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by

M. S. Rangachari

1. Introduction

Let $s(x)$ be bounded and integrable in every finite positive interval of x. Then we may define a generalization of the Laplace transform of $s(x)$ by the integral

$$(1.1)\quad L(t, \alpha) = \frac{\int_0^\infty e^{-tx}x^\alpha s(x)dx}{\int_0^\infty e^{-tx}x^\alpha dx} \equiv \frac{t^{\alpha+1}}{\Gamma(\alpha+1)}\int_0^\infty e^{-tx}x^\alpha s(x)dx,$$

$$\alpha > -1,\ t > 0,$$

whose convergence is assumed to be absolute for $0 < t < \delta$ and so for all $t > 0$. Jakimovski and Rajagopal [4] first employed the transform $L(t, \alpha)$ in the case $\alpha \geqq 0$ to obtain asymptotic versions of known Tauberian theorems for the Laplace transform. Later Jakimovski [5] made a study of the transform $L(t, \alpha)$ even dispensing with the condition that its convergence is absolute, and extending to it some properties of the Laplace transform. Still later Rubel [11] independently treated the transform $L(t, \alpha)$ for any $\alpha > -1$ calling it a Littlewood mean of $s(x)$. Taking $s(x)$ bounded in $(0, \infty)$, he related the behaviour of the mean or transform $L(t, \alpha)$ when $t \to +0$, firstly to the behaviour of the Cesàro mean of order $k > 0$ as usually defined ([2], p. 110—1), viz.

$$(1.2)\quad C(u, k) = \frac{s_k(u)}{u^k} \equiv \frac{k}{u^k}\int_0^u (u-x)^{k-1}s(x)dx,\ k > 0,\ u > 0,$$

when $u \to \infty$, and secondly to the behaviour of what he called the Pólya mean of $s(x)$, say, of index θ, $0 \leqq \theta < 1$, defined as

$$(1.3)\qquad P(u, \theta) = \frac{1}{u-\theta u}\int_{\theta u}^u s(x)dx,\quad 0 \leqq \theta < 1,\ u > 0,$$

[1] The contents of this paper formed part of a thesis approved for the Ph. D. degree of the University of Madras.

when $u \to \infty$. Each of the means in (1.1), (1.2) and (1.3) defines a regular method of summability as follows. First, introducing a notation in consonance with that used to define the means of $s(x)$ in (1.1), (1.2) and (1.3), we write

$$\lim_{t\to+0} \begin{matrix}\sup\\ \inf\end{matrix} L(t, \alpha) = \left.\begin{matrix}\bar{L}(\alpha)\\ \underline{L}(\alpha)\end{matrix}\right\}, \tag{1.4}$$

$$\lim_{u\to\infty} \begin{matrix}\sup\\ \inf\end{matrix} C(u, k) = \left.\begin{matrix}\bar{C}(k)\\ \underline{C}(k)\end{matrix}\right\}, \tag{1.5}$$

$$\lim_{u\to\infty} \begin{matrix}\sup\\ \inf\end{matrix} P(u, \theta) = \left.\begin{matrix}\bar{P}(\theta)\\ \underline{P}(\theta)\end{matrix}\right\}. \tag{1.6}$$

Then we say that $s(x)$ is summable to sum S (finite) by the method (L, α), $\alpha > -1$, or by the method (C, k), $k > 0$, or by the method (P, θ), $0 \leqq \theta < 1$, according as

$$\bar{L}(\alpha) = \underline{L}(\alpha) = S, \text{ or } \bar{C}(k) = \underline{C}(k) = S, \text{ or } \bar{P}(\theta) = \underline{P}(\theta) = S, \tag{1.7}$$

denoting these relations respectively by

$$s(x) \to S(L, \alpha),\ s(x) \to S(C, k),\ s(x) \to S(P, \theta). \tag{1.7'}$$

More generally, according as

$$\begin{cases} -\infty < \underline{L}(\alpha) < \bar{L}(\alpha) < \infty, \text{ or } -\infty < \underline{C}(k) < \bar{C}(k) < \infty, \\ \qquad\text{or } \quad -\infty < \underline{P}(\theta) < \bar{P}(\theta) < \infty, \end{cases} \tag{1.8}$$

we write

$$s(x) = O(1)\ (L, \alpha), \text{ or } s(x) = O(1)\ (C, k), \text{ or } s(x) = O(1)(P, \theta). \tag{1.8'}$$

The following further definition depends on the fact that $\bar{L}(\alpha)$, $\underline{L}(\alpha)$ in (1.4) are monotonic functions of $\alpha > -1$ (Theorem 3.1. *infra*). Restricted to $s(x)$ bounded in $(0, \infty)$, it is implicit in Rubel's paper [11].

$$s(x) \to S(L, -1+0) \text{ if } \lim_{\alpha\to-1+0} \bar{L}(\alpha) = \lim_{\alpha\to-1+0} \underline{L}(\alpha) = S. \tag{1.9}$$

