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On distribution of arithmetical functions on the set prime plus one

by

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1. Introduction

P. Erdös proved the following theorem [1].

Let $f(n)$ be a real valued additive number-theoretical function, and put

$$f^*(n) = \begin{cases} f(n) & \text{for } |f(n)| \leqq 1, \\ 0 & \text{for } |f(n)| > 1. \end{cases}$$

Put

$$F_N(x) = \frac{1}{N} \sum_{\substack{f(n) < x \\ n \leqq N}} 1.$$

Then the distribution-functions $F_N(x)$ tend for $N \to +\infty$ to a limiting distribution function $F(x)$ at all points of continuity of $F(x)$, if the following three conditions are satisfied:

1. $$\sum_p \frac{f^*(p)}{p} \text{ is convergent,}$$

2. $$\sum_p \frac{(f^*(p))^2}{p} < +\infty,$$

3. $$\sum_{|f(p)| > 1} \frac{1}{p} < +\infty.$$

It has been shown also by P. Erdös that $F(x)$ is continuous if and only if the series $\sum_{f(p) \neq 0} 1/p$ diverges.

New proof of this theorem has been given by H. Delange [2] and by A. Rényi [3].

A multiplicative function $g(n)$ is called strongly multiplicative, if for all primes p and all positive integers k it satisfies the condition

$$g(p^k) = g(p).$$

H. Delange proved the following theorem [4].

If $g(n)$ is a strongly multiplicative number-theoretical function such that $|g(n)| \leqq 1$ for $n = 1, 2, \ldots$, and such that the series

$$\sum_p \frac{g(p)-1}{p}$$

converges, then

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} g(n) = M(g)$$

exists and

$$M(g) = \prod_p \left(1 + \frac{g(p)-1}{p}\right).$$

A new proof of this theorem has been given by A. Rényi [5].

Throughout the paper p, q denote primes, and $\sum_p$ and $\prod_p$ denote a sum and a product, respectively, taken over all primes. Let further $\operatorname{li} x = \int_2^x du/\log u$.

The aim of this paper is to prove the following statement.

THEOREM 1. *Let $g(n)$ be a complex-valued multiplicative function such that $|g(n)| \leqq 1$ for $n = 1, 2, \ldots$, and such that the series*

$$\sum_p \frac{g(p)-1}{p} \tag{1.1}$$

converges. Let $N(g)$ denote the product

$$N(g) = \prod_p \left(1 + \sum_{k=1}^{\infty} \frac{g(p^k)-g(p^{k-1})}{p^{k-1}(p-1)}\right). \tag{1.2}$$

Then

$$\lim_{x\to\infty} \frac{1}{\operatorname{li} x} \sum_{p \leqq x} g(p+1) = N(g). \tag{1.3}$$

From this theorem easily follows the

THEOREM 2. *Let $f(n)$ be a real valued additive number-theoretical function which satisfies the conditions* 1, 2, 3, *of the theorem of Erdös.*

Put

$$F_N(y) = \frac{1}{\operatorname{li} N} \sum_{\substack{f(p+1) < y \\ p \leqq N}} 1.$$

Then the distribution-functions $F_N(y)$ tend for $N \to \infty$ to a limiting distribvtron-function $F(y)$ at all points of continuity of $F(y)$.

Further $F(y)$ is a continuous function if and only if

$$\sum_{f(p) \neq 0} \frac{1}{p} = \infty.$$

2. Deduction of Theorem 2 from Theorem 1

In what follows $c, c_1, c_2, \ldots$ denote constants not always the same in different places.

For the proof of Theorem 2 we need to prove only that the sequence of characteristic functions

$$\varphi_N(u) = \frac{1}{\operatorname{li} N} \sum_{p \leq N} e^{iuf(n)} \tag{2.1}$$

converges to a function $\varphi(u)$, which is a continuous one on the real axis.

