COMPOSITIO MATHEMATICA

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Compositio Mathematica, tome 19, nº 3 (1968), p. 213-258
<http://www.numdam.org/item?id=CM_1968__19_3_213_0>

Unitary groups in projective octave planes [1]

by

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0. In this paper we consider groups of projectivities in an octave plane which commute with a polarity in that plane. These groups behave in a way similar to the classical unitary and orthogonal groups. In general, not much can be said if the polarity is elliptic. But if we deal with a hyperbolic polarity, then the corresponding unitary group has a simple normal subgroup generated by unitary transvections. This is also a consequence of Tits' results [29] which are placed in the more general framework of algebraic groups; our results are derived with the methods in use for the description of octave planes and they give some more detailed information.

There are three types of hyperbolic polarities. First, the hermitian linear polarities already considered in [10, II], [21] and [22]; the corresponding unitary groups are simple and of type F_4. Then there exists a nonhermitian linear type of polarities; the corresponding group has a simple normal subgroup generated by the unitary transvections, which is a group of type C_4, already found by Tits [27] in the case of octaves over the reals. Finally, we have a nonlinear type of hyperbolic polarity which gives rise to a quasi-split outer form of the group E_6; these groups have already been considered by Tits [28] who used different techniques.

This paper is divided into three chapters. Ch. I deals with polarities in octave planes and some general results about unitary groups; it generalizes results already found for octaves over the reals by Tits [26, 27]. In II, the structure of these unitary groups is examined in more detail, especially in the case of hyperbolic polarities. In III, methods of algebraic group theory are applied to determine the type of the groups under consideration. For this purpose, roots and rootvectors of some Lie algebras are explicitely computed.

[1] This research was partially supported by the National Science Foundation under Grant No. NSF-GP-4017 during the author's stay at Yale University.

For notations and preliminary results on Jordan algebras and octave planes we refer in general to [22]; as a general background we also need [18, 19, 20 and 21].

There are some deviations from the rule on notations we have just fixed. Thus, linear transformations will, with a few exceptions, be denoted by italic capitals. We shall write $\mathscr{P}(C)$ instead of $\mathscr{P}_C$ for the projective plane over the octave field C. The line of points x such that $(u, x) = 0$, will be denoted by u^* instead of $\bar{u}$.

A $(\gamma_1, \gamma_2, \gamma_3)$-hermitian matrix

$$\begin{pmatrix} \xi_1 & x_3 & \gamma_1^{-1}\gamma_3\bar{x}_2 \\ \gamma_2^{-1}\gamma_1\bar{x}_3 & \xi_2 & x_1 \\ x_2 & \gamma_3^{-1}\gamma_2\bar{x}_1 & \xi_3 \end{pmatrix}$$

will in most cases be denoted by $(\xi_1, \xi_2, \xi_3; x_1, x_2, x_3)$; this notation, of course, does not show explicitely the coefficients $\gamma_1, \gamma_2, \gamma_3$, but these will be equal to 1 in the greater part of this paper.

We shall frequently need the crossed product $x \times y$ of elements x and y of a Jordan algebra of 3×3 hermitian octave matrices; see e.g. [22, p. 418]. For convenience in computations we give here the expression of this product in matrix form in case $\gamma_1 = \gamma_2 = \gamma_3 = 1$, which is easily derived from the formula

$$(x \times y, z) = 3\langle x, y, z\rangle :$$

$$(\xi_1, \xi_2, \xi_3; x_1, x_2, x_3) \times (\eta_1, \eta_2, \eta_3; y_1, y_2, y_3) = (\zeta_1, \zeta_2, \zeta_3; z_1, z_2, z_3),$$

where

$$\zeta_i = \xi_{i+1}\eta_{i+2} + \eta_{i+1}\xi_{i+2} - (x_i, y_i)$$
$$z_i = \bar{x}_{i+2}\bar{y}_{i+1} + \bar{y}_{i+2}\bar{x}_{i+1} - \xi_i y_i - \eta_i x_i,$$

subscripts taken mod 3. Of course, it is easy to derive a similar formula for arbitrary $(\gamma_1, \gamma_2, \gamma_3)$-hermitian matrices, but the above formula will suffice for our purposes.

The author wishes to express his gratitude to Professor T. A. Springer for some critical remarks. In particular simpler proofs for theorems 5.1 and 5.7 are due to him.

I. Polarities, unitary groups

1. Let K be a commutative field of characteristic $\neq 2,3$, C an octave field over K and $\mathscr{P}(C)$ a projective plane over C. $\mathscr{P}(C)$ can be described with the aid of any Jordan algebra $A = A(C; \gamma_1, \gamma_2, \gamma_3)$ of 3×3 $(\gamma_1, \gamma_2, \gamma_3)$-hermitian matrices with entries in C. A choice of such a coordinate algebra A defines a polarity π_0 in $\mathscr{P}(C)$, which maps a point u on the line u^*; we will call π_0 the *standard polarity* with respect to the coordinate algebra A. Any duality of $\mathscr{P}(C)$ can be written as

$$\pi = \pi_0 \ulcorner T \urcorner,$$

where $\ulcorner T \urcorner$ is the collineation of $\mathscr{P}(C)$ induced by the σ-linear transformation T of A satisfying

$$det\, Tx = \nu\sigma\,(det\, x) \qquad \text{for all } x.$$

We shall call

$$\nu = det\, T.$$

Since

$$Tx \times Ty = \nu T'^{-1}\,(x \times y) \qquad \text{(see [21, p. 455])},$$

the action of π on a line u^* is given by

$$\pi u^* = \nu \ulcorner T' \urcorner^{-1} u.$$

Hence π is a polarity if and only if

$$\nu T'^{-1} T = \lambda,$$

for some $\lambda \neq 0$ in K, i.e.,

(1.1) $\quad \pi = \pi_0 \ulcorner T \urcorner$ is a polarity $\Leftrightarrow T = \alpha T',\ \alpha \in K,\ \alpha \neq 0.$

Since T' is σ^{-1}-linear, it follows that $\sigma^2 = 1$. From (1.1) we get by taking the transposed

$$T' = \sigma(\alpha) T,$$

Hence

$$\alpha\sigma(\alpha) = 1.$$

If $\sigma = 1$, this implies $\alpha = \pm 1$. But $\alpha = -1$ would give $T' = -T$, hence $(x, Tx) = 0$ for all x since char. $K \neq 2$. So every point would lie on its polar, which is impossible in a plane (see [22, p. 424]). So in this case we get $\alpha = 1$. If $\sigma \neq 1$, then by Hilbert's "theorem 90" there exists a ρ such that $\alpha = \rho^{-1}\sigma(\rho)$. Replacing T by ρT we find that we may assume $T = T'$. Thus we have shown

(1.2) *π is a polarity if and only if it can be written as*

$$\pi = \pi_0 \ulcorner T \urcorner,$$

where the collineation $\ulcorner T \urcorner$ is induced by a σ-linear transformation T of A satisfying

$$det\,(Tx) = \nu\sigma\,(det\, x),\ \sigma^2 = 1,\ T = T'.$$

We call a point u *isotropic* with respect to π if $u \in \pi u$, otherwise *nonisotropic.*

(1.3) *If the point u_1 is nonisotropic with respect to π, it can be embedded in a polar triangle u_1, u_2, u_3.*

Proof. It suffices to show the existence of a nonisotropic point $u_2 \in \pi u_1$. Assume v and $w \in \pi u_1$ are isotropic. Then $\pi v = (u_1 \times v)^*$, $\pi w = (u_1 \times w)^*$. Take $x \in \pi v$, $\neq u_1$, $\neq v$, $y = \pi x \cap \pi w$, $z = \pi x \cap \pi y$. Then $\pi z = (x \times y)^*$. Take $u_2 = (u_1 \times z)^* \cap \pi u_1$, $u_3 = \pi z \cap \pi u_1$, then $\pi u_2 = (u_1 \times u_3)^*$. v, w, u_2 and u_3 are harmonic, hence $u_2 \neq u_3$ since K has characteristic $\neq 2$.

2. Let u_1, u_2, u_3 form a polar triangle. Three noncollinear points can be transformed into any other three noncollinear points by an element of the little projective group (see [21, p. 461]), hence we may coordinatize $\mathscr{P}(C)$ in such a way that

$$u_1 = (1, 0, 0; 0, 0, 0), u_2 = (0, 1, 0; 0, 0, 0), u_3 = (0, 0, 1; 0, 0, 0).$$

For the polarity $\pi = \pi_0 \ulcorner T \urcorner$ we then have

$$Tu_i = \lambda_i u_i, \qquad i = 1, 2, 3. \tag{2.1}$$

The subspace $R = \langle e - u_1 \rangle + E_0$ in the Peirce decomposition of A with respect to u_1 is spanned by the points $x \in u_1^* = \pi u_1$. For such an x, $u_1 \in \pi x$, hence $Tx \in R$. Consider

$$y = \xi_1 u_1 + x, \qquad x \in R.$$

Then

$$Ty = \lambda_1 \sigma(\xi_1) u_1 + Tx, \qquad Tx \in R.$$

Now $\det y = \xi_1 Q_1(x)$ (see [21, p. 452]) and $\det(Ty) = \nu\sigma(\det y)$, hence we must have

$$\lambda_1 \sigma(\xi_1) Q_1(Tx) = \nu\sigma(\xi_1)\sigma(Q_1(x))$$
$$Q_1(Tx) = \lambda_1^{-1} \nu\sigma(Q_1(x)).$$

Since $Tu_i = \lambda_i u_i$, T must leave invariant the orthogonal complement C (with respect to Q_1) of $\langle u_2 \rangle + \langle u_3 \rangle$ in R. Replacing u_1 by u_2 and u_3 respectively in the above argument, we find that

$$T = [\lambda_1, \lambda_2, \lambda_3; T_1, T_2, T_3], \tag{2.2}$$

which is defined by

$$(\xi_1, \xi_2, \xi_3; x_1, x_2, x_3) \to (\lambda_1\sigma(\xi_1), \lambda_2\sigma(\xi_2), \lambda_3\sigma(\xi_3); T_1x_1, T_2x_2, T_3x_3).$$

The T_i are σ-linear transformations of C.

In this representation of T, the condition

$$\text{(2.3)} \qquad det\,(Tx) = \nu\sigma\,(det\,x)$$

is equivalent to the following conditions (2.4—5—6).

$$\text{(2.4)} \qquad \lambda_1\lambda_2\lambda_3 = \nu,$$

$$\text{(2.5)} \qquad N(T_i x) = \lambda_i^{-1}\nu\sigma\big(N(x)\big) \quad \text{for all } x \in C,$$

$$\text{(2.6)} \quad (T_1x_1, T_2x_2, T_3x_3) = \nu\sigma\big((x_1, x_2, x_3)\big) \text{ for all } x_1, x_2, x_3 \in C.$$

Here we use in (2.6) the notation

$$\text{(2.7)} \qquad (x_1, x_2, x_3) = x_1(x_2x_3) + \overline{x_1(x_2x_3)} = \big(1, x_1(x_2x_3)\big).$$

In the terminology of [12, p. 135], T_1, T_2, T_3 form a *related triple* of *semi-similarities* in C. (2.5) and (2.6) are equivalent to (2.5) and

$$\text{(2.8)} \qquad \hat{T}_i(xy) = \lambda_i^{-1}(T_{i+1}x)(T_{i+2}y),$$

subscripts taken mod 3 (see [12, p. 135] or [17, p. 152]). Here $\hat{T}_i z = \overline{T_i \bar{z}}$. Applying N on both sides of the equation (2.8) one obtains (2.4) (see [16, p. 16]), so (2.4) is a consequence of (2.5—6).

Let T_i' denote the transposed of T_i with respect to the bilinear form $(.,.)$ on C related to N. One easily verifies that

$$\text{(2.9)} \quad [\lambda_1, \lambda_2, \lambda_3; T_1, T_2, T_3]' = [\sigma(\lambda_1), \sigma(\lambda_2), \sigma(\lambda_3); \alpha_1T_1', \alpha_2T_2', \alpha_3T_3']$$

where

$$\alpha_i = \gamma_{i-1}\gamma_{i+1}^{-1}\sigma(\gamma_{i-1}^{-1}\gamma_{i+1}).$$

From now on we coordinatize $\mathscr{P}(C)$ only with Jordan algebras $A(C; \gamma_1, \gamma_2, \gamma_3)$ such that $\gamma_i = \sigma(\gamma_i)$, $i = 1, 2, 3$.

Then $\alpha_i = 1$ in (2.9), hence the condition $T = T'$ in (1.2) becomes equivalent to

$$\text{(2.10)} \qquad \lambda_i = \sigma(\lambda_i), \qquad i = 1, 2, 3$$

$$\text{(2.11)} \qquad T_i = T_i', \qquad i = 1, 2, 3.$$

From (2.4) and (2.10) it follows that $\nu = \sigma(\nu)$. (2.5) implies that (2.11) can be replaced by

$$\text{(2.12)} \qquad T_i^2 = \mu_i = \lambda_i^{-1}\nu, \qquad i = 1, 2, 3.$$

Conversely, assume that a proper semi-similarity T_1 of C is given satisfying (2.5) and (2.12) for $i = 1$. By [3] and [31] we can find T_2 and T_3 such that (2.5) and (2.6) are satisfied for some λ_i $(i = 2, 3)$, if ν is given. T_2 and T_3 are unique up to factors ρ

and ρ^{-1} respectively, $\rho \in K$, $\neq 0$. Is (2.12) satisfied for $i = 2, 3$? $T_1^2 = \mu_1$, T_2^2 and T_3^2 form a related triple, hence $T_2^2 = \alpha_2$, $T_3^2 = \alpha_3$ for some $\alpha_2, \alpha_3 \in K$.

Then

$$\begin{aligned}\mu_1\alpha_2\alpha_3(x_1, x_2, x_3) &= (T_1^2x_1, T_2^2x_2, T_3^2x_3)\\ &= \nu^2(x_1, x_2, x_3).\end{aligned}$$

So

$$\mu_1\alpha_2\alpha_3 = \nu^2 = \mu_1\mu_2\mu_3$$

$$\alpha_2\alpha_3 = \mu_2\mu_3.$$

Furthermore,

$$\alpha_i^2N(x) = N(T_i^2x) = \mu_i^2N(x).$$

$$\alpha_i^2 = \mu_i^2.$$

Thus we find: either $\alpha_2 = \mu_2$, $\alpha_3 = \mu_3$, or $\alpha_2 = -\mu_2$, $\alpha_3 = -\mu_3$. Considering first the case $\sigma \neq 1$, assume that we have T_2 and T_3 such that $T_i^2 = -\mu_i$. If F denotes the fixed field of K under σ, then $K = F(\vartheta)$, $\vartheta^2 \in F$, $\sigma(\vartheta) = -\vartheta$. Replacing T_1, T_2, T_3 by T_1, ϑT_2, $\vartheta^{-1}T_3$ we get a related triple satisfying (2.5), (2.6) and (2.12). (2.5) and (2.12) imply that $\sigma(\mu_i) = \mu_i$, for

$$\mu_i^2N(x) = N(\mu_ix) = N(T_i^2x) = \mu_i\sigma(\mu_i)N(x).$$

T_2 and T_3 are uniquely determined by T_1 up to multiplicative factors ρ, ρ^{-1} respectively. Now $N(\rho T_2x) = \rho^2\mu_2\ N(x)$ and $(\rho T_2)^2 = \rho\sigma(\rho)\mu_2$, hence we must have $\rho = \sigma(\rho)$ in order to satisfy (2.12). Let us now turn to the case $\sigma = 1$. First assume $\mu_1 = \beta_1^2$, $\beta_1 \in K$. Then $U_1 = \beta_1^{-1}T_1$ is a rotation, $U_1^2 = 1$. So U_1 is a product of an even number of reflections in C. If $T_1 = \beta_1$, take $T_2 = \beta_1^{-1}\nu$, $T_3 = 1$. If $T_1 = \beta_1S_aS_b$, S_a and S_b reflections in $a^\perp$ and $b^\perp$ respectively, $(a, b) = 0$, then it follows from [3, p. 159—160] that

$$T_2 : x \to \beta_2N(a)N(b)^{-1}a^{-1}(bx).$$

An easy computation shows that

$$\mu_2 = \beta_2^2N(a)N(b)^{-1}.$$

Making use of the relations (1.2) and (1.4) of [2, p. 408] and (10) of [13, p. 419] — also to be found in [4, p. 210], formula (1.5′) — and taking into account that $(a, b) = 0$, we find

$$\begin{aligned}T_2^2x &= \beta_2^2N(a)^2N(b)^{-2}a^{-1}[b\{a^{-1}(bx)\}]\\ &= \beta_2^2N(a)^2N(b)^{-2}(a^{-1}ba^{-1})(bx)\\ &= -\beta_2^2N(a)^2N(b)^{-2}(\bar{b}\bar{a}^{-1}a^{-1})(bx)\\ &= -\beta_2^2N(a)N(b)^{-1}x.\end{aligned}$$

Hence

$$T_2^2 = -\mu_2.$$

Therefore $\beta_1^{-1}T_1$ cannot be a (2,6)-involution.