The limits in (1.9) are the inner limits of oscillation of the means of the method (L, α), $\bar{L}(\alpha)$ being monotonic increasing and $\underline{L}(\alpha)$ monotonic decreasing. The outer limits of oscillation of the means of the method (L, α) and those of the means of the method (P, θ) may be similarly defined as under:

$$(1.10)\qquad \begin{cases} \bar{L}(\infty) = \lim\limits_{\alpha\to\infty} \bar{L}(\alpha),\ \underline{L}(\infty) = \lim\limits_{\alpha\to\infty} \underline{L}(\alpha), \\ \bar{P}(1) = \lim\limits_{\theta\to 1-0} \bar{P}(\theta),\ \underline{P}(1) = \lim\limits_{\theta\to 1-0} \underline{P}(\theta), \end{cases}$$

provided of course the second pair of limits in (1.10) exists.

It will be recalled that the notion of summability (L, α), $\alpha > -1$, for a function $s(x)$, has an analogue for a sequence s_n $(n = 0, 1, \ldots)$ which Borwein [1] has discussed calling it summability (A_α). However, for a sequence, Borwein's summability (A_{-1}) is different from summability (A_{-1+0}) defined as the analogue of summability $(L, -1+0)$ in (1.9), while, for a function, Rubel does not deal with a method of summability $(L, -1)$ whose definition is analogous to that of the method (A_{-1}) and so different from that of the method $(L, -1+0)$ of (1.9). One object of the present paper is to give a definition of summability $(L, -1)$ for a function exactly analogous to Borwein's definition of summability (A_{-1}) for a sequence and show how the notion of summability $(L, -1)$ naturally and usefully supplements that of (L, α), $\alpha > -1$ (Corollary 3.1). A second object of this paper is to state and prove two results (Theorems 4.1, 4.2) which are major Tauberians for $s(x)$ summable $(L, -1)$ analogous to such theorems given by Jakimovski [5] for $s(x)$ summable (L, α), $\alpha > -1$. A final object of this paper is to show that, though the method (P, θ) is equivalent to the method $(C, 1)$, as seen from Corollary 5.1, there are some points worth noticing about the oscillations of the means of the methods (L, α), (C, k), (P, θ) stated in Theorems 5.2, 5.3.

2. Some preliminaries and lemmas

The transform $(L, -1)$ of $s(x)$ may be defined, on the analogy of the transform (L, α), $\alpha > -1$, in (1.1), by

$$(2.1)\qquad \begin{cases} L(t, -1) = \dfrac{\int_1^\infty (e^{-tx}/x)s(x)dx}{\int_1^\infty (e^{-tx}/x)dx} \qquad (t > 0) \\ \sim \dfrac{\int_1^\infty (e^{-tx}/x)s(x)dx}{\log 1/t} \qquad (t \to +0).\ ^2 \end{cases}$$

Here the integral in the numerator is assumed to be absolutely

[2] The asymptotic relation follows from a well-known limit (see e.g. (3.15) *infra*).

convergent for $0 < t < \delta$ and hence for all $t > 0$; and the lower limit of integration, unity, may be changed to any given $x_0 > 0$ without affecting the asymptotic equality in (2.1). Summability $(L, -1)$ of $s(x)$ to sum S may then be defined, in terms of

$$\lim_{t\to+0} \begin{matrix}\sup\\ \inf\end{matrix} L(t, -1) = \left.\begin{matrix}\bar{L}(-1)\\ \underline{L}(-1)\end{matrix}\right\}, \tag{2.2}$$

as follows:

$$s(x) \to S(L, -1) \text{ if } \bar{L}(-1) = \underline{L}(-1) = S. \tag{2.3}$$

Plainly summability $(L, -1)$ is regular, and its definition may be supplemented by the following:

$$s(x) = O(1)\,(L, -1) \text{ if } -\infty < \underline{L}(-1) < \bar{L}(-1) < \infty. \tag{2.3$'$}$$

Clearly the transform (L, α) of $s(x)$, defined by (1.1) for $\alpha > -1$ and defined by (2.1) for $\alpha = -1$, may be rewritten in terms of $y = 1/t$. And it then becomes a particular case of the function-to-function transform defined, in Hardy's notation ([2], § 3.7) by the absolutely convergent integral

$$\left\{\begin{array}{l} \tau(y) = \int_{x_0}^{\infty} c(y, x)s(x)dx \quad (x_0 \geqq 0,\ y > 0), \\ \text{where} \\ \text{(i) } c(y, x) \geqq 0 \text{ for all } x \text{ and } y \text{ in question,} \\ \text{(ii) } \int_{x_0}^{X} c(y, x)dx \to 0 \text{ when } y \to \infty \text{ for every finite } X, \\ \text{(iii) } \int_{x_0}^{\infty} c(y, x)dx \to 1 \text{ when } y \to \infty, \end{array}\right. \tag{2.4}$$

with the result that the transformation of $s(x)$ to $\tau(y)$ is 'normal' in Hardy's sense ([2], p. 55, Theorem 11), i.e.

$$\liminf_{x\to\infty} s(x) \leqq \lim_{y\to\infty} \begin{matrix}\sup\\ \inf\end{matrix}\, \tau(y) \leqq \limsup_{x\to\infty} s(x),$$

the limits being not necessarily finite.