It is easy to verify that from the conditions 1/2/3 it follows that

$$\sum_p \frac{e^{iuf(p)}-1}{p} \tag{2.2}$$

converges for every real u. Using now Theorem 1 with $g(n) = e^{iuf(n)}$ we obtain that $\varphi_N(u) \to \varphi(u)$, where

$$\varphi(u) = \prod_p \left(1 + \sum_{k=1}^{\infty} \frac{e^{iuf(p^k)}-e^{iuf(p^{k-1})}}{p^{k-1}(p-1)}\right). \tag{2.3}$$

The continuity of (2.3) is guaranteed by the continuity of (2.2), which follows from the conditions 1/2/3 evidently.

For the proof of the continuity of $F(x)$ in the case

$$\sum_{f(p)\neq 0} \frac{1}{p} = \infty$$

we remark the following.

P. Levy proved the following theorem [8].

Let $X_1, X_2, \ldots, X_n, \ldots$ be a sequence of independent random variables with discrete distribution and suppose that there exists the sum

$$\sum_{k=1}^{\infty} X_k = X$$

with probability 1. Let

$$d_k = \sup_x P\ (X_k = x).$$

Then the distribution function of X is continuous if and only if

$$\prod_{k=1}^{\infty} d_k = 0.$$

Let now the X_p's be independent random variables with characteristic functions

$$1+\sum_{k=1}^{\infty} \frac{e^{iuf(p^k)}-e^{iuf(p^{k-1})}}{p^{k-1}(p-1)}$$

and let

$$X = \sum_p X_p.$$

It is evident from (2.3) that X has the distribution function $F(x)$, and so this is continuous if and only if

$$\prod_{f(p)\neq 0}\left(1-\frac{1}{p}\right) = 0, \quad \text{i.e.} \sum_{f(p)\neq 0} \frac{1}{p} = \infty.$$

3. The proof of Theorem 1

We need the following Lemmas.

Lemma 1. *Let $g(p)$ be a complex-valued function defined on the primes, for which $|g(p)| \leqq 1$ and*

(3.1) $$\sum \frac{g(p)-1}{p}$$

converges. Then

(3.2) $$\sum_{|\arg g(p)|>c} \frac{1}{p} < +\infty$$

for every positive constant c, further

(3.3) $$\sum_p \frac{|g(p)-1|^2}{p} < +\infty$$

and

(3.4) $$\sum_{x^{1/2}<p<x} \frac{|g(p)-1|}{p} \to 0 \qquad \text{for } x \to +\infty.$$

Proof. The assertion in (3.4) is an immediate consequence of (3.3) since

$$\sum_{x^{1/2}<p<x} \frac{|g(p)-1|}{p} \leqq \left(\sum_{x^{1/2}<p<x} \frac{1}{p}\right)^{\frac{1}{2}} \left(\sum_{x^{1/2}<p<x} \frac{|g(p)-1|^2}{p}\right)^{\frac{1}{2}},$$

further $\sum_{x^{1/2}<p<x} 1/p$ is bounded, and the second sum on the right hand side tends to zero because of (3.3).

Let us put

$$|g(p)| = r(p) \quad \text{and} \quad \arg g(p) = \vartheta(p),$$

where $-\pi < \vartheta(p) \leqq +\pi$, i.e. we suppose that $g(p) = r(p)e^{i\vartheta(p)}$.

From the convergence of (3.1) it follows that

$$\sum_p \frac{1-\operatorname{Re} g(p)}{p} \; (<+\infty)$$

converges too. This sum has positive terms and the inequality $|\arg g(p)| > c$ involves that $1-\operatorname{Re} g(p) > c_1$ (>0). Hence (3.2) follows.

From the inequality $|a+bi|^2 \leqq 2(|a|^2+|b|^2)$ it follows that

$$\sum_p \frac{|g(p)-1|^2}{p} \leqq 2\sum \frac{|\operatorname{Re}(1-g(p))|^2}{p} + 2\sum \frac{|\operatorname{Im} g(p)|^2}{p}.$$

The first sum on the right hand side evidently converges since

$$\sum_p \frac{|\operatorname{Re}(1-g(p))|^2}{p} < \sum \frac{1-\operatorname{Re} g(p)}{p} + 4 \sum_{|\vartheta(p)|>\frac{1}{2}} \frac{1}{p}.$$

It is sufficient to prove that

$$\sum_{|\vartheta(p)|\leqq\frac{1}{2}} \frac{r^2(p)\sin^2\vartheta(p)}{p} < \infty. \tag{3.5}$$

Using the inequality

$$r^2(p)\sin^2\vartheta(p) \leqq c\vartheta^2(p) \leqq 2c\sin^2\frac{\vartheta(p)}{2} \leqq 1-\cos\vartheta(p)$$
$$\leqq 1-r(p)\cos\vartheta(p),$$

we have (3.5).

