=0}^{(T-T\delta)^*-1} |f(k)|\right\}\Big/ b(T) \\ &\quad + \{\mu b(T)\}^{-1} \sum_{k=(T-T\delta)^*}^{(T+T\delta)^*-1} |f(k)|. \end{aligned} \tag{2.18}$$

The conditions of the theorem are such that lemma 2.4 applies to $|f|$ as well as to f. So, with (1.20) and (1.5), to any sequence $T_k \to \infty$ there is a subsequence $\{T'_j\}$ for which the first and second term on the right in (2.18) converge a.s. to zero, and (2.16) follows with (2.17) and lemma 2.5

From (2.16) and lemma 2.4, with (1.5),

$$\left\{\sum_{k=1}^{[T/\mu]} f(S_k) - \mu^{-1}\sum_{k=0}^{T^*-1} f(k)\right\}\Big/ b(T^*) \xrightarrow{P} 0.$$

Here T^*-1 may be replaced by $[T]$ and by taking $T = n\mu$ and making use of (1.20) or (1.5) we find (2.15).

The assertion for the basic process follows by conditioning with respect to X_1 in the homogeneous process, noting that $P\{X_1 = 0\} > 0$ by (1.15). Cf. the proof of theorem 2.1.

3. Convergence almost sure; X_k integer valued with span 1

We use the technique for proving a well-known theorem on a.s. convergence of series of orthogonal random variables (Doob [6], Ch. IV 4, theorem 4.2).

LEMMA 3.1. Let $\{v(n), n = 0, 1, 2, \cdots\}$ be any complex valued stochastic process. Then for $r = 0, 1, 2, \cdots$,

$$\Big| \sum_{k=2^r+1}^{n} v(k)\Big|^2 \leqq W(r), \qquad 2^r < n \leqq 2^{r+1}, \tag{3.1}$$

with

$$W(r) \overset{\text{df}}{=} (r+1) \sum_{\nu=0}^{r} \sum_{k=0}^{2^{r-\nu}-1} |\eta(\nu, k)|^2, \tag{3.2}$$

$$\begin{aligned} \eta(\nu, k) &= v(2^r+k2^\nu+1) \\ &\quad + v(2^r+k2^\nu+2) + \cdots + v(2^r+(k+1)2^\nu), \\ &\qquad k = 0, 1, \cdots, 2^{r-\nu}-1, \quad \nu = 0, 1, \cdots, r. \end{aligned} \tag{3.3}$$

PROOF. We refer to Doob [6], Ch. IV. 4, proof of lemma IV. 4.1.

LEMMA 3.2. If in lemma 3.1

$$v(k) = c_k w_k, \qquad k = 0, 1, 2, \cdots,$$

where $\{w_k\}$ is a wide sense stationary process with bounded spectral density, then

$$EW(r) \leqq A(r+1)^2 \sum_{j=2^r+1}^{2^{r+1}} |c_j|^2, \qquad r = 0, 1, 2, \cdots, \tag{3.4}$$

with A independent of r.

PROOF. Let g be the spectral density of the process $\{w_k\}$. Then from (3.3), since $g(\lambda) \leqq A_0$, $-\pi \leqq \lambda \leqq \pi$,

$$\begin{aligned} E \sum_{k=0}^{2^{r-\nu}-1} |\eta(\nu, k)|^2 &= \sum_{k=0}^{2^{r-\nu}-1} \int_{-\pi}^{\pi} \Big| \sum_{h=2^r+k2^\nu+1}^{2^r+(k+1)2^\nu} c_h e^{ih\lambda}\Big|^2 g(\lambda) d\lambda \\ &\leqq 2\pi A_0 \sum_{k=0}^{2^{r-\nu}-1} \sum_{h=2^r+k2^\nu+1}^{2^r+(k+1)2^\nu} |c_h|^2 = 2\pi A_0 \sum_{h=2^r+1}^{2^{r+1}} |c_h|^2, \end{aligned}$$

and the inequality (3.4) follows with (3.2).

LEMMA 3.3. If in lemma 3.1

$$v(k) = c_k w_k, \qquad k = 0, 1, 2, \cdots,$$

where $\{w_k\}$ is a wide sense stationary process with spectral density g satisfying

$$g(\lambda) \leqq b|\lambda|^{\theta-1}, \qquad -\pi \leqq \lambda \leqq \pi,$$

with $\theta \in (0, 1)$, then for $r = 0, 1, 2, \cdots$,

$$EW(r) \leqq A(r+1)^2\delta_r^\theta \left\{\sum_{j=2^r+1}^{2^{r+1}} |c_j|\right\}^2 + B(r+1)^2\delta_r^{\theta-1} \sum_{j=2^r+1}^{2^{r+1}} |c_j|^2, \tag{3.5}$$

for any choice of $\delta_r \in (0, \pi)$, the constants A and B not depending on r or δ_r.

Proof. Starting as in the proof of lemma 3.2 we find

$$\begin{aligned} E\sum_{k=0}^{2^{r-\nu}-1} |\eta(\nu, k)|^2 &\leqq \sum_{k=0}^{2^{r-\nu}-1} \left\{\sum_{j=2^r+k2^\nu+1}^{2^r+(k+1)2^\nu} |c_j|\right\}^2 \int_{-\delta_r}^{\delta_r} b|\lambda|^{\theta-1}d\lambda \\ &\quad + \sum_{k=0}^{2^{r-\nu}-1} \int_{\delta_r \leqq |\lambda| \leqq \pi} b\delta_r^{\theta-1} \left|\sum_{h=2^r+k2^\nu+1}^{2^r+(k+1)2^\nu} c_h e^{ih\lambda}\right|^2 d\lambda \\ &\leqq 2b\theta^{-1}\delta_r^\theta \left\{\sum_{k=0}^{2^{r-\nu}-1} \sum_{j=2^r+k2^\nu+1}^{2^r+(k+1)2^\nu} |c_j|\right\}^2 + 2\pi b\delta_r^{\theta-1} \sum_{k=0}^{2^{r-\nu}-1} \sum_{h=2^r+k2^\nu+1}^{2^r+(k+1)2^\nu} |c_h|^2 \\ &= 2b\theta^{-1}\delta_r^\theta \left\{\sum_{j=2^r+1}^{2^{r+1}} |c_j|\right\}^2 + 2\pi b\delta_r^{\theta-1} \sum_{h=2^r+1}^{2^{r+1}} |c_k|^2, \end{aligned}$$

and the inequality (3.5) follows with (3.2).

Theorem 3.1. Under (1.21), (1.23) and (1.27) we have for the homogeneous and the basic renewal process

$$\lim_{n\to\infty} \left\{\sum_{k=0}^{n-1} f(k)z_k - \mu^{-1} \sum_{k=0}^{n-1} f(k)\right\} \Big/ b(n) = 0, \text{ a.s.} \tag{3.6}$$

Proof. First consider the homogeneous process. It will be shown that the series

$$\sum_k f(k)\tilde{z}_k/b(k), \tag{3.7}$$

where $\tilde{z}_k = z_k - \mu^{-1}$, converges a.s. The relation (3.6) then follows with Kronecker's lemma (Loève [12], sec. 16.3, p. 238, Feller [8], p. 238).