From the above computations, however, it follows that for a (4,4)-involution $\beta_1^{-1}T_1$ we get the desired relation $T_2^2 = \mu_2$; for a (6,2)-involution $\beta_1^{-1}T_1$ we obtain again that $T_2^2 = -\mu_2$.

Finally, consider the case $\sigma = 1$, $\mu_1 \notin K^2$. Extend K to $K(\eta)$, $\eta^2 = \mu_1$. The linear extension of T_1 to $C \otimes_K K(\eta) = C'$ satisfies $T_1^2 = \eta^2$. If τ is the K-automorphism of $K(\eta)$ mapping η on $-\eta$, then on C'

$$T_1(1 \otimes \tau) = (1 \otimes \tau)T_1,$$

hence $T_1x = \eta x$ implies $T_1(1 \otimes \tau)x = -\eta(1 \otimes \tau)x$. From this relation one easily concludes that $\eta^{-1}T_1$ is a (4,4)-involution on C'. Hence T_1 can be extended to a related triple T_1, T_2', T_3' on C' such that the relations $N(T_i'x) = \mu_i'N(x)$, $T_i^2 = \mu_i'$, hold. On C we can also find a related triple of similarities T_1, T_2, T_3, such that $T_2^2 = \pm\mu_2$. The linear extension of T_2 to C' must be a multiple of T_2'. $T_2^2 = -\mu_2$ would therefore imply $T_2'^2 = -\mu_2'$, a contradiction. Hence $T_2^2 = \mu_2$ and $T_3^2 = \mu_3$. Thus we have shown

(2.13). *Let $\sigma^2 = 1$, $\nu \in K$, $\nu = \sigma(\nu)$. If T_1 is a σ-similarity of C with $N(T_1x) = \mu_1N(x)$, $T_1^2 = \mu_1$ (hence $\sigma(\mu_1) = \mu_1$), then there exist σ-similarities T_2, T_3 such that T_1, T_2, T_3 form a related triple, $N(T_ix) = \mu_iN(x)$, $T_i^2 = \mu_i$, if and only if one of the following conditions holds.*

(1) $\sigma = 1$, $T = \alpha_1$.

(2) $\sigma = 1$, $\mu_1 \in K^2$, $\sqrt{\mu_1^{-1}}\,T_1$ *is a (4,4)-involution.*

(3) $\sigma = 1$, $\mu_1 \notin K^2$.

(4) $\sigma \neq 1$, T_1 *is a proper σ-similarity.*

Then $\sigma(\mu_i) = \mu_i$ for $i = 1, 2, 3$. T_2, T_3 are unique up to multiplicative factors ρ, ρ^{-1}, respectively, with $\rho = \sigma(\rho)$.

REMARK. The above proposition also holds in case C is a split octave algebra, since in its proof it is of no importance whether C is split or not.

In the case $\sigma \neq 1$ we want to derive conditions for a σ-similarity to be proper. Let $e_1, \ldots, e_8$ be an orthogonal basis of C. We assume that the discriminant

$$det\ (\tfrac{1}{2}(e_i, e_j)) = N(e_1) \ldots N(e_8) = \beta$$

satisfies $\beta = \sigma(\beta)$, i.e., $\beta \in F$, the field of σ-invariant elements of K. $K = F(\vartheta)$, $\vartheta^2 = \alpha \in F$, $\sigma(\vartheta) = -\vartheta$.

$$det\,(\tfrac{1}{2}(Te_i, Te_j)) = \mu^8\sigma(\beta) = \mu^8\beta, \text{ if } N(Tx) = \mu\sigma(N(x)).$$

On the other hand, if

$$det\,T = \lambda \text{ with respect to } e_1, \ldots, e_8,$$

then

$$det\,(\tfrac{1}{2}(Te_i, Te_j)) = \lambda^2\beta.$$

Hence

$$\lambda = \pm\mu^4.$$

Let $f_1, \ldots, f_8$ be another orthogonal basis of C. $f_i = \sum_j \gamma_{ij}e_j$, $det\,(\gamma_{ij}) = \delta$. Then $N(f_1)\ldots N(f_8) = \delta^2\beta = \gamma$. If we require again $\sigma(\delta^2\beta) = \delta^2\beta$, then we must have $\delta^2 \in F$.

With respect to $f_1, \ldots, f_8$ we put

$$det\,T = \lambda',$$

then

$$\lambda' = \sigma(\delta)\delta^{-1}\lambda.$$

Since $\delta^2 \in F$, we have either $\delta \in F$ or $\delta \in \vartheta F$, hence accordingly $\sigma(\delta)\delta^{-1} = 1$ or -1. Now $\beta \in K^2 \cap F = F^2 \cup \alpha F^2$. For $\beta = 1$ we define T to be proper if $\lambda = \mu^4$. Then it follows from the above considerations:

(2.14) *Let $K = F(\vartheta)$, $\vartheta^2 = \alpha \in F$, σ the nontrivial F-automorphism of K. Furthermore assume that C is an octave algebra over K. Let $e_1, \ldots, e_8$ be an orthogonal basis of C with $N(e_1)\ldots N(e_8) = \beta \in F$. Let T be a σ-similarity of C such that $det\,T = \lambda$ with respect to $e_1, \ldots, e_8$, $N(Tx) = \mu\sigma(N(x))$ for all $x \in C$. Then T is proper if and only if either $\beta \in F^2$, $\lambda = \mu^4$ or $\beta \in \alpha F^2$, $\lambda = -\mu^4$.*

REMARK. The proof of (2.14) can also be formulated in the Clifford algebra of the quadratic form N; cf. [30]. It actually works in any space of even dimension.

An important special case is the following one. Let T be a σ-similarity with ratio μ, $T^2 = \mu$, $\mu \in F^2$. Then C decomposes in two F-spaces L_+, L_-:

$$x \in L_+(L_-) \text{ if and only if } Tx = \sqrt{\mu}x \ (-\sqrt{\mu}x),$$

where $\sqrt{\mu}$ is either one of the roots of μ in F; thus our notation depends on this arbitrary choice. Obviously, $C = L_+ \oplus L_-$ as an F-space, $\vartheta L_+ = L_-$, $\vartheta L_- = L_+$, hence F-dim $L_+ = F$-dim $L_- = 8$. For $x \in L_+$

$$\mu N(x) = N(\sqrt{\mu}x) = N(Tx) = \mu\sigma(N(x)).$$

Similarly for $x \in L_-$, hence

$$\sigma(N(x)) = N(x) \text{ for } x \in L_+, L_-.$$

So for any orthogonal F-basis $e_1, \ldots, e_8$ of $L_+(L_-)$, $N(e_1)\ldots N(e_8) \in F$ $Te_i = \sqrt{\mu}e_i$ $(-\sqrt{\mu}e_i)$, hence $\det T = \mu^4$. So

$$T \text{ proper} \Leftrightarrow N(e_1)\ldots N(e_8) \in F^2,$$
$$T \text{ improper} \Leftrightarrow N(e_1)\ldots N(e_8) \in \alpha F^2.$$

3. Let π be a polarity. If there exist isotropic points with respect to π, then we call π *hyperbolic*, otherwise *elliptic*. A line is called *isotropic* if it is incident with its pole, i.e., if it is the polar of an isotropic point. Let u be isotropic. If $v \in \pi u$, $\neq u$, then $u \in \pi v \neq \pi u$, hence $v \notin \pi v$. Therefore

(3.1) *On an isotropic line there is exactly one isotropic point, viz. its pole. Through an isotropic point there is exactly one isotropic line, viz. its polar.*

(3.2) Definition. *If π is a hyperbolic polarity, the set $\mathscr{C}(\pi)$ of all isotropic points is called the conic defined by π. For $u \in \mathscr{C}(\pi)$, πu is called the tangent to $\mathscr{C}(\pi)$ at u. $v \in \mathscr{P}(C)$ is called an outer point of $\mathscr{C}(\pi)$, if $\pi(v) \cap \mathscr{C}(\pi)$ consists of at least two points, and an inner point, if $\pi(v) \cap \mathscr{C}(\pi) = \emptyset$.*

If there is no danger of confusion, we shall often write $\mathscr{C}$ instead of $\mathscr{C}(\pi)$, to be well distinguished from C, which denotes a composition algebra, in particular an octave algebra.

On $\mathscr{C}(\pi)$ we choose two distinct points v and w, on πw a point $u_1 \neq w$. Take $u_2 = \pi v \cap \pi u_1$, $u_3 = \pi u_1 \cap \pi u_2$. After a suitable coordinate transformation (in the middle projective group) we may assume that $u_1 = (1, 0, 0; 0, 0, 0)$, $u_2 = (0, 1, 0; 0, 0, 0)$, $u_3 = (0, 0, 1; 0, 0, 0)$, $v = (1, 0, 1; 0, 1, 0)$, $w = (0, 1, 1; 1, 0, 0)$, where we have coordinatized $\mathscr{P}(C)$ with an algebra $A(C; 1, 1, 1)$. u_1, u_2, u_3 form a polar triangle, hence $\pi = \pi_0{}^\ulcorner T^\urcorner$ with a σ-linear transformation

$$T = [\lambda_1, \lambda_2, \lambda_3; T_1, T_2, T_3].$$

We may take $\lambda_1 = 1$.

$$\pi v = (u_2 \times v)^*, \quad \pi w = (w \times u_1)^*,$$

hence

$$T(1, 0, 1; 0, 1, 0) = \alpha(1, 0, 1; 0, -1, 0),$$
$$T(0, 1, 1; 1, 0, 0) = \beta(0, 1, 1; -1, 0, 0).$$

From these relations and from $\lambda_1 = 1$ it follows that $\lambda_2 = \lambda_3 = 1$, $T_2 1 = -1$, $T_1 1 = -1$.

Since $T_1^2 = \mu_1$, $T_2^2 = \mu_2$, we have $\mu_1 = \mu_2 = 1$. $\nu = \lambda_1\lambda_2\lambda_3 = 1$. Hence $\mu_3 = 1$. $T_1 1 = -1$ implies $\hat{T}_1 = T_1$. Hence by (2.8)

$$T_1(xy) = T_2 x \cdot T_3 y.$$

Putting $x = 1$ respectively $y = 1$ in this formula we conclude that $T_1 = T_2 = -T_3$. Hence there is a σ-automorphism U of C such that $T_3 = U$, $T_1 = T_2 = -U$. From (2.13), (2.14) and the discussion after (2.14) it follows that we have

(3.3) *If π is a hyperbolic polarity, then $\mathscr{P}(C)$ can be coordinatized with an algebra $A(C; 1, 1, 1)$ in such a way that $\pi = \pi_0\ulcorner T\urcorner$, where $T = [1, 1, 1; -U, -U, U]$, U a σ-automorphism of C satisfying any of the following three conditions.*

(1) $\sigma = 1$, $U = 1$.

(2) $\sigma = 1$, *U is a reflection in a quaternion subfield of C.*

(3) $\sigma \neq 1$, $\sigma^2 = 1$, *F = fixed field of K under σ, $C = C_F \otimes_F K$ where C_F is an octave field over F, $U = 1 \otimes \sigma$.*

4. In this section we consider the group $PU(\pi)$ of projectivities of $\mathscr{P}(C)$ which commute with the polarity π. Let $\pi = \pi_0\ulcorner T\urcorner$ for some standard polarity π_0. Then we have the following theorem.

(4.1) (i) *$\ulcorner S\urcorner \in PU(\pi)$ if and only if there exists a λ such that $S'TS = \lambda T$. Such a λ satisfies $\lambda = \sigma(\lambda)$.*

(ii) *Every element of $PU(\pi)$ can be written as $\ulcorner S\urcorner$ where $S'TS = T$. For such an S, $\det S \cdot \sigma(\det S) = 1$.*

(iii) *If $\sigma = 1$,* i.e., *if π is a linear polarity, then every element of $PU(\pi)$ can be written as $\ulcorner S\urcorner$ with $S'TS = T$, $\det S = 1$; such an S is uniquely determined by $\ulcorner S\urcorner$.*

Proof. (i) The action of S on the lines is given by

$$v^* \to (S'^{-1}v)^*,$$

hence $S \in PU(\pi)$ if and only if $TS = \lambda S'^{-1}T$ for some λ, so $S'TS = \lambda T$. Taking the adjoint of both sides of this equation, one gets $S'TS = \sigma(\lambda)T$ since $T = T'$. Hence $\lambda = \sigma(\lambda)$.

(ii) From $S'TS = \lambda T$ it follows that $\det S \cdot \sigma(\det S) = \lambda^3$, since $\det S' = \det S$. [21, p. 455, prop. 3]. Hence $\lambda = \rho \cdot \sigma(\rho)$ with $\rho = \lambda^{-1} \det S$. So $(\rho^{-1}S)'T(\rho^{-1}S) = T$ and

$$\det(\rho^{-1}S)\sigma\big(\det(\rho^{-1}S)\big) = 1.$$

(iii) Let $S \in PU(\pi)$ with $S'TS = T$. Then $(det\, S)^2 = 1$ by (ii). If $det\, S = -1$, take $-S$ instead of S'. If $S'TS = T$, $det\, S = 1$, and $(\lambda S)'T(\lambda S) = T$, $det\,(\lambda S) = 1$, then $\lambda^2 = 1$ and $\lambda^3 = 1$, hence $\lambda = 1$.

II. The unitary groups of hyperbolic polarities

5. Let π be a hyperbolic polarity, $\pi = \pi_0 \ulcorner T \urcorner$, in the plane $\mathscr{P}(C)$. We may suppose that $\mathscr{P}(C)$ is coordinatized by an algebra $A(C; 1, 1, 1)$ in such a way that $T = [1, 1, 1; -U, -U, U]$, where U is a σ-automorphism of C of one of the types enumerated in (3.3).

We now want to determine the central collineations in $PU(\pi)$. First consider a dilatation $\ulcorner S \urcorner$. By [21, prop. 6] we may assume that $S = d_{a,b;\lambda}$. We first point out two corrections of this prop. 6 of [21]. In (b), read $\tilde{d}_{a,b;\lambda} = d_{b,a;\lambda^{-1}}$ instead of $d_{a,b;\lambda} = d_{b,a;\lambda^{-1}}$. (*d*) has to be read as follows: If $u \in \Gamma$ is σ-linear, then $d_{u(a),\tilde{u}(a);\sigma(\lambda)} = u d_{a,b;\lambda} u^{-1}$.

Now $\ulcorner S \urcorner \in PU(\pi)$ if and only if $(Sx, TSy) = \alpha(x, Ty)$, for a fixed α. If $S = d_{a,b;\lambda}$, this means

$$T d_{a,b;\lambda} T^{-1} = \alpha \tilde{d}_{a,b;\lambda},$$

hence

$$d_{Ta,\tilde{T}b;\sigma(\lambda)} = \alpha d_{b,a;\lambda^{-1}}.$$

Hence $Ta = \beta b$ and $\lambda^{-1} = \sigma(\lambda)$. Since $(a, b) \neq 0$, we find $a \notin \pi a$. Thus we have shown

(5.1) *A dilatation $\ulcorner S \urcorner$ belongs to $PU(\pi)$ if and only if one can write $S = d_{a,Ta;\lambda}$ with $a \notin \pi a$, $\lambda\sigma(\lambda) = 1$.*

For $\lambda = -1$ we get *unitary reflections* in this way. If $\sigma = 1$, the unitary reflections are the only unitary dilatations different from the identy.

The existence of unitary reflections has the following consequence. Let a be an isotropic point, b^* a line through a, $\neq \pi a$. Let c be any nonisotropic point on b^*. πc does not pass through a, hence a is not fixed under the unitary reflection with center c and axis πc. Therefore, b^* contains more than one isotropic point. From this it follows:

(5.2) *If a is an isotropic point with respect to π, any line $\neq \pi a$ through a contains an isotropic point $\neq a$, hence any point $\neq a$ on πa is an outer point.*

Finally, we want to determine the transvections in $PU(\pi)$. Let $\ulcorner S \urcorner$ be such a transvection with center c. Since all lines through c are invariant under $\ulcorner S \urcorner$, their poles are fixed under $\ulcorner S \urcorner$. Hence πc must be the axis of $\ulcorner S \urcorner$. If a is any isotropic point not on πc, its image b is also isotropic. Therefore, by [21, prop. 5], $S = t_{a,b;Tc}$. We assert that b may be any isotropic point $\neq c$ on the line $a \times c$.