LEMMA 2. *Let $N_k(x)$ denote the number of solutions of the equation*

$$p+1 = kq, \; p \leqq X$$

in primes p, q. Then

$$N_k(X) < c\,\frac{x}{\varphi(k)\log^2 X/k}$$

for $k < x$, where c is an absolute constant.

For the proof see Prachar's book [6], Theorem 4.6, p. 51.

Let $\pi(x, k, l)$ denote the number of primes in the arithmetical progression $\equiv l \pmod k$ not exceeding x.

LEMMA 3. (*Brun-Titchmarsh*). *For all $k \leqq x^{1-\delta}$, $\delta > 0$*

$$\pi(x, k, l) < c_\delta\,\frac{x}{\varphi(k)\log x},$$

where c_δ is a constant depending on δ only.

For the proof see [6].

LEMMA 4. (*E. Bombieri* [7]).

$$\sum_{D \leqq Y} \max_{\substack{l (\text{mod } D) \\ (l, D) = 1}} \left| \pi(x, D, l) - \frac{\operatorname{li} x}{\varphi(D)} \right| < \frac{cx}{(\log x)^A}$$

where $Y = x^{\frac{1}{2}} (\log x)^{-B}$; $B \geqq 2A+23$, *A arbitrary constant.*

Let $\tau(n)$ be the number of divisors of n.

LEMMA 5.

$$\sum_{n < y} \frac{\tau^2(n)}{\varphi(n)} < c (\log y)^4,$$

where c is a constant.

The proof is very simple and so can be omitted.

Let us define the multiplicative function $g_K(n)$ by putting

$$g_K(p^\alpha) = \begin{cases} g(p^\alpha), & \text{if } p^\alpha \leqq K, \\ 1, & \text{if } p^\alpha > K. \end{cases}$$

By other words we put for any natural number n

$$g_K(n) = \prod_{\substack{p^\alpha \| n \\ p^\alpha \leqq K}} g(p^\alpha).$$

Let us put further

$$h_K(n) = \sum_{d | n} \mu\left(\frac{n}{d}\right) g_N(d),$$

where d runs over all (positive) divisors of n and $\mu(n)$ is the Möbius function. Then $h_K(n)$ is also a multiplicative function, $h_K(p^\alpha) = g_K(p^\alpha) - g_K(p^{\alpha-1})$; $h_K(p) = 0$ for $p > K$; $h_K(p) = 0$ for $p^{\alpha-1} > K$, $\alpha \geqq 2$.

Let further $h(n)$ be defined by

$$h(n) = \sum_{d | n} \mu\left(\frac{n}{d}\right) g(d).$$

Let us introduce the notation

$$I_K(x) = \sum_{p \leqq x} g_K(p+1); \quad I(x) = \sum_{p \leqq x} g(p+1).$$

Choose now $K_1 = (\frac{1}{4} - \varepsilon) \log x$, $K_2 = x^{\frac{1}{4}}$, $K_3 = x^{1-\delta}$, where ε and δ are suitable small positive numbers.

We shall prove the following relations:

$$I_{K_1}(x) = (1+o(1)) \operatorname{li} x \, N(g) \qquad \text{for } x \to \infty, \tag{3.6}$$

$$I_{K_2}(x) - I_{K_1}(x) = o(\operatorname{li} x) \qquad \text{for } x \to \infty, \tag{3.7}$$

(3.8) $I_{K_3}(x) - I_{K_2}(x) = o(c_\delta \operatorname{li} x)$ for $x \to \infty$, uniformly in $\delta\,(>0)$,

(3.9) $I(x) - I_{K_3}(x) = O(\delta \operatorname{li} x)$ for $x \to \infty$.