In lemma 3.1 and 3.2 we take

$$v(k) = c_k w_k, \quad w_k = \tilde{z}_k, \quad c_k = f(k)/b(k), \qquad k = 0, 1, \cdots.$$

The spectral density ρ, given by (2.2), of the process $\{w_k\}$ is bounded on $[-\pi, \pi]$ by lemma 2.2[b]. So

$$E|\sum_{k=M}^{N} c_k w_k|^2 \leqq 2\pi A_0 \sum_{k=M}^{N} |c_k|^2,$$

and with (1.27) it follows that the series (3.7) converges in quadratic mean to a finite limit v. Similarly

$$E|v-\sum_{k=1}^{N} c_k w_k|^2 \leqq 2\pi A_0 \sum_{k=N+1}^{\infty} |c_k|^2$$
$$\leqq 2\pi A_0\{{}^2\log N\}^{-2} \sum_{k=N+1}^{\infty} ({}^2\log k)^2\,|c_k|^2 \leqq A_1\{{}^2\log N\}^{-2},$$

with (1.27). So

$$\sum_{r=1}^{\infty} E|v-\sum_{k=1}^{2^r} c_k w_k|^2 \leqq \sum_{r=1}^{\infty} A_1 r^{-2} < \infty,$$

and therefore

$$\lim_{r\to\infty} \sum_{k=1}^{2^r} c_k w_k = v, \text{ a.s.} \tag{3.8}$$

For $n = 1, 2, \cdots$ let r_n be the integer with

$$2^{r_n}+1 \leqq n \leqq 2^{r_n+1}.$$

Then by lemma 3.1

$$\sum_{k=1}^{n} c_k w_k = \sum_{k=1}^{2^{r_n}} c_k w_k + R(n), \tag{3.9}$$

$$|R(n)|^2 \leqq W(r_n), \qquad n = 1, 2, \cdots, \tag{3.10}$$

the $W(r)$ being given by (3.2) in terms of the $v(k)$ defined above. From lemma 3.2

$$\sum_{r=1}^{\infty} E\{W(r)\} \leqq A_2 \sum_{r=1}^{\infty} (r+1)^2 \sum_{j=2^r+1}^{2^{r+1}} |c_j|^2$$
$$\leqq A_3 \sum_{r=1}^{\infty} \sum_{j=2^r+1}^{2^{r+1}} ({}^2\log j)^2\,|c_j|^2 = A_3 \sum_{j=3}^{\infty} ({}^2\log j)^2\,|c_j|^2 < \infty$$

with (1.27), so that

$$\lim_{r\to\infty} W(r) = 0, \text{ a.s.} \tag{3.11}$$

The a.s. convergence of (3.7) now follows from (3.9), (3.8), (3.10) and (3.11).

For the basic process the theorem follows from the fact that (3.6) holds for the homogeneous process a.s. on the event $\{X_1 = 0\}$ which has positive probability by (1.15).

THEOREM 3.2. If (1.21), (1.20), (1.24), (1.25) and (1.28) hold, then for the homogeneous and the basic renewal process

$$\lim_{n\to\infty} \lambda(n) = 0, \text{ a.s.,} \tag{3.12}$$

with $\lambda(n)$ defined by (1.19).

Proof. First we consider the homogeneous process. From (1.24) and lemma 2.2[a] it follows that the spectral density $\rho(u)$ given by (2.2) satisfies

$$\rho(u) \leqq A_1|u|^{\theta-1}, \qquad -\pi \leqq |u| \leqq \pi. \tag{3.13}$$

So, from (1.19[a]), lemma 2.1 and (1.25)

$$\begin{aligned} E|\lambda(n)|^2 &\leqq A_1\{b(n)\}^{-2}\left\{\sum_{k=0}^{n-1}|f(k)|\right\}^2\int_{-\delta}^{\delta}|u|^{\theta-1}\,du \\ &\quad +A_1\{b(n)\}^{-2}\delta^{\theta-1}\int_{\delta\leqq|u|\leqq\pi}\left|\sum_{k=0}^{n-1}f(k)e^{iuk}\right|^2du \\ &\leqq A_2\delta^\theta+A_3\delta^{\theta-1}\{b(n)\}^{-2}\sum_{k=0}^{n-1}|f(k)|^2. \end{aligned}$$

Taking $\delta=\delta(n)=\varepsilon_1(r)$ for $n=2^r$, with $\varepsilon_1(r)$ as in (1.28), we have with (1.20)

$$\begin{aligned} \sum_{r=1}^{\infty}E|\lambda(2^r)|^2 &\leqq A_2\sum_{r=1}^{\infty}\varepsilon_1^\theta(r)+A_3\sum_{r=1}^{\infty}\{b(2^r)\}^2\varepsilon_1^{\theta-1}(r)\sum_{k=0}^{2^r-1}|f(k)|^2 \\ &\leqq A_2\sum_{r=1}^{\infty}\varepsilon_1^\theta(r)+A_4\sum_{r=1}^{\infty}\varepsilon_1^{\theta-1}(r)2^{-r}\sum_{j=2^r+1}^{2^{r+1}}\{b(j)\}^{-2}\sum_{k=0}^{j-1}|f(k)|^2 \\ &\leqq A_2\sum_{r=1}^{\infty}\varepsilon_1^\theta(r)+A_5\sum_{r=1}^{\infty}\sum_{j=2^r+1}^{2^{r+1}}\varepsilon_1^{\theta-1}({}^2\log j)\{b(j)\}^{-2}j^{-1}\sum_{k=0}^{j-1}|f(k)|^2 \\ &= A_2\sum_{r=1}^{\infty}\varepsilon_1^\theta(r)+A_5\sum_{j=3}^{\infty}\varepsilon_1^{\theta-1}({}^2\log j)\{b(j)\}^{-2}j^{-1}\sum_{k=0}^{j-1}|f(k)|^2<\infty, \end{aligned}$$

by (1.28[a]) and (1.28[b]). So

$$\lim_{r\to\infty}\lambda(2^r)=0,\ \text{a.s.} \tag{3.14}$$

For $n=1,2,\cdots$, let r_n be the integer with

$$2^{r_n}+1\leqq n\leqq 2^{r_n+1}.$$

Then, with (1.5), (1.20) and lemma 3.1,

$$|\lambda(n)|\leqq|\lambda(2^{r_n})|+A_6\{b(2^{r_n+1})\}^{-1}W^{\frac{1}{2}}(r_n), \tag{3.15}$$

where $W(r)$ is defined by (3.2) with $v(k)=f(k)\tilde{z}_k$, $k=0,1,2,\cdots$. By (3.13) lemma 3.3 applies with $c_k=f(k)$, $w_k=\tilde{z}_k$. We take $\delta_r=\varepsilon_2(r)$ in (3.5) with $\varepsilon_2(r)$ the same as in (1.28[c]) and (1.28[d]). Then, with (1.25)

$$\sum_{r=1}^{\infty}\{b(2^{r+1})\}^{-2}\,E\{W_r\} \leqq A_7\sum_{r=1}^{\infty}(r+1)^2\,\varepsilon_2^{\theta}(r)$$
$$+A_8\sum_{r=1}^{\infty}(r+1)^2\,\varepsilon_2^{\theta-1}(r)\{b(2^{r+1})\}^{-2}\sum_{j=2^r+1}^{2^{r+1}}|f(j)|^2,$$

where the first term converges by (1.28[c]) and the second term is majorized by

$$A_8\sum_{r=1}^{\infty}\sum_{j=2^r+1}^{2^{r+1}}({}^2\log j)^2\,\varepsilon_2^{\theta-1}({}^2\log j)|f(j)/b(j)|^2$$
$$= A_8\sum_{j=3}^{\infty}({}^2\log j)^2\,\varepsilon_2^{\theta-1}({}^2\log j)|f(j)/b(j)|^2 < \infty,$$

by (1.28[d]), so that

$$\lim_{r\to\infty}\{b(2^{r+1})\}^{-2}\,W(r) = 0, \text{ a.s.} \tag{3.16}$$

The relation (3.12) follows from (3.15), (3.14) and (3.16).