(5.3) *Let a, b, c be any three collinear isotropic points, $c \neq a$, $\neq b$. Then there exists a unique unitary transvection with center c and axis πc which maps a on b. Any transvection in $PU(\pi)$ is of this form.*

Proof. Only the existence still needs a proof. The action of $t_{a,b;Tc}$ on lines is given by $\tilde{t}_{a,b;Tc}$. Geometrically it is clear that $\tilde{t}_{a,b;Tc}$ is a transvection whose "axis" is c. If p is any point on πc, then $p \times a$ is mapped on $p \times b$. Therefore

$$\tilde{t}_{a,b;Tc} = \alpha t_{p\times a,\, p\times b;\, c}.$$

$t_{a,b;Tc}$ is unitary if and only if

$$T t_{a,b;Tc} T^{-1} = \beta \tilde{t}_{a,b;Tc}.$$

By [21, prop. 5], this is equivalent to

$$(*) \qquad t_{Ta,Tb;c} = \alpha\beta t_{p\times a,\, p\times b;\, c}$$

a and b are isotropic, hence lie on πa and πb. $\pi a \cap \pi b$ is a point of πc; take $p = \pi a \cap \pi b$, then $p \times a = \gamma Ta$, $p \times b = \delta Tb$, hence the above equality (*) holds. This implies that $t_{a,b;c} \in PU(\pi)$.

We shall call the linear transformations $t_{a,b;c}$ as in [21, prop. 5], *normalized transvections*. Every projective transvection is induced by a normalized transvection.

6. In this section we study the unitary transformations leaving an outer point fixed.

Let u_1 be an outer point. We coordinatize as in (3.3) such that $u_1 = (1,0,0;0,0,0)$, $\pi u_1 = u_1^*$. $\pi = \pi_0 \ulcorner T \urcorner$, $T = [1,1,1;-U,-U,U]$. In the Peirce decomposition with respect to u_1 we consider $R = \langle e-u \rangle + E_0$.

x is an isotropic point on u_1^* if and only if $x \in R$, $x \times x = 0$, $(x, Tx) = 0$. Any $x \in R$ can be written as $x = (0, \xi_2, \xi_3, x_1, 0, 0)$. $x \times x = 0$ amounts to

$$\xi_2 \xi_3 - N(x_1) = 0,$$

i.e., to

$$Q_1(x) = 0 \qquad \text{(see [21, p. 452])}.$$

Furthermore,

$$(x,\ Tx) = \xi_2\sigma(\xi_2)+\xi_3\sigma(\xi_3)-(x_1,\ Ux_1).$$

We now distinguish the cases $\sigma = 1$ and $\sigma \neq 1$.

First assume $\sigma = 1$. On R we define quadratic forms Q^+ and Q^- by

$$\begin{aligned} Q^+(x) &= \tfrac{1}{2}\{Q_1(x)+\tfrac{1}{2}(x,\ Tx)\}, \\ Q^-(x) &= \tfrac{1}{2}\{-Q_1(x)+\tfrac{1}{2}(x,\ Tx\}). \end{aligned} \tag{6.1}$$

Then

$$\begin{aligned} Q_1 &= Q^+-Q^-, \\ \tfrac{1}{2}(x,\ Tx) &= Q^+(x)+Q^-(x). \end{aligned} \tag{6.2}$$

We shall also need the bilinear forms related to Q^+ and Q^-;

$$\begin{aligned} Q^+(x,\ y) &= Q^+(x+y)-Q^+(x)-Q^+(y) \\ Q^-(x,\ y) &= Q^-(x+y)-Q^-(x)-Q^-(y). \end{aligned} \tag{6.3}$$

In the subspace $x_1 = 0$ of R we change coordinates:

$$\begin{aligned} \zeta_2 &= \tfrac{1}{2}(\xi_2+\xi_3), \quad \xi_2 = \zeta_2+\zeta_3, \\ \zeta_3 &= \tfrac{1}{2}(\xi_2-\xi_3), \quad \xi_3 = \zeta_2-\zeta_3. \end{aligned} \tag{6.4}$$

In these coordinates we find

$$\begin{aligned} Q_1(x) &= \zeta_2^2-\zeta_3^2-N(x_1), \\ \tfrac{1}{2}(x,\ Tx) &= \zeta_2^2+\zeta_3^2-\tfrac{1}{2}(x_1,\ Ux_1). \end{aligned} \tag{6.5}$$

Let R^+ be the radical of Q^-, R^- that of Q^+. Then

$$R = R^+ \oplus R^-.$$

For $x \in R$, write

$$x = x^+ + x^-, \quad x^+ \in R^+, \quad x^- \in R^-.$$

From (6.2) it follows that for $x \in R$,

$$x \times x = 0 \Leftrightarrow Q^+(x^+) = Q^-(x^-). \tag{6.6}$$

(6.7) x is an isotropic point $\Leftrightarrow Q^+(x^+) = Q^-(x^-) = 0$.

For the two standard forms for U corresponding to $\sigma = 1$ in (3.3) one easily finds Q^+, Q^-, R^+ and R^-:

(6.8) *If* $U = 1$, *then*

$$\begin{aligned} &Q^+(x) = \zeta_2^2-N(x_1), \quad Q^-(x) = \zeta_3^2. \\ &\mathit{dim}\ R^+ = 9, \quad \nu(Q^+|R^+) = 1. \\ &\mathit{dim}\ R^- = 1, \quad \nu(Q^-|R^-) = 0. \end{aligned}$$

If U is the reflection in a quaternion subfield D of C, we write $x \in C$ as $x = x'+x''$ with $x' \in D$, $x'' \perp D$, so $Ux = x'-x''$. Then

$$Q^+(x) = \zeta_2^2 - N(x_1'), \quad Q^-(x) = \zeta_3^2 + N(x_1'').$$
$$dim\ R^+ = 5, \quad \nu(Q^+|R^+) = 1.$$
$$dim\ R^- = 5, \quad \nu(Q^-|R^-) = 0.$$

Here ν denotes the index of a quadratic form. Note that in either of the above cases, *Q^+ is not equivalent with Q^-.*

We can now determine the group of the transformations $\ulcorner S \urcorner \in PU(\pi)$ with $\ulcorner S \urcorner u_1 = u_1$. By (4.1) we may assume that $S'TS = T$, $det\ S = 1$.

(6.9). *$\sigma = 1$. Let u_1 be an outer point.*

(i) *If $\ulcorner S \urcorner \in PU(\pi)$, $S'TS = T$, $det\ S = 1$, $Su_1 = \lambda u_1$, then $Su_1 = u_1$, S leaves R^+ and R^- invariant.*

(ii) *Let $S^+ : R^+ \to R^+$, $S^- : R^- \to R^-$ be linear transformations, $S^+ \oplus S^- : R \to R$ the transformation with restriction $S^+(S^-)$ to $R^+(R^-)$. $S^+ \oplus S^-$ can be extended to a linear transformation S of A leaving det invariant and such that $S'TS = T$, $Su_1 = u_1$, if and only if*

$$S^+ \in O^+\ (Q^+|R^+), \quad S^- \in O^+\ (Q^-|R^-), \quad sp\ (S^+) = sp\ (S^-),$$

sp denoting the spinornorm of a rotation. For given S^+ and S^-, the extension S is unique up to a unitary reflection with u_1 as center and πu_1 as axis.

If G denotes the group of all extendable $S^+ \oplus S^-$, then the following sequence is exact

$$1 \to \Omega\ (Q^+|R^+) \xrightarrow{i} G \xrightarrow{j} O^+(Q^-|R^-) \to 1,$$

where Ω denotes the commutator subgroup of the orthogonal group, i the mapping $S^+ \to S^+ \oplus 1^-$ (1^- = identity on R^-), j the mapping $S^+ \oplus S^- \to S^-$.

Proof. (i) Since

$$(Su_1, TSu_1) = (u_1, Tu_1),$$

we find $\lambda^2 = 1$. Let S_R denote the restriction of S to R. From $det\ S = 1$, and $Su_1 = \lambda u_1$ it follows that

$$Q_1(S_R x) = \lambda^{-1} Q_1(x).$$

Furthermore,

$$\tfrac{1}{2}(S_R x, TS_R x) = \tfrac{1}{2}(x, Tx).$$

For $\lambda = 1$ it follows that S_R leaves Q^+ and Q^- invariant, hence $S_R R^+ = R^+$, $S_R R^- = R^-$.

For $\lambda = -1$, we find that

$$Q^+(S_R x) = Q^-(x), \quad Q^-(S_R x) = Q^+(x).$$

Since Q^+ is not equivalent with Q^-, this is impossible.

(ii) Put $S_R = S^+ \oplus S^-$. If S_R can be extended to an S as desired, S_R has to be a rotation with respect to Q_1; see [19, prop. 3] or [21, p. 456—7]. Hence S^+ and S^- are either both a rotation with respect to Q^+ and Q^- respectively, or both a nonrotation. First assume S^+ and S^- are rotations. Then $S^+ \oplus 1^-$ is an even product of rotations S'_a defined by

$$S'_a = S_a Z,$$

where S_a is the orthogonal (with respect to Q_1 in R) reflection in the hyperplane $a^\perp$, a nonisotropic in R^+, and

$$Z = -S_{e-u_1} : (\zeta_2, \zeta_3, x_1) \rightarrow (\zeta_2, -\zeta_3, -x_1)$$

(see [21, p. 453]). Note that S'_a is a rotation in R, not in R^+ and R^-.

Let us first consider the extension of S'_a to a linear transformation leaving *det* invariant; cf. [21]. If

$$a = (0, \alpha_2, \alpha_3; a_1, 0, 0),$$

the extension S of S'_a to E_1 must be of the form

$$y = (0, 0, 0; 0, y_2, y_3) \rightarrow \tfrac{1}{2}\gamma(0, 0, 0; 0, \alpha_3 y_2 + \bar{a}_1 \bar{y}_3, \alpha_2 y_3 + \bar{y}_2 \bar{a}_1),$$

where $\gamma \in K$, $\gamma \neq 0$ can be arbitrarily chosen.

If $a \in R^+$, then $\alpha_2 = \alpha_3$, $Ua_1 = a_1$. An easy computation yields

$$\begin{aligned}(Sy, TSy) &= \tfrac{1}{4}\gamma^2(\alpha_2^2 - N(a_1))(y, Ty)\\ &= \tfrac{1}{4}\gamma^2 Q^+(a)(y, Ty).\end{aligned}$$

In a similar way, $1 \oplus S^-$ is an even product of rotations S'_a, $a \in R^-$. $a \in R^-$ means $\alpha_2 = -\alpha_3$, $Ua_1 = -a_1$. Hence the extension S of S'_a to E_1 satisfies

$$(Sy, TSy) = \tfrac{1}{4}\gamma^2 Q^-(a)(y, Ty).$$

From these it is clear that $S^+ \oplus S^-$ can be extended to a linear transformation S leaving *det* and $(\cdot, T\,\cdot)$ invariant such that $Su_1 = u_1$ if and only if

$$sp\,(S^+)\, sp\,(S^-) = 1 \pmod{K^{*2}},$$

or

$$sp\,(S^+) = sp\,(S^-).$$

$Q^+|R^+$ has index 1, so the spinornorm on $O^+(Q^+|R^+)$ can assume any value in K^*/K^{*2}. Hence S^- can be any element of $O^+(Q^-|R^-)$; once S^- is chosen, S^+ is unique up to an element of $\Omega(Q^+|R^+)$, since this coincides with the subgroup of elements of $O^+(Q^+|R^+)$ with spinornorm 1.

Next let S^+ and S^- be nonrotations with respect to Q^+ and Q^- respectively such that $S^+ \oplus S^-$ can be extended to an S in the required way. Since for any $S_0^- \in O^+(Q^-|R^-)$ there exists an $S_0^+ \in O^+(Q^+|R^+)$ such that $S_0^+ \oplus S_0^-$ can be extended, we may after multiplication of S^- by a suitable element of $O^+(Q^-|R^-)$ assume that $S^- = -1^-$. This will lead to a contradiction. We distinguish two cases. If $U = 1$, we reason as follows.

$$(0, 1, -1; 0, 0, 0) \in R^-,$$

hence

$$(0, 1, -1; 0, 0, 0) \to (0, -1, 1; 0, 0, 0).$$

Then for $x_2, x_3 \in C$,

$$(0, 1, -1; 0, x_2, x_3) \to (0, -1, 1; 0, y_2\, y_3),$$

with $y_2, y_3 \in C$, The invariance of the cubic form *det* implies

$$N(x_2)-N(x_3) = -N(y_2)+N(y_3).$$

The invariance of $(.\,, T\,.)$ gives

$$-N(x_2)+N(x_3) = -N(y_2)+N(y_3).$$

Combining we get

$$N(x_2)-N(x_3) = -N(x_2)+N(x_3),$$

which obviously cannot hold for all $x_2, x_3 \in C$.

In case U is a reflection in a quaternion subfield D of C, we reason as follows. Choose $x_1 \in C$, $x_1 \perp D$, $N(x_1) \neq 0$.

$(0, 0, 0; x_1, 0, 0) \in R^-$, hence

$$(1, 0, 0; x_1, 0, 0) \to (1, 0, 0; -x_1, 0, 0).$$

The invariance of *det* then yields

$$N(x_1) = -N(x_1),$$

which contradicts the assumption $N(x_1) \neq 0$.

Thus we have shown that it is impossible for S^+ and S^- to be nonrotations.

To complete the proof of the theorem, take $S^+ = 1^+$, $S^- = 1^-$. The only unitary transformations leaving u_1 and all points of πu_1 fixed are the identity and the unitary reflection with center u_1 and axis πu_1.

Now we turn to the case $\sigma \neq 1$. Let $K = F(\vartheta)$, $\sigma|F = 1$, $\sigma(\vartheta) = -\vartheta$, $\vartheta^2 = \alpha \in F$. By (3.3) we may assume

$$C = C_F \otimes_F K, \quad U = 1 \otimes \sigma,$$

C_F being an octave field over F.

In R we introduce again the coordinates ζ_2, ζ_3, x_1, where $\zeta_{2,3}$ are as defined in (6.4). We use the following notation.

$$(6.10) \qquad \begin{aligned} \zeta_i &= \zeta_i' + \vartheta\zeta_i'' \text{ with } \zeta_i', \zeta_i'' \in F, \qquad i = 2, 3. \\ x_1 &= x_1' + \vartheta x_1'' \text{ with } x_1', x_1'' \in C_F. \end{aligned}$$

We split R into a direct sum of two F-linear spaces R^+ and R^-, where

$$(6.11) \qquad \begin{aligned} R^+ &= \{x = (\zeta_2, \zeta_3, x_1) | \zeta_2'' = \zeta_3' = 0, x_1'' = 0\} \\ R^- &= \{x = (\zeta_2, \zeta_3, x_1) | \zeta_2' = \zeta_3'' = 0, x_1' = 0\}. \end{aligned}$$

Furthermore, two F-quadratic forms Q^+ and Q^- on R and an F-bilinear form $\langle .,. \rangle$ between R^+ and R^- are defined by

$$(6.12) \qquad \begin{aligned} Q^+(x) &= \zeta_2'^2 - \alpha\zeta_3''^2 - N(x_1'), \qquad x \in R, \\ Q^-(x) &= \zeta_3'^2 - \alpha\zeta_2''^2 + \alpha N(x_1''), \qquad x \in R, \\ \langle x^+, x^- \rangle &= \zeta_2'\zeta_2'' - \zeta_3'\zeta_3'' - \tfrac{1}{2}(x_1', x_1''), \\ x^+ &= (\zeta_2', \zeta_3'', x') \in R^+, \quad x^- = (\zeta_2'', \zeta_3', x_1'') \in R^-. \end{aligned}$$

R^- is the radical of Q^+, R^+ that of Q^-. Obviously,

$$(6.13) \qquad \begin{aligned} Q_1(x) &= Q^+(x) - Q^-(x) - 2\vartheta\langle x^+, x^- \rangle \quad \text{if} \\ &\quad x = x^+ + x^-, \quad x^+ \in R^+, \quad x^- \in R^-. \\ \tfrac{1}{2}(x, Tx) &= Q^+(x) + Q^-(x). \end{aligned}$$

We now want to consider the linear transformations S of A leaving *det* and $(\cdot, T\,\cdot)$ invariant and such that $Su_1 = u_1$. In this way we do not obtain all $\ulcorner S \urcorner \in PU(\pi)$ with $\ulcorner S \urcorner u_1 = u_1$, i.e. $Su_1 = \lambda u_1$.

(6.14) $\sigma \neq 1$. *Let u_1 be an exterior point.*

(i) *Let $\ulcorner S \urcorner \in PU(\pi)$ with $S'TS = T$, $\det S = 1$, $Su_1 = u_1$. Then S leaves the F-subspace R^+ invariant.*

(ii) *Let S^+ be an F-linear transformation: $R^+ \to R^+$. Then there exists an $\ulcorner S \urcorner \in PU(\pi)$ with $S'TS = T$, $\det S = 1$, $Su_1 = u_1$, such that $S|R^+ = S^+$ if and only if $S^+ \in \Omega(Q^+|R^+)$, Ω denoting the commutator subgroup of the orthogonal group. S is uniquely determined by S^+ up to a unitary reflection with u_1 as center and πu_1 as axis.*

Proof. (i) $\ulcorner S \urcorner$ leaves πu_1 invariant, hence S transforms R into itself. Since S leaves *det* and $(\cdot, T\cdot)$ invariant, its restriction to R leaves Q_1 and $(\cdot, T\cdot)$ invariant, hence also Q^+, Q^- and $\langle\cdot,\cdot\rangle$. Since R^+ and R^- are the radical of Q^- and Q^+ respectively, they are invariant under S.