Theorem 1 follows if we choose $\delta = \delta(x)$ tending to zero so slowly that the right hand side of (3.8) is $o(\operatorname{li} x)$.

First we prove (3.6). We have

$$I_{K_1}(x) = \sum_{p \leqq x} g_{K_1}(p+1) = \sum_{p \leqq x} \sum_{d \mid p+1} h_{K_1}(a) = \sum_{d} h_{K_1}(d)\pi(x, d, -1)$$
$$= \operatorname{li} x \sum_{d} \frac{h_{K_1}(d)}{\varphi(d)} + R,$$

where

$$|R| \leqq \sum_{d} |h_{K_1}(d)| \left| \pi(x, d, -1) - \frac{\operatorname{li} x}{\varphi(d)} \right| = R_1.$$

Using the prime number theorem, we obtain that $h_{K_1}(d) = 0$ for $d \geqq x^{\frac{1}{4}-\varepsilon/2}$ because

$$\prod_{p^\alpha < K_1} p^\alpha < e^{(\frac{1}{4}-\varepsilon/2)\log x},$$

if x is sufficiently large.

Since $|g(n)| \leqq 1$, so $|h_{K_1}(p^\alpha)| \leqq 2$ and $|h_{K_1}(d)| \leqq \tau(d)$.

For the estimation of R_1 we split all of the d's, $d \leqq x^{\frac{1}{4}-\varepsilon/2}$ into two classes $\mathfrak{A}_1$, $\mathfrak{A}_2$ as follows:

d belongs to $\mathfrak{A}_1$ or $\mathfrak{A}_2$ according to that $\tau(d) \leqq (\log x)^5$ or $\tau(d) > (\log x)^5$, respectively.

Using Lemma 3 and Lemma 5 we have

$$\sum_{d \in \mathfrak{A}_2} |h_{K_1}(d)| \left| \pi(x, d, -1) - \frac{\operatorname{li} x}{\varphi(d)} \right| \leqq c \operatorname{li} x \sum_{d \in \mathfrak{A}_2} \frac{\tau(d)}{\varphi(d)}$$
$$\leqq c \operatorname{li} x \,(\log x)^{-5} \sum_{d \leqq x} \frac{\tau^2(d)}{\varphi(d)} < c \frac{x}{\log^2 x}.$$

Otherwise, using the Bombieri's result (Lemma 4), we have that the sum

$$\sum_{d \in \mathfrak{A}_1} |h_{K_1}(d)| \left| \pi(x, d, -1) - \frac{\operatorname{li} x}{\varphi(d)} \right|$$

not exceed

$$(\log x)^5 \sum_{d \leqq x^{1/4}} \left| \pi(x, d, -1) - \frac{\operatorname{li} x}{\varphi(d)} \right| = O\left(\frac{x}{(\log x)^{A-5}}\right) = O\left(\frac{x}{\log^2 x}\right),$$

if $A \geqq 7$.

Hence

$$R = O\left(\frac{x}{\log^2 x}\right).$$

Further we have

$$\sum_{d} \frac{h_{K_1}(d)}{\varphi(d)} = \prod_{p<K_1} \left(1+\sum_{\alpha=1}^{\infty} \frac{g_{K_1}(p^\alpha)-g_{K_1}(p^{\alpha-1})}{p^{\alpha-1}(p-1)}\right).$$

From the convergence of the series $\sum (g(p)-1)/p$ it follows that the product on the right hand side tends to $N(g)$ for $x \to +\infty$. So (3.6) is proved.