For the basic process the proof is the same as in theorem 3.1.

To convert $\lambda(n)$ into a sum of a nonrandom number of terms we need the following two lemmas.

LEMMA 3.4. For any random walk satisfying (1.1) and (1.2), let

$$L(t) \stackrel{\mathrm{df}}{=} \max\{k : S_k < t\}, \qquad t > 0. \tag{3.17}$$

Then, irrespective of the distribution of X_1,

$$\lim_{t\to\infty} t^{-1}L(t) = \mu^{-1}, \text{ a.s.} \tag{3.18}$$

PROOF. By ω we denote points of the probability space on which the X_k are defined. Let

$$A \stackrel{\mathrm{df}}{=} \{\omega : n^{-1}S_n(\omega) \to \mu,\ t^{-1}M(t) \to \mu^{-1}\},$$

where $M(t)$ is defined by (2.5) and (2.6). Since $M(t) \leqq L(t)$,

$$\mu^{-1} \leqq \liminf_{t\to\infty} t^{-1}L(t, \omega), \qquad \omega \in A. \tag{3.19}$$

If for $\omega \in A$ fixed $\limsup_{t\to\infty} t^{-1}L(t, \omega) > \mu^{-1}$, there would be $\delta(\omega) > 0$ and a sequence $t_k(\omega) \to \infty$ with

$$L(t_k(\omega), \omega) \geqq \{\mu^{-1}+\delta(\omega)\}t_k(\omega), \qquad k = 1, 2, \cdots.$$

This implies the existence of integers $\nu_k(\omega)$, $k = 1, 2, \cdots$, with

$$S_{\nu_k(\omega)}(\omega) < t_k(\omega), \tag{3.20}$$

$$\nu_k(\omega) \geqq \{\mu^{-1}+\delta(\omega)\}t_k(\omega). \tag{3.21}$$

Since $S_n(\omega) = n\mu\{1+\varepsilon_n(\omega)\}$ with $\lim_{n\to\infty} \varepsilon_n(\omega) = 0$, the relations (3.20) and (3.21) are contradictory. So

$$\limsup_{t\to\infty} t^{-1}L(t, \omega) \leqq \mu^{-1}, \qquad \omega \in A. \tag{3.22}$$

The lemma follows from (3.19) and (3.22), since $P(A) = 1$ by the strong law of large numbers and lemma 2.3.

LEMMA 3.5. If either

(a) (1.21), (1.23), (1.27) and (1.29) hold,

or

(b) (1.21), (1.20), (1.24), (1.25), (1.28) and (1.29) hold,

then for the homogeneous random walk

$$\lim_{T\to\infty} \left\{ \sum_{k=1}^{M(T)} f(S_k) - \mu^{-1} \sum_{k=0}^{T^*-1} f(k) \right\} \Big/ b(T^*) = 0, \text{ a.s.}$$

Here T^* is defined by (2.11) and $M(T)$ by (2.5) and (2.6).

PROOF. Since the conditions (a) and (b) are such that theorem 3.1 or 3.2 applies, and since $N(-\infty, 0) < \infty$, a.s., it is sufficient to prove that $\lim_{T\to\infty} \zeta(T) = 0$, a.s., where

$$\zeta(T) = \left\{ \sum_{k=0}^{T^*-1} f(k) z_k - \sum_{\substack{k=1 \\ S_k \geqq 0}}^{M(T)} f(S_k) \right\} \Big/ b(T^*).$$

We have

$$\begin{aligned} |\zeta(T)| &\leqq \{b(T^*)\}^{-1} \sum_{\substack{k > M(T) \\ S_k \in [0, T)}} |f(S_k)| \\ &= \left\{ \sum_{k=0}^{T^*-1} |f(k)| z_k - \sum_{\substack{k=1 \\ S_k \geqq 0}}^{M(T)} |f(S_k)| \right\} \Big/ b(T^*). \end{aligned} \tag{3.23}$$

Now

$$\sum_{k=0}^{T^*-1} |f(k)| z_k \leqq \sum_{k=1}^{L(T)} |f(S_k)|, \tag{3.24}$$

where $L(t)$ is defined by (3.17). Lemma 2.3 and lemma 3.4 imply that to every $\delta > 0$ there is $\tau(\delta)$ a.s. finite such that

$$L(t-t\delta) \leqq M(t), \qquad t > \tau(\delta).$$

Therefore, taking into account (3.24), we have for $T > \tau(\delta)$,

$$\sum_{k=0}^{(T-T\delta)^*-1} |f(k)| z_k \leqq \sum_{k=1}^{L(T-T\delta)} |f(S_k)| \leqq \sum_{k=1}^{M(T)} |f(S_k)|.$$

So, with (3.23), since $N(-\infty, 0) < \infty$, a.s.,

$$\limsup_{T\to\infty} |\zeta(T)| \leqq \limsup_{T\to\infty} \{b(T^*)\}^{-1} \sum_{(T-T\delta)^*}^{T^*-1} |f(k)| z_k$$
$$= \limsup_{T\to\infty} \left\{ \sum_{k=0}^{T^*-1} |f(k)|\tilde{z}_k - \sum_{k=0}^{(T-T\delta)^*} |f(k)|\tilde{z}_k + \mu^{-1} \sum_{(T-T\delta)^*}^{T^*-1} |f(k)| \right\} \Big/ b(T^*), \text{ a.s.}$$

The conditions (a) and (b) are such that theorem 3.1 or 3.2 applies to $|f|$. Therefore, with (1.5)

$$\limsup_{T\to\infty} |\zeta(T)| \leqq \limsup_{T\to\infty} \{\mu b(T^*)\}^{-1} \sum_{(T-T\delta)^*}^{T^*-1} |f(k)|, \text{ a.s.}$$

So $\zeta(T) \to 0$, a.s., with (1.29).

THEOREM 3.3. Under (1.21), if either

(a) (1.20), (1.23), (1.27) and (1.29) hold,

or

(b) (1.20), (1.24), (1.25), (1.28) and (1.29) hold,

we have for the homogeneous and the basic random walk

$$\lim_{n\to\infty} \left\{ \sum_{k=1}^{n} f(S_k) - \mu^{-1} \sum_{k=0}^{[n\mu]} f(k) \right\} \Big/ b(n) = 0, \text{ a.s.}$$

COROLLARY 1. Under the conditions of the theorem

$$\sum_{k=0}^{n-1} f(S_k)/b(n) \to \lambda, \text{ a.s.,}$$

with λ necessarily degenerate, if and only if

$$\sum_{k=0}^{[n\mu]} f(k)/\mu b(n) \to \lambda.$$

COROLLARY 2. If (1.24) holds and f is bounded,

$$\lim_{n\to\infty} n^{-1} \left\{ \sum_{k=1}^{n} f(S_k) - \mu^{-1} \sum_{k=0}^{[n\mu]} f(k) \right\} = 0, \text{ a.s.}$$

PROOF. For the homogeneous process the theorem is proved in the same way as theorem 2.2, except that no subsequences need to be taken. Lemma 3.5 holds for f as well as for $|f|$ and plays the same rôle as lemma 2.4 in the proof of theorem 2.2. For the basic process we consider the event $\{X_1 = 0\}$ in the homogeneous process. This event has positive probability by (1.15).