(ii) Assume that S^+ can be extended to S as required. S leaves R^+ and R^- invariant. Now we notice that

$$\vartheta R^+ = R^-, \quad \vartheta R^- = R^+.$$

S is K-linear, hence its restriction S^- to R^- satisfies

$$\vartheta S^+ = S^- \vartheta.$$

Hence

$$S^- = \vartheta S^+ \vartheta^{-1}.$$

The fact that S leaves Q^+, Q^- and $\langle\cdot,\cdot\rangle$ invariant is, because of the last relation, equivalent to

$$Q^+(S^+x) = Q^+(x), \qquad x \in R^+.$$

Let S^+ be written as a product of Q^+-reflections in R^+:

$$S_R = S_{a_1} \cdot S_{a_2} \cdots S_{a_k}, \qquad a_i \in R^+.$$

$S_R = S|R$ is the K-linear extension of S^+, hence

$$S_R = S_{a_1,R} \cdot S_{a_2,R} \cdots S_{a_k,R},$$

where $S_{a_i,R}$ denotes the K-linear extension of S_{a_i} to R. Now we have the relations

$$(6.15)\quad \begin{aligned} &Q^+(\vartheta x^-) = -\alpha Q^-(x^-), \quad x^- \in R^-.\\ &Q^-(\vartheta x^+) = -\alpha Q^+(x^+), \quad x^+ \in R^+.\\ &Q^+(x^+, \vartheta x^-) = 2\alpha\langle x^+, x^-\rangle, \quad x^+ \in R^+, \quad x^- \in R^-, \end{aligned}$$

where $\quad Q^+(x, y) = Q^+(x+y) - Q^+(x) - Q^+(y).$

The fact that S_{a_i} are Q^+-reflections in R^+ therefore implies that $S_{a_i,R}$ are Q_1-reflections in R. Since S_R has to be a Q_1-rotation, it follows that k is even. But then we can write, as in the case $\sigma = 1$,

$$S_R = S'_{b_1} \cdots S'_{b_k}, \qquad b_i \in R^+,$$

where

$$S'_{b_i} = S_{b_i,R} \cdot Z,$$
$$Z : (\zeta_2, \zeta_3, x_1) \to (\zeta_2, -\zeta_3, -x_1).$$

Now a similar computation as in the proof of (6.9), (ii), shows that

S^+ can be extended in the required way if and only if $sp\,(S^+) = 1 \pmod{F^2}$, i.e., if $S^+ \in \Omega(Q^+|R^+)$ since $Q^+|R^+$ has index > 0.

If $S^+ = 1^+$, $\ulcorner S \urcorner$ leaves u_1 and the points of πu_1 fixed, hence S is a unitary dilatation $d_{u_1, u_1; \lambda}$. Since $u_1 = Su_1 = d_{u_1, u_1; \lambda}\, u_1 = \lambda^2 u_1$, we find $\lambda^2 = 1$. Hence S is the identity or a unitary reflection. This completes the proof of (6.14).

We conclude this section with the determination of the isotropic points on the line πu_1, u_1 an outer point. We keep the same notations as above.

First assume $\sigma = 1$. As seen in (6.7), $x = x^+ + x^-$, $x^+ \in R^+$, $x^- \in R^-$, is an isotropic point if and only if $Q^+(x^+) = Q^-(x^-) = 0$. Since $Q^-|R^-$ has index 0 by (6.8), $x^- = 0$. So

(6.16) $\qquad x = (0, \xi_2, \xi_2; x_1, 0, 0), \quad Ux_1 = x_1, \xi_2^2 = N(x_1).$

If $\sigma \neq 1$, the situation is as follows. $x \in R$ is an isotropic point if and only is $Q_1(x) = \frac{1}{2}(x, Tx) = 0$. Hence

$$x = x^+ + x^-, \quad x^+ \in R^+, \quad x^- \in R^-,$$

with

$$Q^+(x^+) = Q^-(x^-) = \langle x^+, x^- \rangle = 0.$$

By (6.15), the latter condition can be written as

$$Q^+(x^+) = Q^+(\vartheta^{-1}x^-) = Q^+(x^+, \vartheta^{-1}x^-) = 0.$$

Hence, if we can prove that $Q^+|R^+$ has index 1, we can conclude that

$$x^- = \rho\vartheta x^+, \quad \rho \in F, \quad Q^+(x^+) = 0.$$

Thus we find that

(6.17) $\qquad x = \lambda(0, \xi_2, \sigma(\xi_2); x_1, 0, 0),$

with

$$x_1 \in C_F, \; \xi_2\sigma(\xi_2) = N(x_1).$$

Remains to show that $Q^+|R^+$ has index 1. Let $\nu = \nu(Q^+|R^+)$. $\nu \geqq 1$ is obvious: see (6.12). Assume $\nu > 1$. Thus the F-quadratic form

$$\eta_1^2 - \alpha\eta_2^2 - \eta_3^2 - N(x_0), \; x_0 \in C_F, \; (1, x_0) = 0,$$

has index > 1. Hence there must exist an isotropic point orthogonal to the hyperbolic plane given by the equations $\eta_2 = 0$, $x_0 = 0$, hence the form

$$\alpha\eta_2^2 + N(x_0)$$

is isotropic. So there exists an $x_0 \in C_F$, $(1, x_0) = 0$, with $\alpha + N(x_0) = 0$. Hence

$$\vartheta^2+N(x_0) = N(\vartheta+x_0) = 0,$$

which contradicts the fact that N is anisotropic on $C = C_F \otimes_F K$. Therefore $\nu = 1$.

7. Let u_1 be an outer point. We need some information about unitary transvections with center on πu_1.

We choose coordinates in the usual way such that

$$u_1 = (1, 0, 0; 0, 0, 0)$$
$$a = (0, 1, 1; -1, 0, 0)$$
$$c = (0, 1, 1; 1, 0, 0)$$

and such that $\pi = \pi_0 \ulcorner T \urcorner$ with

$$T = [1, 1, 1; -U, -U, U]$$

(cf. proof of (3.3)). Then by (6.16) and (6.17)

$$b = (0, \alpha_2, \sigma(\alpha_2); a_1, 0, 0),\ Ua_1 = a_1 \ (\text{so for } \sigma \neq 1 : a_1 \in C_F).$$

Furthermore,

$$Tc = (0, 1, 1; -1, 0, 0).$$

We now apply the results on transvections of [21, p. 458—9].

The transvection S whose axis is $\pi c = (Tc)^*$ and which maps a on b is given by

$$S = 1+\alpha F+\tfrac{1}{2}\alpha^2 F^2, \quad F^3 = 0,$$

where $\alpha = (a, c)^{-1}(b, c)^{-1}$,

$$Fx = 2(Tc)\times(v\times x)-\tfrac{1}{2}(x, c)v,$$
$$v = (b, c)a-(a, c)b.$$

A straigtforward computation yields

$$(7.1) \qquad F(\xi_1, \xi_2, \xi_3; x_1, x_2, x_3) = (\eta_1, \eta_2, \eta_3; y_1, y_2, y_3)$$

with

$$\eta_1 = 0$$
$$\eta_2 = 2(\alpha_2-\sigma(\alpha_2))\xi_2+(-\alpha_2+\sigma(\alpha_2)-(1, a_1))(1, x_1)+2(a_1, x_1)$$
$$\eta_3 = 2(-\alpha_2+\sigma(\alpha_2))\xi_3+(\alpha_2-\sigma(\alpha_2)-(1, a_1))(1, x_1)+2(a_1, x_1)$$
$$y_1 = (\alpha_2-\sigma(\alpha_2)-(1, a_1)+2a_1)\xi_2+(-\alpha_2+\sigma(\alpha_2)-(1, a_1)+2a_1)\xi_3$$
$$\quad -2(1, x_1)a_1+2(a_1, x_1)$$
$$y_2 = (\alpha_2-\sigma(\alpha_2)-(1, a_1)+2\bar{a}_1)\bar{x}_3+(-\alpha_2+\sigma(\alpha_2)-(1, a_1)+2a_1)x_2$$
$$y_3 = x_3(\alpha_2-\sigma(\alpha_2)-(1, a_1)+2a_1)+\bar{x}_2(-\alpha_2+\sigma(\alpha_2)-(1, a_1)+2\bar{a}_1)$$
$$\quad = \bar{y}_2.$$

A tedious but not difficult computation shows that

$$(Fx, Ty)+(x, TFy) = 0 \text{ for all } x, y.$$

From this one easily derives that S leaves $(\cdot, T\cdot)$ invariant, which yields another proof of (5.3).

We want to draw some other conclusions of (7.1) that will be useful in the sequel. It is not hard to verify that in the case $\sigma = 1$ the infinitesimal transvection F maps R^- on 0. Hence for the restriction S_R of S to R we have: $S^- = 1$. Hence the S_R, S a unitary transvection with center on πu_1, belong to a group which is isomorphic to $\Omega(Q^+|R^+)$ (see (6.9)). Since the latter group is simple, it is generated by the S_R; for the S_R generate a normal subgroup of it. In the case $\sigma \neq 1$ the situation is similar. According to (6.14) the restriction of S_R to R^+ belongs to $\Omega(Q^+|R^+)$. This group being simple, it is generated by the restrictions to R^+ of the unitary transvections with center on πu_1.

8. In this section we consider the group generated by the unitary transvections. It will turn out to be a simple group.

(8.1) Definition. *$T(\pi)$ is the group generated by the normalized unitary transvections. $PT = PT(\pi)$ is the subgroup of $PU(\pi)$ generated by the projective unitary transvections.*

Obviously, PT is a normal subgroup of PU.

$$PT(\pi) = \{\ulcorner S \urcorner | S \in T(\pi)\}.$$

(8.2) *Let $\ulcorner S \urcorner$ be a unitary reflection in an outer point. Then $\ulcorner S \urcorner \in PT$.*

Proof. Let u_1 be the center of $\ulcorner S \urcorner$. Take coordinates as in the previous sections. So the axis of $\ulcorner S \urcorner$ is $\pi u_1 = u_1^*$. Choose the following isotropic points on u_1^*:

$$a = (0, \alpha_2, \sigma(\alpha_2); a_1, 0, 0)$$
$$b = (0, \sigma(\alpha_2), \alpha_2; \bar{a}_1, 0, 0)$$

where $a_1 \neq \bar{a}_1$, $Ua_1 = a_1$, $\alpha_2\sigma(\alpha_2) = N(a_1)$.

$$c = (0, 1, 1; 1, 0, 0)$$
$$d = (0, 1, 1; -1, 0, 0).$$

Let S_1, S_2 and S_3 be unitary transvections:

S_1 has center a and maps c on d
S_2 has center b and maps d on c
S_3 has center c and maps S_2S_1d on d.

Let $x \in \pi c = (Tc)^*$, then we may assume

$$x = (1, N(x_2), N(x_2); N(x_2), x_2, \bar{x}_2).$$
$$S_2 S_1 x = \rho Tc \times (b \times (Td \times (a \times x))).$$

A straightforward computation yields

$$S_2 S_1 x = \lambda(1, N(z_2), N(z_2); N(z_2), z_2, \bar{z}_2)$$

with

$$\begin{aligned} z_2 &= (\alpha_2 + \sigma(\alpha_2) - (1, a_1))^{-1} (\alpha_2 + \sigma(\alpha_2) + (1, a_1))^{-1} \\ &\quad \cdot (\sigma(\alpha_2) - \alpha_2 + a_1 - \bar{a}_1)(\alpha_2 - \sigma(\alpha_2) + a_1 - \bar{a}_1) x_2 \\ &= -((\alpha_2 + \sigma(\alpha_2))^2 - (1, a_1)^2)^{-1} N(\alpha_2 + \sigma(\alpha_2) + a_1 - \bar{a}_1) x_2 \\ &= -x_2. \end{aligned}$$

Hence

$$\ulcorner S_3 S_2 S_1 \urcorner x = \ulcorner S \urcorner x \text{ for } x \in \pi c,$$
$$\ulcorner S_3 S_2 S_1 \urcorner d = d = \ulcorner S \urcorner d.$$

But from (5.1) it follows that there are no unitary dilatations with center d, axis c^*, except the identity. Hence $\ulcorner S \urcorner = \ulcorner S_3 S_2 S_1 \urcorner$.

(8.3) *Let $c \in \mathscr{C}$. Then PT_c, the stabilizer of c in PT, is transitive on the set of points $\neq c$ on πc and transitive on the set of lines $\neq \pi c$ through c.*

Proof. Let $x, y \in \pi c$, both $\neq c$, $x \neq y$. Let u_1 be the fourth harmonic of c with respect to x and y. u_1 is an outer point, so the unitary reflection $\ulcorner S \urcorner$ with center u_1 belongs to PT. $\ulcorner S \urcorner c = c$, $\ulcorner S \urcorner x = y$. The second statement is deduced from the first one by considering the polars of the points on πc.

(8.4) *PT is doubly transitive on $\mathscr{C}$ and transitive on the set of outer points.*

Proof. On any line containing two isotropic points there exists a third isotropic point. Hence (5.3) insures transitivity of PT on $\mathscr{C}$. (8.3) and (5.3) imply that PT_c is transitive on the points of $\mathscr{C} \neq c$. The remainder of the proof is obvious if one keeps in mind that the polar of an outer point contains isotropic points.

Before we prove the simplicity of PT we need one more lemma.

(8.5) *Let $\ulcorner S \urcorner$ be a unitary transvection with center c. Then there exists a unitary transvection $\ulcorner V \urcorner$ with center c such that $\ulcorner V \urcorner^2 = \ulcorner S \urcorner$.*

Proof. Take an isotropic point $a \neq c$. $\ulcorner S \urcorner a = b$. Let $\ulcorner V \urcorner$ be a transvection with center c which maps a on d. Then $\ulcorner V \urcorner^2 = \ulcorner S \urcorner$

if and only if c and d are harmonic with respect to a and b. So it suffices to show: If a, b, c and d are four harmonic points and a, b and c are isotropic, then d is isotropic. Take coordinates as usual such that

$$\begin{aligned} a &= (0, 1, 1; -1, 0, 0) \\ b &= (0, 1, 1; 1, 0, 0) \\ c &= (0, \alpha_2, \sigma(\alpha_2); a_1, 0, 0). \end{aligned}$$

The (not unitary) reflection with center a and axis a^* maps c on d (see proof of prop. 7, p. 95 in [19]). From [21, p. 460, formula (17)] one derives that

$$d = \big(0, \sigma(\alpha_2), \alpha_2; \bar{a}_1, 0, 0\big).$$

Hence d is isotropic.

Now we have the tools to prove the following theorem.

(8.6). *PT is a simple group.*

Proof. We apply lemma 4, p. 39 of [6]. Consider PT as a permutation group of $\mathscr{C}$; it is clear that this is a faithful representation. PT is doubly transitive on $\mathscr{C}$, hence primitive.

For $c \in \mathscr{C}$, let H_c be the group of unitary transvections with center c. H_c is a commutative group; see e.g. [1, theorem 2.8, p. 57] or [19, prop. 5, p. 93]. All groups H_c, $c \in \mathscr{C}$, are conjugate in PT and they generate PT. H_c is a normal subgroup of the stabilizer of c. So there remains to be shown that PT is its own commutator subgroup. It suffices to show that every unitary transvection $\ulcorner S \urcorner$ is in the commutator subgroup PT' of PT.

Let u_1 be an exterior point such that the center of $\ulcorner S \urcorner$ lies on πu_1. By (8.5) there exists a transvection $\ulcorner V \urcorner$ with the same center as $\ulcorner S \urcorner$ such that $\ulcorner S \urcorner = \ulcorner V \urcorner^2$. At the end of section 7 we have seen that V_R belongs to a simple noncommutative group generated by transvections; hence V_R is a product of commutators in that group. From this it follows that either $\ulcorner V \urcorner \in PT'$ or that $\ulcorner V \urcorner = \ulcorner V_1 \urcorner \ulcorner W \urcorner$, $\ulcorner V_1 \urcorner \in PT'$, $\ulcorner W \urcorner =$ unitary reflection with axis πu_1. In the former case $\ulcorner S \urcorner \in PT'$, in the latter case

$$\begin{aligned} \ulcorner S \urcorner &= \ulcorner V_1 \urcorner \ulcorner W \urcorner \ulcorner V_1 \urcorner \ulcorner W \urcorner \\ &= \ulcorner V_1 \urcorner^2 \ulcorner V_1 \urcorner^{-1} \ulcorner W \urcorner^{-1} \ulcorner V_1 \urcorner \ulcorner W \urcorner, \text{ since } \ulcorner W \urcorner^{-1} = \ulcorner W \urcorner, \\ &\in PT'. \end{aligned}$$

An immediate consequence of (8.2) and (8.6) is the following proposition.