Let now $\bar{g}(n)$ be a multiplicative function defined by

$$\bar{g}(p^\alpha) = \begin{cases} g(p^\alpha), & \text{if } p \leqq K_1, \\ g(p), & \text{if } p > K_1. \end{cases}$$

It is evident that $\bar{g}(n) = g(n)$ except eventually those n's for which there exists a q, $q > K_1$, such that $q^2|n$. So

(3.10)

$$\sum_{p \leqq x} |g(p+1)-\bar{g}(p+1)| \leqq 2 \sum_{q>K_1} \sum_{\substack{p+1\equiv 0(q^2) \\ p<q}} 1 < 2c \operatorname{li} x \sum_{K_1<q<x^{1/2}} \frac{1}{q(q-1)}$$
$$+x \sum_{q>x^{1/2}} \frac{1}{q^2} = o(\operatorname{li} x).$$

From (3.10)

$$|I_{K_2}(x)-I_{K_1}(x)| \leqq \sum_{p\leqq x} |\bar{g}_{K_2}(p+1)-\bar{g}_{K_1}(p+1)|+o(\operatorname{li} x)$$
$$\leqq \sum_{p\leqq x} \Big| \prod_{\substack{q|p+1 \\ K_1<q\leqq K_2}} g(q)-1\Big|+o\ (\operatorname{li} x) = V+o\ (\operatorname{li} x)$$

follows. Using the formulas

$$\log\ (1+z) = z+O(|z|^2);\ \exp\ (z+O(|z|^2)) = 1+z+O(|z|^2)$$

for $|z| \leqq 1$, $|\arg z| \leqq \pi/2$, we have that

$$(3.11) \qquad \prod_{\substack{q|p+1 \\ K_1<q\leqq K_2}} g(q)-1 = \sum_{\substack{q|p+1 \\ K_1<q\leqq K_2}} h(q)+O\Big(\sum_{\substack{q|p+1 \\ K_1<q\leqq K_2}} |h^2(q)|\Big),$$

if all primdivisor q of $p+1$ in the interval $K_1 < q \leqq K_2$ satisfies the relation $|\arg g(q)| \leqq \pi/2$. Let $\mathfrak{A}_3$ denote the set of the p's possessing this property, and $\mathfrak{A}_4$ the other p's.

We can easily estimate the sum

$$V_1 = \sum_{p\in\mathfrak{A}_4} \Big| \prod_{\substack{q|p+1 \\ K_1<q\leqq K_2}} g(q)-1\Big|,$$

since

$$V_1 < 2 \sum_{\substack{K_1 < q \leqq K_2 \\ |\arg g(q)| \geqq \pi/2}} \pi(x, q, -1) < c \operatorname{li} x \sum_{\substack{|\arg g(q)| > \pi/2 \\ K_1 < q < K_2}} \frac{1}{q}$$

and by (3.2)

$$V_1 = o(\operatorname{li} x).$$

Let

$$V_2 = \sum_{p \in \mathfrak{A}_3} \Big| \prod_{\substack{q | p+1 \\ K_1 < q \leqq K_2}} g(q) - 1 \Big|.$$

From (3.11) we have that

$$V_2 \leqq \sum_p \Big| \sum_{\substack{q | p+1 \\ K_1 < q \leqq K_2}} h(q) \Big| + O\Big(\sum_p \sum_{\substack{q | p+1 \\ K_1 < q \leqq K_2}} |h^2(q)|\Big) = V_3 + O(V_4).$$

Using (3.3) in Lemma 1 and Lemma 3 we have

$$V_4 < \sum_{K_1 < q \leqq K_2} |h^2(q)| \, \pi(x, q, -1) < c \sum_{q > K_1} \frac{|h^2(q)|}{q-1} = o(\operatorname{li} x).$$

Further, from the Cauchy's inequality

$$V_3 < c (\operatorname{li} x)^{\frac{1}{2}} \Big\{ \sum_{\substack{K_1 < q_1, q_2 < K_2 \\ q_1 \neq q_2}} h(q_1) \bar{h}(q_2) \pi(x, q_1 q_2, -1) + \sum_{K_1 < q \leqq K_2} |h(q)|^2 \pi(x, q, -1) \Big\}^{\frac{1}{2}}.$$

Using Bombieri's result we have that

$$V_3 < c (\operatorname{li} x)^{\frac{1}{2}} \left| \sum_{K_1 < q \leqq K_2} \frac{h(q)}{q-1} \right| (\operatorname{li} x)^{\frac{1}{2}} + O\left(\frac{x}{\log^2 x}\right) = o(\operatorname{li} x),$$

since

$$\sum_{K_1 < q \leqq K_2} \frac{h(q)}{q-1} = \sum_{K_1 < q \leqq K_2} \frac{h(q)}{q} + O\left(\sum_{K_1 < q} \frac{1}{q^2}\right) = o(1).$$

So we proved that

$$V_2 = V_3 + O(V_4) = o(\operatorname{li} x); \quad V_1 = o(\operatorname{li} x); \quad V = V_1 + V_2 = o(\operatorname{li} x),$$

whence (3.7) follows.