4. Convergence in probability; F nonlattice.

Throughout this section we assume that $|f|^2$ and therefore $|f|$ is integrable over finite subintervals of $[0, \infty)$, an assumption implicit in (1.31).

We make use of the Fourier representation, given by Feller and Orey [9], of the symmetrized basic renewal function, defined by

$$(4.1) \qquad K(b)-K(a) = K[a, b) \stackrel{\text{df}}{=} U_0[a, b)+U_0(-b, -a],$$

where

$$(1.16) \qquad U_0[a, b) \stackrel{\text{df}}{=} \sum_{m=1}^{\infty} F^{(m)}[a, b), \qquad -\infty < a < b < \infty.$$

LEMMA 4.1. Let F be a nonlattice distribution function satisfying (1.1) and (1.2). If $g \in L_1$ is continuous and vanishes outside some finite interval, and if

$$\gamma(u) \stackrel{\text{df}}{=} \int \exp(iux)g(x)dx, \qquad -\infty < u < \infty,$$

is nonnegative and belongs to L_1, then

$$(4.2) \quad g(0)+\int g(x)dK(x) = \mu^{-1}\gamma(0)+(2\pi)^{-1}\int \gamma(u)\sigma_1(u)du,$$

where

$$\sigma_1(u) \stackrel{\text{df}}{=} \{1-|\varphi(u)|^2\}|1-\varphi(u)|^{-2}, \qquad -\infty < u < \infty,$$

with φ the characteristic function of F.

PROOF. Since our assumptions differ slightly from those in [9], we sketch the proof. Let the distribution functions K_r, $0 < r < 1$, be defined by

$$K_r(b)-K_r(a) = \sum_{m=1}^{\infty} r^m\{F^{(m)}[a, b)+F^{(m)}[-b, -a)\}.$$

Integrating the relation

$$(4.3) \qquad g(-x-t) = (2\pi)^{-1}\int \gamma(u) \exp(iux+iut)du$$

with respect to $dK_r(x)$, applying Fubini's theorem and using (4.3) with $x = 0$, we find

$$g(-t)+\int g(x-t)dK_r(x) = (2\pi)^{-1}\int \gamma(u)\,\frac{1-r^2|\varphi(u)|^2}{|1-r\varphi(u)|^2}\,e^{iut}du.$$

With the Lévy-Cramér continuity theorem it follows that for $r \uparrow 1$ the measure with density function

$$(2\pi)^{-1}\gamma(u)\{1-r^2|\varphi(u)|^2\}|1-r\varphi(u)|^{-2}$$

converges completely to a finite measure m_1 with characteric function

$$\chi(t) = g(-t)+\int g(x-t)dK(x). \tag{4.4}$$

The measure m_1 is absolutely continuous on $(-\infty, 0) \cup (0, \infty)$ with density $(2\pi)^{-1}\gamma(u)\sigma_1(u)$ and

$$m_1(\{0\}) = \lim_{T\to\infty} (2T)^{-1}\int_{-T}^{T} \chi(t)dt = \mu^{-1}\gamma(0).$$

So

$$\chi(t) = \mu^{-1}\gamma(0)+(2\pi)^{-1}\int \gamma(u)\sigma_1(u) \exp (itu)du, \tag{4.5}$$

and (4.2) follows from (4.4) and (4.5) with $t = 0$.

LEMMA 4.2. If F is nonlattice and if the function $h \in L_2$ vanishes outside some finite subinterval of $[0, \infty)$, we have for the homogeneous renewal process

$$E|\int h(t)dN(t)|^2 = \mu^{-1}g(0)+\mu^{-1}\int g(y)dK(y), \tag{4.6}$$

with K defined by (4.1) and

$$g(y) \overset{\text{df}}{=} \int h(x)\bar{h}(x+y)dx, \qquad -\infty < y < \infty. \tag{4.7}$$

Moreover

$$E|\int h(t)d\tilde{N}(t)|^2 = \int |\hat{h}(u)|^2\sigma(u)du, \tag{4.8}$$

with

$$\hat{h}(u) \overset{\text{df}}{=} \int h(x) \exp (iux)dx, \qquad -\infty < u < \infty, \tag{4.9}$$

$$\sigma(u) \overset{\text{df}}{=} \frac{1-|\varphi(u)|^2}{2\pi\mu|1-\varphi(u)|^2}, \qquad -\infty < u < \infty. \tag{4.10}$$

PROOF. Let F_m be the distribution function of S_m and $F^{(m)}$ be the m-fold convolution of F. Then

$$E\left|\int hdN\right|^2 = E\left|\sum_{j=1}^{\infty} h(S_j)\right|^2$$

$$= \sum_{j=1}^{\infty}\int |h|^2 dF_j + \sum_{j=1}^{\infty}\sum_{k=j+1}^{\infty}\iint h(x)\bar{h}(x+y)dF_j(x)dF^{(k-j)}(y)$$

$$+\sum_{k=1}^{\infty}\sum_{j=k+1}^{\infty}\iint \bar{h}(x)h(x+y)dF_k(x)dF^{(j-k)}(y)$$

$$= \int |h|^2 dU + \iint \{h(x)\bar{h}(x+y)+\bar{h}(x)h(x+y)\}dU(x)dU_0(y).$$

The relation (4.6) follows with (1.11), since $h(x) = 0$, $x < 0$, and with $\bar{g}(y) = g(-y)$.

The function g defined by (4.7) satisfies the conditions of lemma 4.1. We note that

$$\gamma(u) \overset{\text{df}}{=} \int \exp(iuy)g(y)dy = |\hat{h}(-u)|^2.$$

So from (4.6) and (4.2)

$$E\left|\int hdN\right|^2 = \mu^{-2}\gamma(0)+(2\pi\mu)^{-1}\int \gamma(u)\sigma_1(u)du$$

$$= \left|\mu^{-1}\int h(x)dx\right|^2 + \int |\hat{h}(u)|^2\sigma(u)du,$$

and (4.8) follows with $E\{\int hdN\} = \mu^{-1}\int h(x)dx$.

Lemma 4.3. Let $\sigma(u)$ be defined by (4.10)

(a) If (1.22) holds, $\sigma(u)$ is bounded for $0 < \varepsilon \leqq |u| < \infty$, $\varepsilon > 0$.

(b) If (1.22) and (1.23) hold, $\sigma(u)$ is bounded.

(c) If (1.22) holds, $\sigma(u)$ is bounded away from zero.

We refer to the proof of lemma 2.2.

Theorem 4.1. Let $\Lambda(T)$ be defined by (1.18). If either

(a) (1.22), (1.23) and (1.31) hold,

or

(b) (1.22), (1.30) and (1.31) hold,

we have for the homogeneous renewal process

$$\lim_{T\to\infty} E|\Lambda(T)|^2 = 0. \tag{4.11}$$

Proof. From lemma 4.2

$$E|\Lambda(T)|^2 = \{b(T)\}^{-2}\int |\hat{f}_T(u)|^2\sigma(u)du,$$

with $\sigma(u)$ given by (4.10) and

$$\hat{f}_T(u) = \int f_T(t) \exp(iut)dt, \qquad -\infty < u < \infty,$$

$$f_T(t) = f(t), \quad t \in [0, T), \quad f_T(t) = 0, \qquad t \notin [0, T).$$

The assertion (a) follows with lemma 4.3[b] and Parseval's formula.