(8.7) *PT is generated by reflections in outer points.*

9. This section is devoted to a study of the full unitary group $PU(\pi)$ of a hyperbolic polarity and its relation to the subgroup $PT(\pi)$ generated by unitary transvections.

(9.1) *Let π be a hyperbolic hermitian polarity; so we may assume $\sigma = 1$, $U = 1$. Then $PU(\pi) = PT(\pi)$, i.e., $PU(\pi)$ is simple.*

Proof. This is a special case of a theorem proved by N. Jacobson in [10, II, section 11]. We shall give here a simple geometric proof. Let $\ulcorner S \urcorner \in PU$. Since PT is transitive on the set of outer points (see (8.4)), we may after multiplication of S by elements of PT assume that $\ulcorner S \urcorner u_1 = u_1$, u_1 an outer point. Then $S_R = S^+ \oplus S^-$. Now $\dim R^- = 1$, $Q^-|R^- = \zeta_3^2$ (see 6.9)), so $S^-|R^- = \pm 1$. If $S^-|R^- = -1$, we multiply S by a unitary reflection in u_2, which belongs to PT since u_2 is an outer point. Hence we may assume that $S^-|R^- = 1$. From the last paragraph of section 7 it follows that $\ulcorner S \urcorner$ is a product of an element of PT and possibly a unitary reflection in u_1. Since u_1 is an outer point, the latter reflection belongs to PT, hence so does $\ulcorner S \urcorner$.

(9.2) *Let π be a hyperbolic nonhermitian linear polarity, i.e., we may assume $\sigma = 1$, U a reflection in a quaternion subfield D of C. Then $PU(\pi)$ is generated by unitary reflections.*

Proof. Assume $\ulcorner S \urcorner \in PU$. After multiplication of $\ulcorner S \urcorner$ by elements of PT, we may again assume that $Su_1 = u_1$, u_1 an outer point. We coordinatize again as in (3.3) and in section 6. Then, as in section 6,

$$S_R = S^+ \oplus S^-.$$

S^- can be an arbitrary rotation in R^- with respect to Q^-, hence it is an even product of reflections in R^-.

Consider any nonisotropic point $a \in u_1^*$. The unitary reflection in a can easily be computed from [21, formula (17), p. 460]. Its restriction to R^-, S^-, turns out to be $-S_{a^-}$, where $a = a^+ + a^-$, $a^+ \in R^+$, $a^- \in R^-$ and S_{a^-} is the orthogonal reflection (with respect to $Q^-|R^-$) in $(a^-)^\perp$. Since a^- can be any nonzero element of R^-, it follows that the possible restrictions to R^- of unitary reflections in nonisotropic points of u_1^* generate $O^+(Q^-|R^-)$. Therefore, after multiplication of $\ulcorner S \urcorner$ by a product of unitary reflections, we may assume that

$$Su_1 = u_1, \quad S_R = S^+ \oplus 1^-.$$

But then we see as in the previous theorem that S^+ is the restriction of a transformation V, $\ulcorner V \urcorner \in PT$, to R^+, so $\ulcorner S \urcorner$ is a product of that $\ulcorner V \urcorner$ and, possibly, the unitary reflection in u_1. This completes the proof.

Finally, we consider the case that π is a hyperbolic nonlinear polarity, hence $\sigma \neq 1$. All coordinate systems in the remainder of this section are supposed to be chosen as in section 6, so $C = C_F \otimes_F K$, $U = 1 \otimes \sigma$, $T = [1, 1, 1; -U, -U, U]$. We shall prove that $PT(\pi)$ is the subgroup consisting of the $\ulcorner S \urcorner \in PU(\pi)$ with $S'TS = T$, $det\, S = 1$. First we need a definition and a lemma.

(9.3) Definition. *Let π be a hyperbolic nonlinear polarity, u_1 an outer point with respect to π. If $\ulcorner S \urcorner \in PU(\pi)$ leaves the point u_1 fixed and S is normalized such that $S'TS = T$, then $\gamma(S) \in K^*$ is defined by $Su_1 = \gamma(S)u_1$.*

Since S leaves $(\cdot, T\,\cdot)$ invariant, $\gamma(S)\sigma(\gamma(S)) = 1$.

(9.4) *The possible values of $\gamma(S)$ for $S \in T(\pi)$ are the $\beta\sigma(\beta)^{-1}$, where $\beta\sigma(\beta) = N(x)$ for some $x \in C_F$, $x \neq 0$.*

Proof. Let $S \in T(\pi)$, $Su_1 = \gamma(S)u_1$. The products of unitary transvections with center on $u_1^* = \pi(u_1)$ are doubly transitive on the isotropic points of u_1^* and are induced by $V \in T(\pi)$ with $Vu_1 = u_1$. Hence we may assume that $\ulcorner S \urcorner$ leaves two isotropic points v and w on u_1^* fixed. Since $(v, Tw) \neq 0$, the invariance of $(\cdot, T\,\cdot)$ under S implies that

$$Sv = \rho v, \quad Sw = \sigma(\rho)^{-1}w.$$

Since $det\, S = 1$, $\gamma(S) = \rho^{-1}\sigma(\rho)$. We shall determine the possible values of ρ. S is a product of normalized transvections. For any two isotropic points x and y there exists a normalized unitary transvection W with $Wx = \eta y$. Hence we can write S in the form

$$S = S_1 S_2 \cdots S_k,$$

where each S_i is a product of at most three transvections with $S_i v = \rho_i v$ for some ρ_i. So we consider the case that S is a product of at most three normalized unitary transvections, $Sv = \rho v$; such an S need not leave the point u_1 invariant, of course.

(i) S is a transvection. Then v must be its center, hence $Sv = v$.

(ii) $S = S_2 S_1$; S_1 and S_2 transvections with centers a and b, respectively; $S_1 v = x$. We choose coordinates in $\mathscr{P}(C)$ such that

$$v = (0, 1, 1; 1, 0, 0) \text{ and } x = (0, 1, 1; -1, 0, 0).$$

Then

$$a = (0, \alpha, \sigma(\alpha); a_1, 0, 0) \text{ and } b = (0, \beta, \sigma(\beta); b_1, 0, 0),$$

where $a_1, b_1 \in C_F$ and $\alpha\sigma(\alpha) = N(a_1)$, $\beta\sigma(\beta) = N(b_1)$.

By [21, prop. 5],

$$S_2 S_1 v = (v, Ta)(x, Ta)^{-1}(x, Tb)(v, Tb)^{-1} v = \rho v.$$

Now

$$\begin{aligned}(v, Ta) &= \alpha+\sigma(\alpha)-(1, Ua_1)\\ &= \alpha+\sigma(\alpha)-(1, a_1).\end{aligned}$$

Since

$$\sigma(1, a_1) = \sigma(U1, a_1) = (1, Ua_1) = (1, a_1),$$
$$(v, Ta) \in F.$$

Similarly for the other factors, hence $\rho \in F$.

(iii) $S = S_3 S_2 S_1$; S_1, S_2 and S_3 normalized transvections with centers a, b and c, respectively.

Call $S_1 v = x$, $S_2 x = y$. We may coordinatize so that

$$v = (0, 1, 1; 1, 0, 0),\ \ x = (0, \xi, \sigma(\xi); x_1, 0, 0),\ \ y = (1, 0, 1; 0, 1, 0)$$

with $x_1 \in C_F$, $\xi\sigma(\xi) = N(x_1)$.

$$S_3 S_2 S_1 v = (v, Ta)(x, Ta)^{-1}(x, Tb)(y, Tb)^{-1}(y, Tc)(v, Tc)^{-1} v.$$

A computation as in case (ii) yields that

$$(v, Ta)(x, Ta)^{-1} \in F.$$

Now we want to compute $(x, Tb)(y, Tb)^{-1}$. In the proof of (8.3) we have seen that there exists a unitary reflection V which transforms the line through x and y in that through x and v. The pole of the former line is $r = \big(\sigma(\xi), \xi, \sigma(\xi); x_1, \sigma(\xi), \bar{x}_1\big)$, that of the latter line is $t = (1, 0, 0; 0, 0, 0)$. Hence the centre s of V must be the fourth harmonic of x with respect to t and r, which is $s = (4\sigma(\xi), \xi, \sigma(\xi); x_1, 2\sigma(\xi), 2\bar{x}_1)$. By [21, formula (17)],

$$Vy = d_{s, Ts; -1}\, y = -y+2(s, Ts)^{-1}(y, Ts)s+8(s, Ts)^{-1} Ts\times(s\times c).$$

For $y = (1, 0, 1; 0, 1, 0)$ this yields

$$Vy = \tfrac{1}{4}\sigma(\xi)^{-1}(0, \xi, \sigma(\xi); -x_1, 0, 0) = \tfrac{1}{4}\sigma(\xi)^{-1} y'.$$

Call $Vb = b'$, then

$$(x, Tb)(y, Tb)^{-1} = 4\sigma(\xi)(x, Tb')(y', Tb')^{-1}.$$

Again

$$(x, Tb')(y', Tb')^{-1} \in F.$$

In the same way we can compute $(y, Tc)(v, Tc)^{-1}$. To this end, x has to be replaced by v in the above computations, hence ξ by 1; thus we find

$$(y, Tc)(v, Tc)^{-1} \in F.$$

Therefore

$$\rho = \sigma(\xi)\rho_1, \quad \rho_1 \in F, \quad \xi\sigma(\xi) = N(x_1), \qquad x_1 \in C_F.$$

Combining (i), (ii) and (iii) we find that for $S \in T(\pi)$ with $Su_1 = \gamma(S)u_1$,

$$\gamma(S) = \beta\sigma(\beta)^{-1} \text{ with } \beta\sigma(\beta) = N(b_1), \qquad b_1 \in C_F.$$

Finally, we have to show that any value of this form can be taken. Let β be given with $\beta\sigma(\beta) = N(b_1)$. From (iii) it is clear that there exists an $S_1 \in T(\pi)$ with $S_1 v = \rho v$, $\rho = \beta\rho_1$ (take $x_1 = b_1$, $\xi = \sigma(\beta)$ in (iii)), with $\rho_1 \in F$. By (8.3) there exists a unitary reflection $S_2 \in T(\pi)$ with $S_2 v = v$ which maps the line through v and $S_1 w$ on the line through v and w. Finally, a suitable transvection S_3 with center v (hence $S_3 v = v$) will transform $S_2 S_1 w$ into a multiple of w. Then $\gamma(S_3 S_2 S_1) = \beta\sigma(\beta)^{-1}$.

Remark. It is not hard to show that the possible values of $\gamma(S)$ in (9.4) are the $\lambda \in K$ satisfying $\lambda\sigma(\lambda) = 1$, $\lambda = N(m)$ for an $m \in C$ with $\bar{m} \cdot Um = 1$. But the form for $\gamma(S)$ of (9.4) will be more convenient for later computations.

(9.5) *Let π be a hyperbolic nonlinear polarity. Then*

(i) $PT(\pi) = \{\ulcorner S \urcorner \in PU(\pi) \mid \mathit{det}\, S = 1\}$;

(ii) $PU(\pi)/PT(\pi) \cong N/N^3$, *where N is the multiplicative group of the $\lambda \in K$ with $\lambda\sigma(\lambda) = 1$ and N^3 consists of all third powers of elements of N;*

(iii) $PU(\pi)$ *is generated by unitary transvections and dilatations.*

(iv) $PT(\pi)$ *is the commutator subgroup of* $PU(\pi)$.

Proof. (i) Every $\ulcorner S \urcorner \in PT$ is induced by a product of normalized transvections, hence by an S with $\mathit{det}\, S = 1$. Conversely assume $\ulcorner S \urcorner \in PU$, $\mathit{det}\, S = 1$. After multiplication of S by a product of normalized transvections, we may assume that

$$Su_1 = \gamma(S)u_1, \quad Sv = \rho v, \quad Sw = \sigma(\rho)^{-1}w.$$

Here u_1 is an outer point and v and w are isotropic points on $\pi u_1 = u_1^*$. We choose coordinates in such a way that

$$u_1 = (1, 0, 0; 0, 0, 0), \quad v = (0, 1, 1; 1, 0, 0), \quad w = (0, 1, 1; -1, 0, 0).$$

We use the notations of section 6 for the case $\sigma \neq 1$.

In R we have

$$Q_1(Sx) = \rho\sigma(\rho)^{-1}Q_1(x),$$
$$(Sx, TSx) = (x, Tx).$$

Consider $S^* = \sigma(\rho)S$. Then

$$Q_1(S^*x) = \rho\sigma(\rho)Q_1(x),$$
$$(S^*x, TS^*x) = \rho\sigma(\rho)(x, Tx).$$

Hence

$$Q^+(S^*x) = \rho\sigma(\rho)Q^+(x),$$
$$Q^-(S^*x) = \rho\sigma(\rho)Q^-(x),$$
$$\langle S^*x^+, S^*x^-\rangle = \rho\sigma(\rho)\langle S^*x^+, S^*x^-\rangle.$$

This implies that S^* leaves R^+ and R^- invariant. With ζ_2', ζ_2'', ζ_3', ζ_3'' as defined in (6.10), $x^+ \in R^+$ can be denoted by $x^+ = (\zeta_2', \zeta_3''; x_1')$, $x_1' \in C_F$. Then

$$Q^+(x^+) = \zeta_2'^2 - \alpha\zeta_3''^2 - N(x_1').$$

Now $S^*v = \rho\sigma(\rho)v$, $S^*w = w$. v and $w \in R^+$; in the coordinates introduced for R^+ we have

$$v = (1, 0; 1) \text{ and } w = (1, 0; -1).$$

Let $L = v^\perp \cap w^\perp$ in R^+, hence

$$L = \{(0, \zeta_3''; x_1') \mid x_1' \in C_F, (1, x_1') = 0\}.$$

$S^*L = L$. If $L_1 = \{(0, 0; x_1' \mid x_1' \in C_F, (1, x_1') = 0\}$, dim $L_1 = 7$, dim $L = 8$ (dim over F), hence $S^*L_1 \cap L_1 \neq 0$. So there exists an element $(0, 0; x_1') \neq 0$ of L_1 with $S(0, 0; x_1') = (0, 0; y_1') \in L_1$. Then

$$Q^+(0, 0; y_1') = \rho\sigma(\rho)Q^+(0, 0; x_1'),$$

i.e.,

$$N(y_1') = \rho\sigma(\rho)N(x_1'),$$
$$\rho\sigma(\rho) = N(b_1),\ b_1 \in C_F.$$

But

$$\gamma(S) = \rho^{-1}\sigma(\rho),$$

so by lemma (9.4) there exists an $S_1 \in T(\pi)$ with $\gamma(S_1) = \gamma(S)^{-1}$. Therefore $\ulcorner SS_1 \urcorner \in PU$, *det* $(SS_1) = 1$, $\gamma(SS_1) = 1$.

From the last paragraph of section 7 it follows that $\ulcorner SS_1 \urcorner$ is an element of PT, up to a unitary dilation $\ulcorner W \urcorner$ with center u_1. But we may assume that $\gamma(W) = 1$, hence W is a unitary reflection in an outer point, which belongs to PT.

(ii) If $\ulcorner S \urcorner \in PU$, $S'TS = T$, then *det* $S \in N$. Hence

$$\ulcorner S \urcorner \to \mathit{det}\ S \bmod N^3$$

is a homorphism of PU into N/N^3. By (i), its kernel is PT. Let $\lambda \in N$. Then there exists a unitary dilatation $\ulcorner S \urcorner$ with $S'TS = T$, *det* $S = \lambda^4$, viz. $d_{u_1, u_1; \lambda^2}$ if $\pi u_1 = u_1^*$. Hence *det* $S = \lambda$ mod N^3.

(iii) Let $\ulcorner S \urcorner \in PU$, $S'TS = T$. As seen in the proof of (ii), there exists a unitary dilatation S_1 with $\det S_1 = \det S \bmod N^3$. Hence $S = S_1 S_2$, $\ulcorner S_2 \urcorner \in PU(\pi)$, $\det S_2 = 1 \bmod N^3$, hence $\ulcorner S_2 \urcorner \in PT$, which is generated by transvections.

(iv) PT is a simple noncommutative group, hence PT is contained in the commutator subgroup of PU. On the other hand, PU/PT is commutative by (ii), hence PT is the commutator subgroup of PU.

III. Applications of algebraic group theory to the groups $PU(\pi)$

10. If π is a linear polarity, then the linear transformations S with $S'TS = T$, $\det S = 1$ form a linear algebraic group. It can be shown that K is a field of definition, but we shall not give the proof here. $PU(\pi)$ is isomorphic with this group, hence is also a linear algebraic group. If π is a nonlinear polarity, we shall see that $PT(\pi)$ is an F-form of an algebraic group.