Similarly we have

$$|I_{K_3}(x) - I_{K_2}(x)| \leqq \sum_{p \leqq x} \Big| \sum_{\substack{q | p+1 \\ K_2 < q \leqq K_3}} h(q) \Big| + c \sum_{p \leqq x} \sum_{\substack{q | p+1 \\ K_2 < q \leqq K_3}} |h(q)|^2 + c \sum_{\substack{K_2 < q \leqq K_3 \\ |\arg g(p)| \geqq \pi/2}} \pi(x, q, -1) = V_5 + cV_6 + cV_7.$$

Using Lemma 3 and (3.4) in Lemma 1 we have that

$$V_5 \leqq \sum_{K_2 < q \leqq K_3} |h(q)| \pi(x, q, -1) < c_\delta \operatorname{li} x \sum_{K_2 < q \leqq K_3} \frac{|h(q)|}{q} = o(c_\delta \operatorname{li} x).$$

Further using (3.3) and (3.2) we obtain that

$$V_6 \leqq \sum_{K_2 < q \leqq K_3} |h^2(q)| \pi(x, q, -1) < c_\delta \operatorname{li} x \sum_{K_2 < q \leqq K_3} \frac{|h^2(q)}{q} = o(c_\delta \operatorname{li} x),$$

$$V_7 \leqq c_\delta \operatorname{li} x \sum_{\substack{q > K_2 \\ |\arg g(p)| \geqq \pi/2}} \frac{1}{q} = o(c_\delta \operatorname{li} x).$$

Hence (3.8) follows.

Finally using Lemma 2 we have

$$|I(x) - I_{K_3}(x)| \leqq 2 \sum_{K_3 < q < x} \pi(x, q, -1) \leqq \sum_{j \leqq x^\delta} N_j(x)$$
$$\leqq c \sum_{j \leqq x^\delta} \frac{x}{\varphi(j) \log^2 x/j} < c \frac{x}{\log^2 x} \sum_{j < x^\delta} \frac{1}{\varphi(j)} < c\delta \frac{x}{\log x},$$

because

$$\sum_{j \leqq y} \frac{1}{\varphi(j)} < c \log y.$$

So the inequality (3.9) is proved, and from (3.6)—(3.9) our theorem follows.

4. Some remarks

1. From our Theorem 2 it follows evidently that if $g(n)$ is a positive valued multiplicative number-theoretical function such that

1. $$\sum_p \frac{((\log g(p))^*}{p} \text{ is convergent,}$$

2. $$\sum_p \frac{(\log g(p))^{*2}}{p} < +\infty,$$

3. $$\sum_{|\log g(p)| > 1} \frac{1}{p} < +\infty,$$

then putting

$$F_N(y) = \frac{1}{\operatorname{li} N} \sum_{g(p+1)<y} 1$$

the distribution functions $F_N(y)$ tend for $N \to +\infty$ to a limiting distribution function $F(y)$ at all points of continuity of $F(y)$.

Hence it follows especially that the functions

$$\frac{\varphi(p+1)}{p+1}, \quad \frac{\sigma(p+1)}{p+1}$$

($\sigma(n)$ denotes the sum of the divisors of n) have limiting distribution functions.

2. Recently M. B. Barban, A. I. Vinogradov, B. V. Levin proved that all results of J. P. Kubilius theory (see [9]) are valid for strongly additive arithmetic functions belonging to the class H, when the argument runs through "shıffed" primes $\{p-l\}$, (see [10], [11]).

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