To prove (b) we note that $\sigma(u)$ is bounded for $|u| \geqq a > 0$ by lemma 4.3[a] and therefore, with Parseval's formula

$$E|A(T)|^2 \leqq \{b(T)\}^{-2} \int_{-a}^{a} |\hat{f}_T(u)|^2 \sigma(u)du + c_1\{b(T)\}^{-2} \int_0^T |f(t)|^2 dt.$$

Here the second term tends to zero for $T \to \infty$ by (1.31). The proof that the first term tends to zero is similar to the proof of theorem 2.1[b].

We note that under (1.22) the condition (1.31) is necessary for (4.11). This follows from lemma 4.3[c].

Theorem 4.2. If either F_1 is absolutely continuous, or, F_1 being arbitrary, some $F^{(m)}$ has an absolutely continuous component, we have under the conditions of theorem 4.1

$$A(T) \xrightarrow{P} 0.$$

Here $F^{(m)}$ denotes the m-fold convolution of F.

Proof. First consider the case that X_1 has an absolutely continuous distribution of special type, viz.

$$F_1(x) = G_0(x) = \sum_{k=-\infty}^{+\infty} \alpha_k F_L(x-ka), \tag{4.12}$$

where the α_k are positive and have sum 1 and F_L denotes the distribution function given by (1.12). Since F_L has positive density on some interval, we may and do take a so that G_0 has positive density on $(-\infty, +\infty)$.

Whether the process is homogeneous or not, we have for $T \geqq t_r$ finite

$$A(T) = \{b(T)\}^{-1} \sum_{k=1}^{r} f(S_k) + Y_r(T), \tag{4.13}$$

$$Y_r(T) \stackrel{\text{df}}{=} \left\{ \int_{[-S_r, T-S_r)} f(S_r+t)dN_0(t) - \mu^{-1} \int_0^T f dt \right\} \Big/ b(T), \tag{4.14}$$

where $\{N_0(t), -\infty < t < \infty\}$, the renewal process defined on $X_{r+1}, X_{r+1}+X_{r+2}, \cdots$, is independent of S_r. The first term in the right-hand side of (4.13) tends to zero for $T \to \infty$, irrespective of the distribution of X_1.

Let $\{T_k\}$ be any sequence with $T_k \to \infty$. By theorem 4.1 there is a subsequence $\{\tau_j\}$ with $\lim_{j\to\infty} \Lambda(\tau_j) = 0$, a.s., for the homogeneous process, and therefore $\lim_{j\to\infty} Y_r(\tau_j) = 0$, a.s., for every r. So the event $A_r \overset{\text{df}}{=} \{\lim_{j\to\infty} Y_r(\tau_j) = 0\}$ in the product space of S_r and the N_0-process has probability 1 if the distribution of S_r is $F_L * F^{(r-1)}$, where F_L is given by (1.12) and $*$ denotes convolution.

Now let $|| \; ||$ denote total variation and U_c the unit step function with jump at c. Then

$$||G_0 * F^{(n)} - F_L * F^{(n)}|| \leqq \sum_k \alpha_k ||F_L * U_{ka} * F^{(n)} - F_L * F^{(n)}||,$$

where the right-hand side tends to zero for $n \to \infty$ since

$$||F_L * U_{ka} * F^{(n)} - F_L * F^{(n)}||$$

tends to zero as $n \to \infty$ (Stam [18], theorem 5).

Therefore, if X_1 has the distribution (4.12), we may take $r = r(\varepsilon)$ so large that $P(A_r) > 1-\varepsilon$. But by (4.13) this implies that $P\{\lim_{j\to\infty} \Lambda(\tau_j) = 0\} > 1-\varepsilon$. Since this holds for every $\varepsilon > 0$, we proved the existence of a subsequence $\{\tau_j\}$ of $\{T_k\}$ with $\Lambda(\tau_j) \to 0$, a.s. So $\Lambda(T)$ converges to zero in probability if $F_1 = G_0$.

For general absolutely continuous F_1 the theorem follows from the fact that F_1 is absolutely continuous with respect to G_0.

If some $F^{(m)}$ has an absolutely continuous component, the theorem follows by an argument similar to the first part of the proof, now starting from the convergence of $\Lambda(T)$ for $F_1 = G_0$. The relation $P(A_r) > 1-\varepsilon$ now follows from the fact that we may take r so large that the component of $F_1 * F^{(r-1)}$ that is absolutely continuous with respect to $G_0 * F^{(r-1)}$, is larger than $1-\varepsilon$.

LEMMA 4.4. If either

(a) (1.22), (1.23) and (1.31) hold,

or

(b) (1.22), (1.30) and (1.31) hold,

then for the homogeneous random walk

$$\left\{\sum_{k=1}^{M(T)} f(S_k) - \mu^{-1} \int_0^T f(t)dt\right\} \Big/ b(T) \xrightarrow{P} 0.$$

Here $M(T)$ is defined by (2.5) and (2.6).

PROOF. By theorem 4.1 it is sufficient to show that $E(T) \xrightarrow{P} 0$, where

$$E(T) \stackrel{\mathrm{df}}{=} \left\{\int_{[0,T)} f(t)dN(t) - \sum_{k=1}^{M(T)} f(S_k)\right\} \Big/ b(T)$$
$$= -\{b(T)\}^{-1} \sum_{\substack{k=1 \\ S_k<0}}^{M(T)} f(S_k) + \{b(T)\}^{-1} \sum_{\substack{k>M(T) \\ S_k\in[0,T)}} f(S_k).$$

Here the first term tends to zero a.s. since $N(-\infty, 0) < \infty$, a.s. For the second term, denoted by $D(T)$, we have

$$D(T) \infty \{b(T)\}^{-1} \sum_{S_h\in[-T,0)} f(S_h+T), \tag{4.15}$$

where $X \infty Y$ means that the random variables X and Y have the same distribution. Also

$$\Big|\sum_{S_h\in[-T,0)} f(S_h+T)\Big| \leqq N(-\infty, 0) \max_{S_k\in[-T,0)} |f(S_k+T)|, \tag{4.16}$$

$$\max_{S_k\in[-T,0)} |f(S_k+T)| \infty \max_{\substack{j>M(T) \\ S_j\in[0,T)}} |f(S_j)|, \tag{4.17}$$

$$\max_{\substack{j>M(T) \\ S_j\in[0,T)}} |f(S_j)| \leqq \left\{\int_{[0,T)} |f|^2 dN\right\}^{\frac{1}{2}}. \tag{4.18}$$

Since $E\{\int_{[0,T)} |f|^2 dN\} = \mu^{-1} \int_0^T |f|^2 dt$, it follows from (1.31) and (4.15)—(4.18) that $D(T) \xrightarrow{P} 0$.

THEOREM 4.3. If either

(a) (1.22), (1.20), (1.23), (1.31) and (1.34) hold,

or

(b) (1.22), (1.20), (1.30), (1.31) and (1.34) hold,

then for the homogeneous random walk, as $n \to \infty$,

$$\left\{\sum_{k=1}^{n} f(S_k) - \mu^{-1} \int_0^{n\mu} f\,dt\right\} \Big/ b(n) \xrightarrow{P} 0.$$

COROLLARY 1. Under the conditions of the theorem

$$\sum_{k=1}^{n} f(S_k)/b(n) \xrightarrow{P} \lambda,$$

necessarily degenerate, if and only if $\int_0^{n\mu} f\,dt/b(n) \to \lambda\mu$.