11. We want to consider the linear groups of transformations S with $\det S = 1$, $(Sx, TSx) = (x, Tx)$ for all x, over a field K such that C is split and $T = [1, 1, 1; -U, -U, U]$, where U is as in (3.3). Thus we have included all K-linear extensions of L-linear transformations T which correspond to a hyperbolic polarity, if C is split over K but not split over L. In this way we will be able to determine the algebraic type of the groups $PU(\pi)$ for $\sigma = 1$, $PT(\pi)$ for $\sigma \neq 1$, π being a hyperbolic polarity. In case $\sigma = 1$ we actually may include the elliptic polarities π, since we can prove

(11.1) *Let $\sigma = 1$, T_1, T_2, T_3 as in* (2.13). *There exists a finite separable extension L of K over which C is split and such that the L-linear extensions of T_1, T_2, T_3 can be brought in either of the following forms by suitable coordinate transformation in the algebra $A(C; 1, 1, 1)$.*

(1) $-T_1 = -T_2 = T_3 = 1$.
(2) $-T_1 = -T_2 = T_3 =$ *reflection in a split quaternion subalgebra of C_L.*

Proof. Consider a coordinate transformation in $A(C; 1, 1, 1)$ of the form $[\beta_1, \beta_2, \beta_3; B_1, B_2, B_3]$. This changes $T = [\lambda_1, \lambda_2, \lambda_3; T_1, T_2, T_3]$ into $[\lambda_1\beta_1^2, \lambda_2\beta_2^2, \lambda_3\beta_3^2; B_1'T_1B_1, B_2'T_2B_2, B_3'T_3B_3]$.

(1) Assume first that $T = [\nu\alpha_1^{-2}, \nu\alpha_2^{-2}, \nu\alpha_3^{-2}; \alpha_1, \alpha_2, \alpha_3]$, where $\det T = \nu = \alpha_1\alpha_2\alpha_3$. After repeated quadratic extensions of K we

may assume that $\alpha_1^{-1} = -\beta_1^2$, $\alpha_2^{-1} = -\beta_2^2$, $\alpha_3^{-1} = \beta_3^2$ for $\beta_1, \beta_2, \beta_3 \in K$. Call $\beta_1\beta_2\beta_3 = \mu$. By a coordinate transformation $[\mu\beta_1^{-2}, \mu\beta_2^{-2}, \mu\beta_3^{-2}; \beta_1, \beta_2, \beta_3]$ we get T in the form $[1, 1, 1; -1, -1, 1]$. Another quadratic extension may be needed to split C.

(2) $T_1 \neq \alpha_1$ for any $\alpha_1 \in K$. Let $T_i^2 = \mu_i$. After repeated quadratic extension of K we may assume that the μ_i are squares. By a similar coordinate change as in case (1) we can get $\mu_1 = \mu_2 = \mu_3 = 1$. Then $-T_1$ must be a reflection in a 4-dimensional subspace V of C by (2.13). $N|V$ is nondegenerate, so after quadratic extensions of the coordinate field K we may assume that $N|V$ is a hyperbolic form. By Witt's theorem, V can be transformed in a split quaternion subalgebra D of C by a rotation B_1. Extending B_1 to a related triple B_1, B_2, B_3 we find a coordinate transformation $[\ldots; B_1, B_2, B_3]$, which brings T in the form $[\ldots; -U, -\lambda U, \lambda^{-1}U]$, U being the reflection in D. Applying a similar argument as in case (1) we get $T = [1, 1, 1; -U, -U, U]$.

12. For the study of the algebraic groups mentioned in the previous section we need some facts about certain Lie algebras of type D_4, F_4 and E_6. We recall some results of Freudenthal [8] in a form adapted to our present needs; we also refer to Jacobson [11] and Soda [16, 17].

Let C be a *split* octave algebra. We choose a *normal basis* in C [16, p. 1.4]: $e, x_0, x_1, x_2, \bar{e}, y_0, y_1, y_2$, where the matrix of the bilinear form $(\cdot, \cdot)$ on C with respect to this basis is

$$\begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix},$$

0 denoting the 4×4 0-matrix, I the 4×4 identity matrix, and where $e+\bar{e} = 1$, $e^2 = e$, $\bar{e}^2 = \bar{e}$, $e\bar{e} = \bar{e}e = 0$, $ex_i = x_i$, $ey_i = 0$, $\bar{e}x_i = 0$, $\bar{e}y_i = y_i$, $x_ix_{i+1} = y_{i+2}$, $y_iy_{i+1} = x_{i+2}$. A complete multiplication table is to be found at the end of this paper, table 1.

We first consider the Lie algebra of skew linear transformations X of C, i.e. X has to satisfy

$$(Xx, y)+(x, Xy) = 0 \text{ for all } x, y.$$

This is a split Lie algebra of type D_4; in the sequel we will denote it by $\boldsymbol{D_4}$. According to [11, p. 140, 141] a splitting Cartan subalgebra $\boldsymbol{H}$ of $\boldsymbol{D_4}$ is given by

$$\boldsymbol{H} = \big\{\{\omega_0, \omega_1, \omega_2, \omega_3, -\omega_0, -\omega_1, -\omega_2, -\omega_3\} \,|\, \omega_i \in K\big\},$$

where $\{\lambda_1, \ldots, \lambda_8\}$ denotes the 8×8 diagonal matrix with $\alpha_{ii} = \lambda_i$. The roots with respect to $\boldsymbol{H}$ are $\pm\omega_i\pm\omega_j$, $i<j$; as simple roots

we may take $\alpha_0 = \omega_1 - \omega_2$, $\alpha_1 = \omega_0 - \omega_1$, $\alpha_2 = \omega_2 - \omega_3$, $\alpha_3 = \omega_2 + \omega_3$. In table 2, at the end of this paper, one finds a complete list of roots α and rootvectors X_α. There we use the following notation:

for $a \in C$,
$$L_a : X \to ax, \quad x \in C,$$
$$R_a : x \to xa, \quad x \in C,$$
$$T_a = L_a + R_a.$$

For $(a, 1) = (b, 1) = (a, b) = 0$ one has

$$L_a L_b = -L_b L_a, \quad R_a R_b = -R_b R_a.$$

From the alternativity of C one derives

$$[R_b, L_a] = [L_b, R_a].$$

Table 2 is derived from [16, p. 5.6]. It is not hard to check that $X_\alpha^2 = 0$ for all roots α.

A triple L_1, L_2, L_3 of elements of $\boldsymbol{D}_4$ is called *Lie related* or a *Lie triple* if

$$(L_1 x_1, x_2, x_3) + (x_1, L_2 x_2, x_2) + (x_1, x_2, L_3 x_3) = 0$$

for all x_1, x_2, x_3 in C; $(.,.,.)$ as in (2.7). L_1, L_2, L_3 are Lie related if and only if they satisfy the *principle of local triality* in C:

$$\hat{L}_i(xy) = (L_{i+1}x)y + x(L_{i+2}y), \text{ (subscripts mod 3)},$$

where $\hat{L}_i z = \overline{L_i \bar{z}}$. Any one of L_1, L_2, L_3 determines the other two uniquely. If L_1, L_2, L_3 are Lie related, then so are $L_{p(1)}$, $L_{p(2)}$, $L_{p(3)}$ for even permutations p, and $\hat{L}_{p(1)}$, $\hat{L}_{p(2)}$, $\hat{L}_{p(3)}$ for odd permutations p. The mapping

$$\lambda : L_i \to L_{i+1}$$

is an exterior automorphism of $\boldsymbol{D}_4$. For $(1, a) = 0$ we get (see [8, p. 15] or [16, p. 5.2])

$$\lambda L_a = -T_a, \quad \lambda R_a = L_a, \quad \lambda T_a = -R_a. \tag{12.1}$$

We want to determine the action of λ on the Cartan subalgebra $\boldsymbol{H}$ of $\boldsymbol{D}_4$. If we denote $\{\omega_0, \omega_1, \omega_2, \omega_3, -\omega_0, -\omega_1, -\omega_2, -\omega_3\}$ by $(\omega_0, \omega_1, \omega_2, \omega_3)$ for brevity, then a basis of $\boldsymbol{H}$ is given by

$$\begin{aligned} \tfrac{1}{2} T_{e-\bar{e}} &= (1, 0, 0, 0) \\ [L_{y_0}, L_{x_0}] &= (-1, -1, 1, 1) \\ [L_{y_1}, L_{x_1}] &= (-1, 1, -1, 1) \\ [L_{y_2}, L_{x_2}] &= (-1, 1, 1, -1). \end{aligned}$$

This is easily verified using table 1. With (12.1) and table 1 the action of λ on these vectors is easily determined. It turns out that $\lambda \boldsymbol{H} = \boldsymbol{H}$ and that, in coordinates $(\omega_0, \omega_1, \omega_2, \omega_3)$, we get for λ the matrix

$$\lambda = \tfrac{1}{2}\begin{pmatrix} -1 & -1 & -1 & -1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & 1 & -1 \\ 1 & -1 & -1 & 1 \end{pmatrix} \tag{12.2}$$

(cf. [8, p. 13]). For convenience we mention the formula for λ^2:

$$\lambda^2 = \tfrac{1}{2}\begin{pmatrix} -1 & 1 & 1 & 1 \\ -1 & 1 & -1 & -1 \\ -1 & -1 & 1 & -1 \\ -1 & -1 & -1 & 1 \end{pmatrix}, \tag{12.3}$$

i.e., the transposed of λ.

13. We now consider the Lie algebra of derivations of the Jordan algebra $A = A(C; \gamma_1, \gamma_2, \gamma_3)$. This Lie algebra is of type F_4; we shall call it $\boldsymbol{F}_4$ in the sequel. If we have

$$u_1 = (1, 0, 0; 0, 0, 0), \quad u_2 = (0, 1, 0; 0, 0, 0), \quad u_3 = (0, 0, 1; 0, 0, 0),$$

then $D \in \boldsymbol{F}_4$, $Du_1 = Du_2 = Du_3 = 0 \Leftrightarrow D = [0, 0, 0; D_1, D_2, D_3]$, where D_1, D_2, D_3 are a Lie triple, i.e. $D_2 = \lambda D_1$, $D_3 = \lambda^2 D_1$. We shall use the notation

$$[0, 0, 0; D_1, \lambda D_1, \lambda^2 D_1] = \Lambda D_1 \text{ for } D_1 \in \boldsymbol{D}_4.$$

Λ is a representation of $\boldsymbol{D}_4$ which is the sum of three nonequivalent representations. Any $D \in \boldsymbol{F}_4$ can be written in a unique way as

$$D = D' + a_d, \tag{13.1}$$

where $D'u_i = 0$ $(i = 1, 2, 3)$, i.e. $D' \in \Lambda \boldsymbol{D}_4$, and where a_d is the derivation of A defined by

$$a_d : x \to [a, x],$$

a being a skew hermitian octave matrix with zeros in the main diagonal. Obviously

$$\begin{aligned} [a_d, b_d] &= [a, b]_d, \\ [\Lambda D, a_d] &= (\Lambda D a)_d, \quad D \in \boldsymbol{D}_4. \end{aligned}$$

As a Cartan subalgebra for $\boldsymbol{F}_4$ we take $\Lambda \boldsymbol{H}$, $\boldsymbol{H}$ the Cartan subalgebra of $\boldsymbol{D}_4$ we have introduced. As rootvectors with respect to

$\boldsymbol{AH}$ one finds the images under A of the rootvectors of $\boldsymbol{H}$ in $\boldsymbol{D}_4$, and the a_d with

$$a = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & a_1 \\ 0 & -\gamma_3^{-1}\gamma_2\bar{a}_1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & -\gamma_1^{-1}\gamma_3\bar{a}_2 \\ 0 & 0 & 0 \\ a_2 & 0 & 0 \end{pmatrix}$$

$$\text{or} \begin{pmatrix} 0 & a_3 & 0 \\ -\gamma_2^{-1}\gamma_1\bar{a}_3 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

with a_1, a_2, a_3 running over the normal basis e, x_0, x_1, x_2, $\bar{e}$, y_0, y_1, y_2 of C. As roots we get those of $\boldsymbol{D}_4$ and

$$\pm\omega_i,\ \tfrac{1}{2}(\pm\omega_0\pm\omega_1\pm\omega_2\pm\omega_3),$$

all possible combinations of $+$ and $-$ being permitted. For the above rootvectors a_d, $a_d^2 \neq 0$, $a_d^3 = 0$. For proofs we refer to [8, p. 21—25].

14. In this section we consider the Lie algebra of linear transformations T of $A = A(C; \gamma_1, \gamma_2, \gamma_3)$ satisfying

$$\langle Tx, x, x\rangle = 0 \text{ for all } x,$$

where $\langle .\,, .\,, .\rangle$ is the symmetric trilinear form associated to the cubic form *det*; cf [8, p. 36—40]. We shall see that this is a Lie algebra of type E_6; we shall call it $\boldsymbol{E}_6$.

Let $t \in A$. By [18, formulas 15,2 and 7]

$$\langle tx, x, x\rangle = \tfrac{1}{3}(t, e)\, det\, x.$$

Hence, for $(t, e) = 0$, the mapping

$$t_m : x \to tx$$

belongs to $\boldsymbol{E}_6$. For arbitrary $T \in \boldsymbol{E}_6$, consider $t = Te$.

$$(t, e) = (t, e\times e) = 3\langle t, e, e\rangle = 3\langle Te, e, e\rangle = 0.$$

Hence

$$T_1 = T - t_m$$

belongs to $\boldsymbol{E}_6$ and $T_1 e = 0$. Therefore, T_1 is a derivation of A, so $T_1 \in \boldsymbol{F}_4$. Consequently each $T \in \boldsymbol{E}_6$ can uniquely be written as

$$T = T_1 + t_m, \quad T_1 \in \boldsymbol{F}_4, \quad t \in A,\ (t, e) = 0. \tag{14.1}$$

Define for any two 3×3 matrices x and y with entries in C,

$$xy = \tfrac{1}{2}(x\cdot y + y^*\cdot x^*),$$

where $\cdot$ denotes the ordinary matrix product and where * is the mapping

$$(x_{ij})^* = (\gamma_i^{-1}\gamma_j \bar{x}_{ji}).$$

For $x, y \in A$ we have the usual Jordan product in A. For $x \in A$, $t = -t^*$,

$$tx = \tfrac{1}{2}[t, x] = \tfrac{1}{2}t_d x.$$

For an arbitrary 3×3 octave matrix t, define

$$t_m : x \to tx \text{ for } x \in A.$$

By (13.1) and (14.1), the $T \in \boldsymbol{E}_6$ are the transformations of the form

$$T = D + t_m. \tag{14.2}$$

$D \in \Lambda \boldsymbol{D}_4$, $t = (t_{ij})$ a 3×3 octave matrix with $t_{ii} \in K$ for $i = 1, 2, 3$, $t_{11} + t_{22} + t_{33} = 0$.

Let $\boldsymbol{H}$ be the Cartan subalgebra of $\boldsymbol{D}_4$ we used in the previous section. Let $\boldsymbol{I}$ be defined by

$$\boldsymbol{I} = \left\{ s_m \middle| s = \begin{pmatrix} \xi_1 & 0 & 0 \\ 0 & \xi_2 & 0 \\ 0 & 0 & \xi_3 \end{pmatrix}, \ \xi_i \in K, \ \sum_i \xi_i = 0 \right\}.$$

A Cartan subalgebra of $\boldsymbol{E}_6$ is

$$\boldsymbol{J} = \Lambda \boldsymbol{H} + \boldsymbol{I}. \tag{14.3}$$

For $D \in \boldsymbol{D}_4$ and $s_m \in \boldsymbol{I}$,

$$[\Lambda D, s_m] = (\Lambda D s)_m = 0.$$

Therefore, the images under Λ of the rootvectors of $\boldsymbol{D}_4$ with respect to $\boldsymbol{H}$ are also rootvectors in $\boldsymbol{E}_6$ with respect to $\boldsymbol{J}$. Let a_{pq} denote the 3×3-matrix (x_{ij}) with $x_{ij} = \delta_{ip}\delta_{jq}a$, $a \in C$, and $a_{pq,m} = (a_{pq})_m$.

Let p, q, r denote any permutation of 1, 2, 3. For $D \in \boldsymbol{D}_4$ and $s_m \in \boldsymbol{I}$ we have

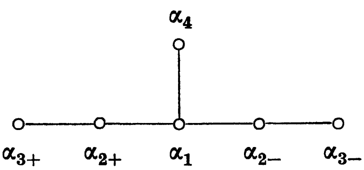
$$[\Lambda D + s_m, a_{pq,m}] = \left((\lambda^{r-1} D a)_{pq} - \tfrac{1}{2}(\xi_p - \xi_q) a_{pq}\right)_m.$$

Hence, for $a = e, x_0, x_1, x_2, \bar{e}, y_0, y_1, y_2$, we find rootvectors $a_{pq,m}$. For the computation of the corresponding roots we need the matrices of λ and λ^2 given in (12.2) and (12.3). The result is given in table 3 at the end of this paper. For convenience in later computations we have chosen $a = -e, -x_0, \ldots, -y_2$ in case of a root of the form $l(\omega)-\frac{1}{2}(\xi_2-\xi_3)$, $l(\omega)-\frac{1}{2}(\xi_3-\xi_1)$, $l(\omega)-\frac{1}{2}(\xi_1-\xi_2)$, $l(\omega)$ being a linear function in $\omega_0, \omega_1, \omega_2, \omega_3$.

As a simple root system we take

$$\begin{aligned}
\alpha_1 &= \omega_2-\omega_3 \\
\alpha_{2+} &= \omega_3+\tfrac{1}{2}(\xi_2-\xi_3) \\
\alpha_{2-} &= \omega_3-\tfrac{1}{2}(\xi_2-\xi_3) \\
\alpha_{3+} &= \tfrac{1}{2}(\omega_0-\omega_1-\omega_2-\omega_3)+\tfrac{1}{2}(\xi_1-\xi_2) \\
\alpha_{3-} &= \tfrac{1}{2}(\omega_0-\omega_1-\omega_2-\omega_3)-\tfrac{1}{2}(\xi_1-\xi_2) \\
\alpha_4 &= \omega_1-\omega_2.
\end{aligned}$$

The corresponding Dynkin diagram is



A straightforward computation yields that

$$(a_{pq,m})^2 = 0 \text{ if } N(a) = 0.$$

From this it follows that all rootvectors of $\boldsymbol{E}_6$ we have written up in table 3 have square 0, which was already proved for the rootvectors of $\boldsymbol{D}_4$.

15. Let $X \in \boldsymbol{E}_6$, $X^2 = 0$. Then

$$S = \exp X = 1+X$$

leaves the cubic form *det* invariant, for

$$\langle Xx, y, z\rangle+\langle x, Xy, z\rangle+\langle x, y, Xz\rangle = 0,$$

for all x, y, z, hence

$$\begin{aligned}
\langle(1+X)x, (1+X)x, (1+X)x\rangle &= \langle x, x, x\rangle+3\langle Xx, x, x\rangle \\
&\quad +3\langle Xx, Xx, x\rangle+\langle Xx, Xx, Xx\rangle = \langle x, x, x\rangle.
\end{aligned}$$

Let G be the group of linear transformations S of $A(C; \gamma_1, \gamma_2, \gamma_3)$ such that $\det Sx = \det x$, and PG the factorgroup of G over its (finite) center. The adjoint representation of PG in $\boldsymbol{E}_6$ is faithfull, hence the transformations exp (tX_α), where $t \in K$ and X_α any rootvector of $\boldsymbol{E}_6$, generate a subgroup of G whose image in PG is isomorphic to the Chevalley group of type E_6 (see [5]).

Since PG is simple (see [10, III]), PG is isomorphic to the Chevalley group of type E_6 (cf. also [15]). This is also an isomorphism in the sense of algebraic groups, since it induces an isomorphism in the Lie algebras, which are $\cong \boldsymbol{E}_6$.

From now on we assume that $\gamma_1 = \gamma_2 = \gamma_3 = 1$ and that C is split. Furthermore we assume that T is of the type considered in section 11, i.e., $T = [1, 1, 1; -U, -U, U]$, where any of the three following conditions holds. (i) $U = 1$; (ii) U is a reflection in a split quaternion subalgebra of C; (iii) K is a quadratic extension of F with Galoisgroup $\langle\sigma\rangle$, $C = C_F \otimes_F K$ with a split octave algebra C_F, $U = 1 \otimes \sigma$. We consider the projective group PU^+ of those $\ulcorner S \urcorner$ for which

$$det\,(Sx) = det\,x,\ (Sx,\, TSx) = (x,\, Tx), \text{ for all } x.$$

This includes groundfield extensions of the groups $PU(\pi)$, π a linear polarity, and $PT(\pi)$, π a hyperbolic nonlinear polarity. PU^+ is the subgroup of elements of PG fixed under the automorphism

$$\varphi : S \to TS'^{-1}T^{-1}. \tag{15.1}$$

If $S = \exp X = 1+X$, $X \in \boldsymbol{E}_6$, $X^2 = 0$, then

$$TS'^{-1}T^{-1} = 1-TX'T^{-1}.$$

So we must compute the action of

$$\varphi : X \to -TX'T^{-1}$$

on the Lie algebra $\boldsymbol{E}_6$.We take T as in section 11.

(i) $\sigma = 1$, $T = [1, 1, 1; -1, -1, 1]$.

Let $X = D+t_m$, as in (14.2). Since $D \in \boldsymbol{D}_4$, $D' = -D$. t_m' is computed as follows

$$(tx,\, y) = (x,\, t^*y).$$

Hence $(t_m)' = (t^*)_m$.

For $t = a_{ij}$, $t^* = \bar{a}_{ji}$, if we assume $\gamma_1 = \gamma_2 = \gamma_3 = 1$. So we can determine X_α' for all rootvectors X_α, making use of table 3. The result can be expressed in the following way. Define the mapping τ of the rootsystem of $\boldsymbol{E}_6$ by $\tau\alpha = \alpha$ if α is a root of $\boldsymbol{D}_4$,

$$\tau(l(\omega)+m(\xi)) = l(\omega)-m(\xi),$$

where

$$l(\omega) = l(\omega_0,\, \omega_1,\, \omega_2,\, \omega_3)$$

and

$$m(\xi) = m(\xi_1,\, \xi_2,\, \xi_3)$$

are linear forms in the ω_i and ξ_j respectively. Then

$$X'_\alpha = -X_{\tau\alpha}.$$

So

$$\varphi : S \to TS'^{-1}T^{-1}$$

maps $\exp(X_\alpha)$ on $\exp(X_{\tau\alpha})$. Hence φ is the outer automorphism of PG defined by the diagram automorphism

$$\alpha_1 \to \alpha_1,\ \alpha_{2+} \leftrightarrow \alpha_{2-},\ \alpha_{3+} \leftrightarrow \alpha_{3-},\ \alpha_4 \to \alpha_4.$$

In this way one obtains the wellknown result that in this case PU^+ is the Chevalley group of type F_4. For the details of the remainder of the proof one has to make use of Steinberg's results in [23] and [25]. The technique is the same as in the following two cases, where we shall give complete proofs.

(ii) $T = [1, 1, 1; -U, -U, U]$, U a reflection in a split quaternion subalgebra D of C. For C we take a normal basis e, x_0, x_1, x_2, $\bar{e}$, y_0, y_1, y_2 as in section 12. Since any two split quaternion subalgebras of C are conjugate under an inner automorphism of C by [9], we may assume that e, $\bar{e}$, x_0, y_0 form a basis for D. Hence

$$\begin{aligned} U : e \to e,\ \bar{e} \to \bar{e},\ x_0 \to x_0,\ y_0 \to y_0 \\ x_1 \to -x_1,\ x_2 \to -x_2,\ y_1 \to -y_1,\ y_2 \to -y_2. \end{aligned}$$

Let X_α be a rootvector of $\boldsymbol{E}_6$. First assume $\alpha = \omega_i - \omega_j$, a root of $\boldsymbol{D}_4$. Then $X'_\alpha = -X_\alpha$ and

$$\varphi X_\alpha = -TX'_\alpha T^{-1} = TX_\alpha T^{-1} = (-1)^{m_0} X_\alpha$$

if

$$\alpha = m_0\alpha_0 + m_1\alpha_1 + m_2\alpha_2 + m_3\alpha_3,$$

as is easily computed with the aid of table 2.

Hence

$$\varphi(\exp X_\alpha) = \varphi(1+X_\alpha) = 1+(-1)^{m_0}X_\alpha.$$

Now let $X_\alpha = a_{ij,m}$ as in table 3. Then

$$\varphi X = -TX'T^{-1} = -T\bar{a}_{ji,m}T^{-1} = -(U\bar{a})_{ji,m}.$$

Making use of the mapping τ as defined in case (i), we find: For

$$\begin{aligned} X_\alpha &= a_{ij,m}, \quad a = e, \bar{e}, x_0, y_0, \\ \varphi X_\alpha &= X_{\tau\alpha}. \end{aligned}$$

For

$$\begin{aligned} X_\alpha &= a_{ij,m}, \quad a = x_1, x_2, y_1, y_2, \\ \varphi X_\alpha &= -X_{\tau\alpha}. \end{aligned}$$

The action of φ on the Cartan subalgebra $\boldsymbol{J}$ of $\boldsymbol{E}_6$ is computated in the following way (see section 14).

$$\boldsymbol{J} = \Lambda\boldsymbol{H} \oplus \boldsymbol{I}$$

$$\boldsymbol{I} = \left\{s_m | s = \begin{pmatrix} \xi_0 & 0 & 0 \\ 0 & \xi_1 & 0 \\ 0 & 0 & \xi_2 \end{pmatrix}, \ \xi_i \in K, \ \sum_i \xi_i = 0\right\}.$$

$\boldsymbol{H}$ is the Cartan subalgebra of $\boldsymbol{D}_4$. As a basis of $\boldsymbol{H}$ we have $T_{e-\bar{e}}$, $[L_{y_i}, L_{x_i}]$, $i = 0, 1, 2$. For $D \in \boldsymbol{D}_4$, $D' = -D$. Furthermore,

$$T\Lambda T_{e-\bar{e}}T^{-1} = \Lambda T_{Ue-U\bar{e}} = \Lambda T_{e-\bar{e}},$$
$$T\Lambda[L_{y_i}, L_{x_i}]T^{-1} = \Lambda[L_{Uy_i}, L_{Ux_i}] = \Lambda[L_{y_i}, L_{x_i}].$$

Thus we see that

$$\varphi X = X \text{ for } X \in \Lambda\boldsymbol{H}.$$

For $s_m \in \boldsymbol{I}$,

$$\varphi s_m = -T(s_m)'T^{-1} = (-s)_m.$$

Define the Lie algebra

$$\boldsymbol{L} = \{X \in \boldsymbol{E}_6 | \varphi X = X\}.$$

Then

$$\boldsymbol{L} = \Lambda\boldsymbol{H} \oplus \bigoplus_\alpha K(X_\alpha + \varphi X_\alpha).$$

$\boldsymbol{L}$ has $\Lambda\boldsymbol{H}$ as a Cartan subalgebra. As rootsystem we find $\{\pm\alpha\}$, where α runs over

$$\omega_0-\omega_1, \ \omega_2-\omega_3, \ \omega_2+\omega_3, \ \omega_0+\omega_1, \ \omega_i \qquad (i = 0, 1, 2, 3),$$
$$\tfrac{1}{2}(\omega_0 \pm \omega_1 \pm \omega_2 \pm \omega_3) \text{ (all combinations of signs possible).}$$

This is a simple Lie algebra of type C_4. As primitive roots we may take

$$\beta_1 = \tfrac{1}{2}(\omega_0-\omega_1-\omega_2-\omega_3), \ \ \beta_2 = \omega_3, \ \ \beta_3 = \omega_2-\omega_3, \ \ \beta_4 = \omega_1.$$

Define

$$\boldsymbol{M} = \{X \in \boldsymbol{E}_6 | \varphi X = -X\}.$$

Then

$$\boldsymbol{E}_6 = \boldsymbol{L} \oplus \boldsymbol{M}.$$

$\boldsymbol{L} \perp \boldsymbol{M}$ with respect to the Killing-Cartan form.

We have

$$[\boldsymbol{L}, \boldsymbol{L}] \subseteqq \boldsymbol{L}, \ [\boldsymbol{L}, \boldsymbol{M}] \subseteqq \boldsymbol{M}, \ [\boldsymbol{M}, \boldsymbol{M}] \subseteqq \boldsymbol{L}.$$

Now $S \in PG$ is fixed under φ if and only if the automorphism

$$AdS : X \to SXS^{-1}, \ X \in \boldsymbol{E}_6,$$

commutes with φ (on $\boldsymbol{E}_6$). Hence $\varphi S = S$ if and only if S leaves $\boldsymbol{L}$ invariant, hence also $\boldsymbol{M}$ since $\boldsymbol{M} = \boldsymbol{L}^{\perp}$. So for $\varphi S = S$, $AdS|\boldsymbol{L} \in Aut(\boldsymbol{L})$. Assume $AdS|\boldsymbol{L} = 1$. We shall prove that this implies $S = 1$.

$AdS\,\boldsymbol{M} = \boldsymbol{M}$. Since

$$\boldsymbol{I} = \{X \in \boldsymbol{M} \mid [\Lambda\boldsymbol{H}, X] = 0\},$$

we have

$$AdS\,\boldsymbol{I} = \boldsymbol{I}.$$

So AdS leaves the Cartan subalgebra $\boldsymbol{J}$ of $\boldsymbol{E}_6$ invariant, hence it transforms rootvectors in scalar multiples of rootvectors. The vectors

$$X_{l(\omega)+m(\xi)}+X_{l(\omega)-m(\xi)},$$

which belong to $\boldsymbol{L}$, are fixed under AdS. Hence

$$AdS\,X_{l(\omega)+m(\xi)} = X_{l(\omega)\pm m(\xi)},$$
$$AdS\,X_{l(\omega)-m(\xi)} = X_{l(\omega)\mp m(\xi)}.$$

From this we can compute the action of φ on $\boldsymbol{I}$.

Denote

$$\begin{pmatrix} \xi_1 & 0 & 0 \\ 0 & \xi_2 & 0 \\ 0 & 0 & \xi_3 \end{pmatrix}_m \in \boldsymbol{I}$$

by (ξ_1, ξ_2, ξ_3), then AdS must act on $\boldsymbol{I}$ as follows:

$$\begin{aligned} (1, -1, \ \ 0) &\to \varepsilon_1(1, -1, \ \ 0), \varepsilon_1 = \pm 1 \\ (1, \ \ 0, -1) &\to \varepsilon_2(1, \ \ 0, -1), \varepsilon_2 = \pm 1 \\ (0, \ \ 1, -1) &\to \varepsilon_3(0, \ \ 1, -1), \varepsilon_3 = \pm 1. \end{aligned}$$

Since the sum of the vectors on the left hand side is zero, so must be the sum on the right hand side. Hence

$$\varepsilon_1 = \varepsilon_2 = \varepsilon_3 = \pm 1 = \varepsilon.$$

Then

$$AdS\,(X_{l(\omega)+m(\xi)}) = X_{l(\omega)+\varepsilon m(\xi)}.$$

Hence

$$AdS\,(X_{l(\omega)+m(\xi)}-X_{l(\omega)-m(\xi)}) = \varepsilon(X_{l(\omega)+m(\xi)}-X_{l(\omega)-m(\xi)}).$$

Let $\alpha = \pm\omega_i\pm\omega_j$, with $(i, j) \neq (0, 1), \neq (2, 3)$. Then $X_\alpha \in \boldsymbol{M}$. There exists a root $l(\omega)+m(\xi)$ such that $\alpha+l(\omega)+m(\xi)$ is again a root, say $l'(\omega)+m(\xi)$. Then, for some $N \neq 0$, we have

$$[X_\alpha, X_{l(\omega)+m(\xi)}+X_{l(\omega)-m(\xi)}] = N(X_{l'(\omega)+m(\xi)}-X_{l'(\omega)-m(\xi)}),$$

since $[\boldsymbol{M}, \boldsymbol{L}] \subseteqq \boldsymbol{M}$. Let AdS act on both sides of the equation; then we find

$$AdS(X_\alpha) = \varepsilon X_\alpha.$$

Therefore

$$AdS|M = \varepsilon.$$

So $AdS = 1$ or φ. Now φ is a product of the outer automorphism

$$X \to -X'$$

and an inner automorphism, hence φ is an outer automorphism. Therefore $AdS = \varphi$ is impossible, hence $AdS = 1$. This means that the mapping

$$S \to AdS|\boldsymbol{L}$$

is an isomorphism of PU^+ into $Aut(\boldsymbol{L})$.

We have seen that $\boldsymbol{L}$ is a Lie algebra of type C_4. The rootvectors of $\boldsymbol{L}$ with respect to $\Lambda\boldsymbol{H}$ are the $X_\alpha + \varphi X_\alpha \neq 0$. Since X_α and φX_α are rootvectors of $\boldsymbol{E}_6$, $\exp\left(t(X_\alpha + \varphi X_\alpha)\right)$, $t \in K$, belongs to the Chevalley group correspondong to $\boldsymbol{E}_6$, i.e., to PG. This implies that $AdPG|\boldsymbol{L}$ contains $Aut(\boldsymbol{L})'$, the commutatorsubgroup of $Aut(\boldsymbol{L})$, which is the Chevalley group of type C_4.

Now assume K algebraically closed. Then by [25,4.6 and 4.7], $Aut(\boldsymbol{L})$ is simple and $AdPG \mid \boldsymbol{L} = Aut(\boldsymbol{L})$. $Aut(\boldsymbol{L})$ has a Lie algebra $\cong \boldsymbol{L}$ and PU^+ has a Lie algebra which contains an algebra $\cong \boldsymbol{L}$. Hence the isomorphism $S \to AdS \mid \boldsymbol{L}$ induces a surjective transformation in the Lie algebras, hence is also an isomorphism in the sense of algebraic groups. [14, th. 1] implies that $AdPG|\boldsymbol{L} = Aut(\boldsymbol{L})$ over any field K. By [25, 4.6], $Aut(\boldsymbol{L})/Aut(\boldsymbol{L})' \cong K^*/K^{*2}$.