COROLLARY 2. For the homogeneous random walk, if (1.22) holds and f is bounded,

$$n^{-1}\left\{\sum_{k=1}^{n} f(S_k)-\mu^{-1}\int_0^{n\mu} f\,dt\right\} \xrightarrow{P} 0.$$

PROOF. The theorem is proved by the same method as theorem 2.2 for the homogeneous process. Lemma 4.4, 2.3 and 2.5 are applied in the same way as lemma 2.4, 2.3 and 2.5 in the proof of theorem 2.2.

THEOREM 4.4. If either F_1 is absolutely continuous, or, F_1 being arbitrary, some $F^{(m)}$ has an absolutely continuous component, theorem 4.3 and its corollaries continue to hold for the nonhomogeneous random walk.

Here $F^{(m)}$ denotes the m-fold convolution of F.

PROOF. The theorem follows from theorem 4.3 in the same way as theorem 4.2 follows from theorem 4.1. The relations (4.13) and (4.14) are replaced by

$$\left\{\sum_{k=1}^{n} f(S_k)-\mu^{-1}\int_0^{n\mu} f\,dt\right\}\Big/ b(n) = \{b(n)\}^{-1}\sum_{k=1}^{r} f(S_k)$$
$$+\left\{\sum_{h=1}^{n-r} f(S_r+X_{r+1}+\cdots+X_{r+h})-\mu^{-1}\int_0^{n\mu} f\,dt\right\}\Big/ b(n).$$

If (1.22) does not hold, the proofs given above fail. Some results on convergence in probability however may be obtained under additional conditions on f. A function f on $(-\infty, +\infty)$ will be called here a strict step function on $[0, \infty)$ if for some $d > 0$

$$f(x) = c_k, \quad kd \leqq x < (k+1)d, \quad k = 0, 1, 2, \cdots. \tag{4.19}$$

THEOREM 4.5. Let $\Lambda(T)$ be defined by (1.18). If F is nonlattice and if either

(a) (1.20ª), (1.30) and (1.31) hold and f is a strict step function on $[0, \infty)$,

or

(b) (1.20ª), (1.30) and (1.31) hold and to every $\varepsilon > 0$ there is a strict step function f_ε on $[0, \infty)$ with

$$T\{b(T)\}^{-2}\int_0^T |f-f_\varepsilon|^2\,dt < \varepsilon, \qquad T \geqq T_1, \tag{4.20}$$

with T_1 independent of ε, then for the homogeneous renewal process

$$\lim_{T\to\infty} E|\Lambda(T)|^2 = 0. \tag{4.21}$$

Proof of (a). Let f satisfy (4.19). Then with (4.8), (4.9), (4.10)

$$E|\Delta(Nd)|^2 = \int 4u^{-2}\sin^2 \tfrac{1}{2}ud \left|\sum_{k=0}^{N-1} c_k e^{iukd}\right|^2 \sigma(u)du \Big/ b^2(Nd),$$

with $\sigma(u)u^{-2}\sin^2 \frac{1}{2}ud \in L_1$ by lemma 4.1. With the Riemann-Lebesgue theorem, in the same way as in the proof of theorem 2.1[b], we may show that $\lim_{N\to\infty} E|\Delta(Nd)|^2 = 0$ and then (4.21) follows with (1.5), (1.20[a]) and (1.31).

Proof of (b). First we show that any f_ε in (4.20) satisfies (1.30) and (1.31). With $f_\varepsilon = f+(f_\varepsilon - f)$ and Minkowski's inequality (1.31) follows for f_ε. With the Schwartz inequality and (4.20)

$$\{b(T)\}^{-1}\int_0^T |f-f_\varepsilon|dt \leqq \varepsilon^{\frac{1}{2}}, \qquad T \geqq T_1, \tag{4.22}$$

and (1.30) follows for f_ε.

Then with Minkowski's inequality, f_η satisfying (4.20) with $\varepsilon = \eta$,

$$E^{\frac{1}{2}}|\Delta(T)|^2 \leqq E^{\frac{1}{2}}\left|\int_{[0,T)} f_\eta d\tilde{N}/b(T)\right|^2 + G^{\frac{1}{2}}(\eta, T), \tag{4.23}$$

where

$$G(\eta, T) = E\left|\int_{[0,T)} (f-f_\eta)d\tilde{N}/b(T)\right|^2.$$

Using (4.6) and (4.7) and then (4.20), the Schwartz inequality and (4.22), we find

$$G(\eta, T) \leqq (\mu T)^{-1}\eta + \mu^{-1}K[-T, T]\int_0^T |f-f_\eta|^2 dt/b^2(T) + \mu^{-2}\eta,$$

with K given by (4.1). Since $K[-T, T] \leqq c_1 T$ for $T \geqq T_2$, theorem 4.5[b] follows from (4.23), (4.20) and theorem 4.5[a].

From theorem 4.5 we may derive results analogous to lemma 4.4, theorem 4.3 and theorem 4.4. We note that if the conditions of theorem 4.5 apply to f, they also apply to $|f|$.

5. Convergence almost sure if $\limsup_{|u|\to\infty} |\varphi(u)| < 1$

Throughout this section we assume that $|f|^2$ and therefore $|f|$ is integrable over finite subintervals of $[0, \infty)$, an assumption implicit in (1.32), (1.33) etc. We use the same methods as in section 3.

LEMMA 5.1. If (1.22) and (1.23) hold and if $\alpha \in L_2(a, b)$ for

$0 \leqq a < b < \infty$, then there is $W(r)$, $r = 0, 1, 2, \cdots$, such that for the homogeneous renewal process

$$\left| \int_{[2^r, n)} \alpha(t) d\tilde{N}(t) \right|^2 \leqq W(r), \qquad \begin{matrix} n = 2^r+1, \cdots, 2^{r+1}, \\ r = 0, 1, 2, \cdots, \end{matrix} \tag{5.1}$$

and

$$E\{W(r)\} \leqq A(r+1)^2 \int_{2^r}^{2^{r+1}} |\alpha(t)|^2 dt, \qquad r = 0, 1, 2, \cdots. \tag{5.2}$$

Proof. From lemma 3.1 with $v(0) = 0$ and

$$v(h) = \int_{[h-1, h)} \alpha(t) d\tilde{N}(t), \qquad h = 1, 2, \cdots, \tag{5.3}$$

it follows that in (5.1) we may take

$$W(r) = (r+1) \sum_{\nu=0}^{r} \sum_{k=0}^{2^{r-\nu}-1} |\eta(\nu, k)|^2, \tag{5.4}$$

with $\eta(\nu, k)$ defined by (3.3), so that

$$\eta(\nu, k) = \int_{[2^r+k2^\nu, 2^r+(k+1)2^\nu)} \alpha(t) d\tilde{N}(t), \qquad \begin{matrix} k = 0, 1, \cdots, 2^{r-\nu}-1, \\ \nu = 0, 1, \cdots, r. \end{matrix} \tag{5.5}$$

From (4.8), (4.9), (4.10), where $\sigma(u)$ is bounded by lemma 4.3[b], it follows with Parseval's formula that

$$E|\eta(\nu, k)|^2 \leqq c \int_{2^r+k2^\nu}^{2^r+(k+1)2^\nu} |\alpha(t)|^2 dt,$$

and (5.2) follows with (5.4).