Remark: In [25, 4.6] it is said that $f = 3$ if Σ is of type C_n; this must be $f = 2$.

Summarizing our results we have shown

(15.2) *Let C be a split octave algebra over a field K, $T = [1, 1, 1; -U, -U, U]$, U a reflection in a split quaternion subalgebra of C. Let PU^+ be the group of transformations $\ulcorner S \urcorner$ with $\det(Sx) = \det x$, $(Sx, TSx) = (x, Tx)$ for all x, and let $PU^{+\prime}$ be the commutatorsubgroup of PU^+. Then $PU^{+\prime}$ is isomorphic with the Chevalley group of type C_4 and $PU^+/PU^{+\prime} \cong K^*/K^{*2}$.*

(iii) $\sigma \neq 1$, $T = [1, 1, 1; -U, -U, U]$, $U = 1 \otimes \sigma$. The action of

$$\varphi : S \to TS'^{-1}T^{-1}$$

on the $\exp(tX_\alpha)$, X_α a rootvector of $\boldsymbol{E}_6$, is the same as in case (i) except for a "twist" with σ:

$$\exp(tX_\alpha) = \exp\big(\sigma(t)X_{\tau\alpha}\big).$$

This means that φ acts on the Lie algebra $\boldsymbol{E}_6$ as the Steinberg automorphism. PU^+, the subgroup of elements of PG fixed under φ, is a group G^φ which contains the Steinberg group G^1 of type $\boldsymbol{E}_6$; see [23, p. 891]. G^φ/G^1 is isomorphic to the group of self-conjugate characters on P_r extendable to P, modulo the subgroup of self-conjugate characters on P; here P denotes the free abelian group generated by the weights on a Cartan subalgebra of $\boldsymbol{E}_6$, P_r the subgroup generated by the roots. By an argument as used in [25, p. 1126, last paragraph of section 6] one obtains the result

$$G^\varphi/G^1 \cong K^{*3} \cap F/F^3.$$

Now $K = F(\vartheta)$, $\vartheta^2 = \alpha \in F$. Assume $x = (\xi+\eta\vartheta)^3 \in F$, for $\xi, \eta \in F$. Then

$$3\xi^2\eta + \eta^3\alpha = 0.$$

Hence $\eta = 0$, so $x \in F^3$, or $-3 = \alpha\rho^2$, $\rho \in F$. Then $\eta = \pm\rho\xi$, hence

$$x = \xi^3 + 3\alpha\xi\eta^2 = -8\xi^3 = (-2\xi)^3 \in F^3.$$

So $K^{*3} \cap F = F^3$, therefore $G^\varphi = G^1$.

Thus we have shown

(15.3) *Assume* $K = F(\vartheta)$, $\vartheta^2 \in K$, σ *the* F-*automorphism* $\neq 1$ *of* K. *Let* C_F *be a split octave algebra over* F, $C = C_F \otimes_F K$, $U = 1 \otimes \sigma$, $T = [1, 1, 1; -U, -U, U]$. *The group* PU^+ *of transformations* $\ulcorner S \urcorner$ *with* $\det Sx = \det x$, $(Sx, TSx) = (x, Tx)$ *for all* x, *is a simple group, viz., the group of rational points (over* F*) of the Steinberg group of type* E_6^1, *i.e. the quasi-split outer form of the group* E_6 *over* F.

16. We conclude this paper with some remarks.

Let π be a hyperbolic nonhermitian linear polarity. So we may assume $\pi = \pi_0 T$, $T = [1, 1, 1; -U, -U, U]$, U a reflection in a quaternionsubfield D of C. We have seen that $PU(\pi)$ is generated by reflections and that $PT(\pi)$ is simple. Nothing has been said about $PU(\pi)/PT(\pi)$. If one extends the ground field such that D and C are split, then $PU(\pi)$ extends to PU^+, which has a simple normal subgroup $PU^{+\prime}$ such that $PU^+/PU^{+\prime} \cong K^*/K^{*2}$. $PU^{+\prime}$ is generated by the $\exp(tX_\alpha)$, $t \in K$, X_α a rootvector, which are transvections. It seems likely that in the case C non split, $PU(\pi)/$

$PT(\pi) \cong K^*/K^{*2}$, which isomorphism would be induced by a kind of spinornorm; then we would have $PT(\pi) = PU(\pi)'$, the commutatorsubgroup of $PU(\pi)$. But we have not been able to furnish a complete proof of these conjectures.

In (11.1) we showed that if $\sigma = 1$, any polarity can be brought in one of two standard-forms over a sufficiently large extension field. If $\sigma \neq 1$, a similar result does not hold. Actually, if one considers a polarity different from the type considered in (15.3), one finds a unitary group which may be another form of E_6. For instance, if $F = \boldsymbol{R}$, $K = \boldsymbol{C}$, $U = U_1 \otimes \sigma$, where U_1 is a reflection in a split quaternion subalgebra of the split octave algebra C_F, then the corresponding unitary group is a real form of E_6 corresponding to an outer automorphism of the complex form, different from the real form we have found in (15.3). There seem to be some interesting problems in this direction; undoubtedly there is a close relationship with the work on Lie algebras of type E_6 done by J. Ferrar in his Yale dissertation [7].

In this paper we have paid no attention to the geometry of reflections with respect to a polarity, as was done in [22] for the case of hermitian linear polarities. In case of a linear non-hermitian polarity it seems that the theory will be much more complicated than in the hermitian case; this makes it doubtful whether the geometry of a group of type C_4, which can be described in other ways, is worth so much energy. In case $\sigma \neq 1$, π as in (9.3), the corresponding geometry of reflections shows similar features as in the hermitian linear case. The generalized theorem of three reflections [22, 27.1] seems to hold in the following form: a product of three reflections S_{v_1}, S_{v_2}, S_{v_3} is involutory if and only if v_1, v_2, v_3 lie on one and the same F-line, where F is the subfield of σ-invariants in K. It seems that the proper algebraic apparatus to cope with this kind of problems still has to be developed.

It is very likely that the results of section 15 can also be derived by working in the algebraic groups themselves without making use of Lie algebras. On the other hand, section 15 shows that there is quite a lot in the theory of algebraic groups that can be done with Lie algebras.

TABLE 1. Multiplication $x \cdot y$ for a normal basis

$x \backslash y$	e	x_0	x_1	x_2	$\bar{e}$	y_0	y_1	y_2
e	e	x_0	x_1	x_2	0	0	0	0
x_0	0	0	y_2	$-y_1$	x_0	$-e$	0	0
x_1	0	$-y_2$	0	y_0	x_1	0	$-e$	0
x_2	0	y_1	$-y_0$	0	x_2	0	0	$-e$
$\bar{e}$	0	0	0	0	$\bar{e}$	y_0	y_1	y_2
y_0	y_0	$-\bar{e}$	0	0	0	0	x_2	$-x_1$
y_1	y_1	0	$-\bar{e}$	0	0	$-x_2$	0	x_0
y_2	y_2	0	0	$-\bar{e}$	0	x_1	$-x_0$	0

TABLE 2. Roots and rootvectors for D_4

α		X_α
α_0	$= \omega_1-\omega_2$	$\frac{1}{2}[R_{y_1}, R_{x_0}] = R_{y_1}R_{x_0}$
α_1	$= \omega_0-\omega_1$	$\frac{1}{2}[L_{x_1}, L_{x_2}] = L_{x_1}L_{x_2}$
α_2	$= \omega_2-\omega_3$	$\frac{1}{2}[R_{x_1}, R_{y_2}] = R_{x_1}R_{y_2}$
α_3	$= \omega_2+\omega_3$	$\frac{1}{2}[T_{x_1}, T_{x_2}] = L_{x_1}L_{x_2}+R_{x_1}R_{x_2}+[R_{x_1}, L_{x_2}]$
$\alpha_0+\alpha_1$	$= \omega_0-\omega_2$	$\frac{1}{2}[L_{x_2}, L_{x_0}] = L_{x_2}L_{x_0}$
$\alpha_0+\alpha_2$	$= \omega_1-\omega_3$	$\frac{1}{2}[R_{x_0}, R_{y_2}] = R_{x_0}R_{y_2}$
$\alpha_0+\alpha_3$	$= \omega_1+\omega_3$	$\frac{1}{2}[T_{x_2}, T_{x_0}] = L_{x_2}L_{x_0}+R_{x_2}R_{x_0}+[R_{x_2}, L_{x_0}]$
$\alpha_0+\alpha_1+\alpha_2$	$= \omega_0-\omega_3$	$\frac{1}{2}[L_{x_0}, L_{x_1}] = L_{x_0}L_{x_1}$
$\alpha_0+\alpha_2+\alpha_3$	$= \omega_1+\omega_2$	$\frac{1}{2}[T_{x_0}, T_{x_1}] = L_{x_0}L_{x_1}+R_{x_0}R_{x_1}+[R_{x_0}, L_{x_1}]$
$\alpha_0+\alpha_3+\alpha_1$	$= \omega_0+\omega_3$	$\frac{1}{2}[R_{y_0}, R_{y_1}] = R_{y_0}R_{y_1}$
$\alpha_0+\alpha_1+\alpha_2+\alpha_3$	$= \omega_0+\omega_2$	$\frac{1}{2}[R_{y_2}, R_{y_0}] = R_{y_2}R_{y_0}$
$2\alpha_0+\alpha_1+\alpha_2+\alpha_3$	$= \omega_0+\omega_1$	$\frac{1}{2}[R_{y_1}, R_{y_2}] = R_{y_1}R_{y_2}$

For $\alpha > 0$, $X_{-\alpha}$ is deduced from X_α by replacing x_i by y_i and y_i by x_i.

TABLE 3. Roots and rootvectors for E_6

root	rootvector	a
$\omega_i-\omega_j$ $(0 \leqq i, j \leqq 3, i \neq j)$	$[0, 0, 0; D, \lambda D, \lambda^2 D]$, with $D = X_{\omega_i-\omega_j}$ as in table 2	
$\omega_0-\frac{1}{2}(\xi_2-\xi_3)$	$a_{23,m}$	$-e$
$\omega_1-\frac{1}{2}(\xi_2-\xi_3)$	$a_{23,m}$	$-x_0$
$\omega_2-\frac{1}{2}(\xi_2-\xi_3)$	$a_{23,m}$	$-x_1$
$\omega_3-\frac{1}{2}(\xi_2-\xi_3)$	$a_{23,m}$	$-x_2$
$-\omega_0-\frac{1}{2}(\xi_2-\xi_3)$	$a_{23,m}$	$-\bar{e}$
$-\omega_1-\frac{1}{2}(\xi_2-\xi_3)$	$a_{23,m}$	$-y_0$

TABLE 3 (*continued*)

root	rootvector	a
$-\omega_2-\frac{1}{2}(\xi_2-\xi_3)$	$a_{23,m}$	$-y_1$
$-\omega_3-\frac{1}{2}(\xi_2-\xi_3)$	$a_{23,m}$	$-y_2$
$-\omega_0+\frac{1}{2}(\xi_2-\xi_3)$	$a_{32,m}$	e
$\omega_1+\frac{1}{2}(\xi_2-\xi_3)$	$a_{32,m}$	x_0
$\omega_2+\frac{1}{2}(\xi_2-\xi_3)$	$a_{32,m}$	x_1
$\omega_3+\frac{1}{2}(\xi_2-\xi_3)$	$a_{32,m}$	x_2
$\omega_0+\frac{1}{2}(\xi_2-\xi_3)$	$a_{32,m}$	$\bar{e}$
$-\omega_1+\frac{1}{2}(\xi_2-\xi_3)$	$a_{32,m}$	y_0
$-\omega_2+\frac{1}{2}(\xi_2-\xi_3)$	$a_{32,m}$	y_1
$-\omega_3+\frac{1}{2}(\xi_2-\xi_3)$	$a_{32,m}$	y_2
$\frac{1}{2}(\omega_0+\omega_1+\omega_2+\omega_3+\xi_3-\xi_1)$	$a_{13,m}$	e
$\frac{1}{2}(\omega_0+\omega_1-\omega_2-\omega_3+\xi_3-\xi_1)$	$a_{13,m}$	x_0
$\frac{1}{2}(\omega_0-\omega_1+\omega_2-\omega_3+\xi_3-\xi_1)$	$a_{13,m}$	x_1
$\frac{1}{2}(\omega_0-\omega_1-\omega_2+\omega_3+\xi_3-\xi_1)$	$a_{13,m}$	x_2
$\frac{1}{2}(-\omega_0-\omega_1-\omega_2-\omega_3+\xi_3-\xi_1)$	$a_{13,m}$	$\bar{e}$
$\frac{1}{2}(-\omega_0-\omega_1+\omega_2+\omega_3+\xi_3-\xi_1)$	$a_{13,m}$	y_0
$\frac{1}{2}(-\omega_0+\omega_1-\omega_2+\omega_3+\xi_3-\xi_1)$	$a_{13,m}$	y_1
$\frac{1}{2}(-\omega_0+\omega_1+\omega_2-\omega_3+\xi_3-\xi_1)$	$a_{13,m}$	y_2
$\frac{1}{2}(-\omega_0-\omega_1-\omega_2-\omega_3-(\xi_3-\xi_1))$	$a_{31,m}$	$-e$
$\frac{1}{2}(\omega_0+\omega_1-\omega_2-\omega_3-(\xi_3-\xi_1))$	$a_{31,m}$	$-x_0$
$\frac{1}{2}(\omega_0-\omega_1+\omega_2-\omega_3-(\xi_3-\xi_1))$	$a_{31,m}$	$-x_1$
$\frac{1}{2}(\omega_0-\omega_1-\omega_2+\omega_3-(\xi_3-\xi_1))$	$a_{31,m}$	$-x_2$
$\frac{1}{2}(\omega_0+\omega_1+\omega_2+\omega_3-(\xi_3-\xi_1))$	$a_{31,m}$	$-\bar{e}$
$\frac{1}{2}(-\omega_0-\omega_1+\omega_2+\omega_3-(\xi_3-\xi_1))$	$a_{31,m}$	$-y_0$
$\frac{1}{2}(-\omega_0+\omega_1-\omega_2+\omega_3-(\xi_3-\xi_1))$	$a_{31,m}$	$-y_1$
$\frac{1}{2}(-\omega_0+\omega_1+\omega_2-\omega_3-(\xi_3-\xi_1))$	$a_{31,m}$	$-y_2$
$\frac{1}{2}(-\omega_0+\omega_1+\omega_2+\omega_3-(\xi_1-\xi_2))$	$a_{12,m}$	$-e$
$\frac{1}{2}(-\omega_0+\omega_1-\omega_2-\omega_3-(\xi_1-\xi_2))$	$a_{12,m}$	$-x_0$
$\frac{1}{2}(-\omega_0-\omega_1+\omega_2-\omega_3-(\xi_1-\xi_2))$	$a_{12,m}$	$-x_1$
$\frac{1}{2}(-\omega_0-\omega_1-\omega_2+\omega_3-(\xi_1-\xi_2))$	$a_{12,m}$	$-x_2$
$\frac{1}{2}(\omega_0-\omega_1-\omega_2-\omega_3-(\xi_1-\xi_2))$	$a_{12,m}$	$-\bar{e}$
$\frac{1}{2}(\omega_0-\omega_1+\omega_2+\omega_3-(\xi_1-\xi_2))$	$a_{12,m}$	$-y_0$
$\frac{1}{2}(\omega_0+\omega_1-\omega_2+\omega_3-(\xi_1-\xi_2))$	$a_{12,m}$	$-y_1$
$\frac{1}{2}(\omega_0+\omega_1+\omega_2-\omega_3-(\xi_1-\xi_2))$	$a_{12,m}$	$-y_2$
$\frac{1}{2}(\omega_0-\omega_1-\omega_2-\omega_3+\xi_1-\xi_2)$	$a_{21,m}$	e
$\frac{1}{2}(-\omega_0+\omega_1-\omega_2-\omega_3+\xi_1-\xi_2)$	$a_{21,m}$	x_0
$\frac{1}{2}(-\omega_0-\omega_1+\omega_2-\omega_3+\xi_1-\xi_2)$	$a_{21,m}$	x_1
$\frac{1}{2}(-\omega_0-\omega_1-\omega_2+\omega_3+\xi_1-\xi_2)$	$a_{21,m}$	x_2
$\frac{1}{2}(-\omega_0+\omega_1+\omega_2+\omega_3+\xi_1-\xi_2)$	$a_{21,m}$	$\bar{e}$
$\frac{1}{2}(\omega_0-\omega_1+\omega_2+\omega_3+\xi_1-\xi_2)$	$a_{21,m}$	y_0
$\frac{1}{2}(\omega_0+\omega_1-\omega_2+\omega_3+\xi_1-\xi_2)$	$a_{21,m}$	y_1
$\frac{1}{2}(\omega_0+\omega_1+\omega_2-\omega_3+\xi_1-\xi_2)$	$a_{21,m}$	y_2

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(Oblatum 26-6-67)

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