Lemma 5.2. If (1.22) and (1.24) hold and if $\alpha \in L_2(a, b)$ for $0 \leqq a < b < \infty$, then there is $W(r)$, $r = 0, 1, 2, \cdots$, such that for the homogeneous renewal process (5.1) holds and

$$\begin{aligned} E\{W(r)\} &\leqq A(r+1)^2 \delta_r^\theta \left\{ \int_{2^r}^{2^{r+1}} |\alpha(t)| dt \right\}^2 \\ &\quad + B(r+1)^2 \delta_r^{\theta-1} \int_{2^r}^{2^{r+1}} |\alpha(t)|^2 dt, \qquad r = 0, 1, 2, \cdots, \end{aligned} \tag{5.6}$$

for any $\delta_r \in (0, 1)$, $r = 0, 1, 2, \cdots$, the constants A and B not depending on r or δ_r. The constant θ in (5.6) is the same as in (1.24).

Proof. By lemma 3.1 we may take $W(r)$ as in (5.4) and (5.5). By (1.22), (1.24) and lemma 4.3[a]

$$\sigma(u) \leqq c_1 |u|^{\theta-1}, \ |u| \leqq 1, \ \sigma(u) \leqq c_2, \ |u| \geqq 1,$$

and so from (4.8), (4.9), (4.10) we have

$$E|\eta(\nu, k)|^2 \leqq c_1 \int_{-\delta_r}^{\delta_r} |u|^{\theta-1} du \left\{\int_{2^r+k2^\nu}^{2^r+(k+1)2^\nu} |\alpha(t)|dt\right\}^2$$
$$+c_2 \delta_r^{\theta-1} \int \left|\int_{2^r+k2^\nu}^{2^r+(k+1)2^\nu} \alpha(t)e^{iut}dt\right|^2 du$$
$$= c_3 \delta_r^\theta \left\{\int_{2^r+k2^\nu}^{2^r+(k+1)2^\nu} |\alpha(t)|dt\right\}^2 + 2\pi c_2 \delta_r^{\theta-1} \int_{2^r+k2^\nu}^{2^r+(k+1)2^\nu} |\alpha(t)|^2 dt.$$

The inequality (5.6) now follows with (5.4), with the aid of the relation

$$\sum_{k=0}^{2^{r-\nu}-1} \left\{\int_{2^r+k2^\nu}^{2^r+(k+1)2^\nu} |\alpha(t)|dt\right\}^2 \leqq \left\{\sum_k \int_{\dots}^{\dots} \cdots\right\}^2 = \left\{\int_{2^r}^{2^{r+1}} |\alpha(t)|dt\right\}^2.$$

In the proof of theorem 5.1 below we need a continuous version of the Kronecker lemma used in proving theorem 3.1.

LEMMA 5.3. Let H be of bounded variation on finite subintervals of $[0, \infty)$, let $|g|$ be integrable with respect to $|dH|$ over finite subintervals of $[0, \infty)$ and let $b(t)$ be nonnegative and non-decreasing on $[0, \infty)$. Then, if

$$\lim_{t\to\infty} b(t) = +\infty, \quad \lim_{T\to\infty} \int_{[0,T)} g dH = s,$$

with s finite, we have

$$\lim_{T\to\infty} \{b(T)\}^{-1} \int_{[0,T)} b(t)g(t)dH(t) = 0.$$

PROOF. Let $S(t) \overset{\mathrm{df}}{=} \int_{[0,t)} g dH$, $t > 0$, $S(t) = 0$, $t \leqq 0$. The lemma follows from the relation

$$\int_{[0,T)} b(t)dS(t) = b(T^-)S(T) - A - \int_{[0,T)} S^*(t)db(T),$$

where A is a constant, $S^*(t) = q(t)S(t) + \{1-q(t)\}S(t^+)$, with $0 \leqq q(t) \leqq 1$ and A and $q(t)$ depend on the behaviour of $b(t)$ at its discontinuities.

THEOREM 5.1. Let $A(T)$ be defined by (1.18). If (1.22), (1.23) and (1.32) hold, we have for the homogeneous renewal process

$$\lim_{T\to\infty} A(T) = 0, \text{ a.s.}$$

PROOF. By lemma 5.3 it is sufficient to show that $\lim_{T\to\infty} \beta(T)$ exists and is finite, a.s., where

$$\beta(T) \overset{\mathrm{df}}{=} \int_{[0,T)} \{f(t)/b(t)\} d\tilde{N}(t).$$

By methods similar to those used in the proof of theorem 3.1 it may be shown that

$$(5.7) \qquad \lim_{n\to\infty} \beta(n) = w \text{ (finite), a.s.}$$

Instead of lemma 2.1, 3.1 and 3.2 one has to apply lemma 4.2, where $\sigma(u)$ is bounded by lemma 4.3[b], and lemma 5.1 with $\alpha = f/b$.

Finally let n_T denote the integer with $n_T \leqq T < n_T+1$. Then

$$(5.8) \qquad \beta(T) = \beta(n_T) + \int_{[n_T, T)} (f/b)d\tilde{N},$$

$$(5.9) \qquad \left|\int_{[n_T, T)} (f/b)d\tilde{N}\right| \leqq \varepsilon(n_T),$$

where, for $n = 0, 1, 2, \cdots$,

$$\varepsilon(n) \overset{\mathrm{df}}{=} \int_{[n,n+1)} |f/b|\, |d\tilde{N}| = \int_{[n,n+1)} |f/b| d\tilde{N} + 2\mu^{-1}\int_n^{n+1} |f/b| dt.$$

Here the first term tends to zero a.s. by (5.7) applied to $|f|$, where it is noted that the conditions of the theorem also apply to $|f|$. The second term tends to zero by (1.32), since

$$(5.10) \qquad \int_n^{n+1} |f/b| dt \leqq \left\{\int_n^{n+1} |f/b|^2 dt\right\}^{\frac{1}{2}}.$$

It follows now from (5.8), (5.7) and (5.9) that $\beta(T) \to w$, a.s.

THEOREM 5.2. Let $\Lambda(T)$ be defined by (1.18). If (1.22), (1.20), (1.24), (1.30) and (1.33) hold, then for the homogeneous renewal process

$$\lim_{T\to\infty} \Lambda(T) = 0, \text{ a.s.}$$

PROOF. From (1.24) and lemma 4.3[a]

$$\sigma(u) \leqq c_1|u|^{\theta-1}, \ |u| \leqq 1, \ \sigma(u) \leqq c_2, \ |u| \geqq 1.$$

By methods similar to those used in the proof of theorem 3.2, applying lemma 4.2 with Parseval's formula and lemma 5.2 with $\alpha = f$, it may be shown that

$$(5.11) \qquad \lim_{n\to\infty} \Lambda(n) = 0, \text{ a.s.}$$

Finally, let n_T be the integer with $n_T \leqq T < n_T+1$. Then, with (1.5) and (1.20),

$$(5.12) \qquad \begin{aligned} &|\Lambda(T)| \leqq |\Lambda(n_T)| + c_3\xi(n_T), \\ &\xi(n) = \{b(n+1)\}^{-1}\int_{[n,n+1)} |f(t)|\, |d\tilde{N}(t)|, \quad n = 0, 1, 2, \cdots \end{aligned}$$

We have

$$\xi(n) = \{b(n+1)\}^{-1}\int_{[n,n+1)} |f|d\tilde{N} + 2\{\mu b(n+1)\}^{-1}\int_n^{n+1} |f|dt.$$

Here the first term tends to zero a.s. by (5.11) applied to $|f|$, and the second term by (1.5), (5.10) and (1.33^{d}). The theorem now follows with (5.12) and (5.11).

Theorem 5.3. If either F_1 is absolutely continuous, or, F_1 being arbitrary, some $F^{(m)}$ has an absolutely continuous component, theorems 5.1 and 5.2 continue to hold for the nonhomogeneous renewal process.

Proof. The theorem follows from theorem 5.1 and 5.2 in the same way as theorem 4.2 follows from theorem 4.1, except that no subsequences need to be taken.

Lemma 5.4. If either

(a) (1.22), (1.23), (1.32) and (1.34) hold,

or

(b) (1.22), (1.20), (1.24), (1.30), (1.33) and (1.34) hold,

then for the homogeneous random walk

$$\lim_{T\to\infty}\left\{\sum_{k=1}^{M(T)} f(S_k) - \mu^{-1}\int_0^T f(t)dt\right\}\Big/ b(T) = 0, \text{ a.s.,}$$

where $M(T)$ is defined by (2.5) and (2.6).

Proof. The lemma follows from theorem 5.1 and 5.2 in the same way as lemma 3.5 follows from theorem 3.1 and 3.2, the main difference being that $\sum_{k=0}^{T^*-1} f(k)z_k$ is replaced by $\int_{[0,T)} f dN$.

Theorem 5.4. If either

(a) (1.22), (1.20), (1.23), (1.32) and (1.34) hold,

or

(b) (1.22), (1.20), (1.24), (1.30), (1.33) and (1.34) hold,

then for the homogeneous random walk

$$\lim_{n\to\infty}\left\{\sum_{k=1}^{n} f(S_k) - \mu^{-1}\int_0^{n\mu} f dt\right\}\Big/ b(n) = 0, \text{ a.s.}$$

Corollary 1. Under the conditions of the theorem

$$\sum_{k=1}^{n} f(S_k)/b(n) \to \lambda, \text{ a.s.,}$$

with λ necessarily degenerate, if and only if $\int_0^{n\mu} f dt/b(n) \to \lambda\mu$.

COROLLARY 2. If (1.22) and (1.24) hold and f is bounded,

$$\lim_{n\to\infty} n^{-1}\left\{\sum_{k=1}^{n} f(S_k)-\mu^{-1}\int_0^{n\mu} f\,dt\right\} = 0, \text{ a.s.}$$

PROOF. The theorem follows from lemma 5.4 by essentially the same methods as theorem 2.2 from lemma 2.4. Since the theorem is on almost sure convergence it is not necessary to consider subsequences.

THEOREM 5.5. If either F_1 is absolutely continuous, or, F_1 being arbitrary, some $F^{(m)}$ has an absolutely continuous component, theorem 5.4 and its corollaries continue to hold for the nonhomogeneous random walk.

PROOF. The theorem follows from theorem 5.4 in the same way as theorem 4.2 from theorem 4.1 ,but without taking subsequences and with the same modification as in the proof of theorem 4.4.

The first part of theorem 5.5 also may be formulated as follows: For the basic random walk

$$\lim_{n\to\infty}\left\{\sum_{k=1}^{n} f(y+S_k)-\mu^{-1}\int_0^{n\mu} f\,dt\right\}\Big/ b(n) = 0, \text{ a.s.,}$$

for almost all y with respect to Lebesgue measure.

6. Recurrence

Here we consider the theorem of Chung and Derman on recurrent sets. (Chung and Derman [3], Derman [5]).

Throughout this section we assume that $f \in L_2(0, T)$, $T > 0$. As before, the conditions

$$\int |x|dF < \infty, \tag{1.1}$$

$$\mu \overset{\text{df}}{=} \int x\,dF > 0, \tag{1.2}$$

essential for our results, will be assumed to hold.

THEOREM 6.1. Under (1.22), if f is nonnegative on $[0, \infty)$ and if

$$\int_0^\infty f(t)dt = +\infty, \tag{6.1}$$

$$\lim_{T\to\infty}\left\{\int_0^T f^2 dt\right\}\left\{\int_0^T f\,dt\right\}^{-2} = 0, \tag{6.2}$$

then for the homogeneous random walk

$$\sum_{k=1}^{\infty} f(S_k) = +\infty, \text{ a.s.}$$

COROLLARY. Let A be a Lebesgue measurable subset of $[0, \infty)$. Under (1.22), for the homogeneous random walk, $P\{S_k \in A, \text{i.o.}\}$ is zero or one according as A has finite or infinite Lebesgue measure.

i.o.: infinitely often.

PROOF. We put

$$Y(T) \overset{\text{df}}{=} \int_{[0,T)} f dN, \qquad T > 0, \tag{6.3}$$

$$b(T) \overset{\text{df}}{=} E\{Y(T)\} = \mu^{-1} \int_0^T f dt, \qquad T > 0. \tag{6.4}$$

By Chebychev's inequality, for $b(T) > c$,

$$P\{Y(T) \leqq c\} \leqq \{b(T) - c\}^{-2} E|\tilde{Y}(T)|^2,$$

where for $T \to \infty$ the right-hand side tends to zero by theorem 4.1[b], since (1.31) and (1.30) follow from (6.2) and (6.4), respectively. Now $\lim_{T\to\infty} Y(T)$ exists, finite or $+\infty$. By what was shown above, the limit must be $+\infty$, a.s., and the theorem follows by taking into account that $N(-\infty, 0) < \infty$, a.s.

PROOF OF COROLLARY. That $P\{S_k \in A \text{ i.o.}\} = 1$ if A has infinite Lebesgue measure follows from theorem 6.1 by taking for f the indicator function of A. Then (6.1) implies (6.2).

THEOREM 6.2. If f is nonnegative on $[0, \infty)$ and if (1.22), (6.1) and (6.2) hold, then for the basic random walk

$$\sum_{k=1}^{\infty} f(y+S_k) = +\infty, \text{ a.s.,}$$

for almost all y with respect to Lebesgue measure. If moreover some $F^{(m)}$ has an absolutely continuous component,

$$\sum_{k=1}^{\infty} f(S_k) = +\infty, \text{ a.s.}$$

PROOF. From theorem 6.1, by the same method applied to derive theorem 4.2 from theorem 4.1, but without taking subsequences. Cf. the proof of theorem 4.4.

In the same way, from the corollary to theorem 6.1:

THEOREM 6.3. Let A be a Lebesgue measurable subset of $[0, \infty)$. Under (1.22) for the basic random walk $P\{y+S_k \in A \text{ i.o.}\}$ is zero or one for almost all y with respect to Lebesgue measure, according as A has finite or infinite Lebesgue measure. If moreover some $F^{(m)}$ has an absolutely continuous component, a similar assertion holds for $P\{S_k \in A \text{ i.o.}\}$.

If the X_k are integer-valued with span 1, the theorem of Chung and Derman may be derived from theorem 2.1[b]. The counterpart of theorem 6.1 now may be simplified to:

$$\sum f(S_k) < \infty, \text{ a.s., if } \sum_0^\infty f(k) < \infty,$$

$$\sum f(S_k) = \infty, \text{ a.s., if } \sum_0^\infty f(k) = \infty.$$

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