For the function-to-function transform $\tau(y)$ in (2.4), there is an analogue as follows, of the essentials of a theorem for sequence-to-sequence transforms, due in principle to Vijayaraghavan, but actually formulated by Hardy ([2], p. 306, Theorem 238).

Lemma 1. *Let $s(x)$ be as stated at the outset and $\tau(y)$ as in (2.4). Suppose that $\Phi(x)$ is a positive, strictly increasing, unbounded function of $x \geqq x_0$ satisfying the conditions*:

(i) *if* $M \to \infty$, $y \to \infty$, $\Phi(y) - \Phi(M) \to \infty$, *then*

$$\int_{x_0}^{M} c(y, x)dx \to 0,$$

and if $N \to \infty$, $y \to \infty$, $\Phi(N) - \Phi(y) \to \infty$, *then*

$$\int_{N}^{\infty} c(y, x)dx \to 0, \int_{N}^{\infty} c(y, x)\{\Phi(x) - \Phi(N)\}dx \to 0;$$

(ii) *there are positive constants a and b such that*

$$s(v) - s(u) > -a\{\Phi(v) - \Phi(u)\} - b \text{ for } v > u > x_0. \tag{2.5}$$

Then

$$\tau(y) = O(1)(y \to \infty) \text{ implies } s(x) = O(1) \quad (x \to \infty).$$

The proof of Lemma 1 is omitted, as it is exactly like that of its analogue for a sequence-to-sequence transform formulated by Hardy and referred to just before Lemma 1.

LEMMA 2. *Let $s(x)$ be as stated at the outset.*

(a) *Suppose that $\Lambda(x)$ is a positive, continuous, strictly increasing, unbounded function of x such that for u and v subject to the condition $\Lambda(v) = \lambda \Lambda(u)$ for a $\lambda > 1$,*

$$\liminf_{u \to \infty} \underset{u < u' < v}{\text{lower bd}} \{s(u') - s(u)\} = -w(\lambda) > -\infty. \tag{2.6}$$

Then there are positive constants a and b such that

$$s(v) - s(u) > -a\{\log \Lambda(v) - \log \Lambda(u)\} \\ -b \text{ for } v > u > x_0 \geqq 0. \tag{2.7}$$

(b) *Suppose that* (2.6) *holds, with the implication that $w(\lambda)$ exists as a monotonic increasing function of λ in some neighbourhood of λ to the right of $\lambda = 1$. Suppose further that*

$$w(\lambda) \uparrow 0 \text{ as } \lambda \downarrow 1+0.$$

Then

$$\frac{1}{\Lambda(u)} \int_{x_0}^{u} s(x)d\{\Lambda(x)\} \to S(u \to \infty) \text{ implies } s(x) \to S(x \to \infty).$$

The principle underlying Lemma 2(a) is well-known (see, for instance, [2], p. 307, Theorem 239). Lemma 2(a), in the actual form stated, is given by Karamata and recalled with proof by Rajagopal ([7], Lemma 2A).

Lemma 2(b) is also given by Karamata ([6], p. 36, Théorème V).

Lemma 3. *If $g(u)$ is bounded for $0 \leqq u \leqq 1$, and furthermore continuous on the left at $u = 1$, and if $\varepsilon > 0$ is given (however small), then there are polynomials $h(u)$, $H(u)$ such that*

$$h(u) \leqq g(u) \leqq H(u) \quad (0 \leqq u \leqq 1),$$
$$h(1) \geqq g(1)-\varepsilon,\ H(1) \leqq g(1)+\varepsilon.$$

Lemma 3 is a corrected form of a result in the original draft of this paper. For the lemma and its proof given below the author is indebted to Dr. B. Kuttner.

Proof. Suppose that $g(u) \leqq M$ $(0 \leqq u \leqq 1)$. Since $g(u)$ is given as continuous on the left at $u = 1$, it follows that, given ε, there is a δ such that

$$g(u) \leqq g(1)+\varepsilon/3 \quad (1-\delta \leqq u \leqq 1). \tag{2.8}$$

Now define

$$\begin{cases} f(u) = M+2\varepsilon/3 \\ f(1) = g(1)+2\varepsilon/3 \end{cases} (0 \leqq u \leqq 1-\delta) \tag{2.9}$$

and define $f(u)$ in the interval $1-\delta < u \leqq 1$, so that it is continuous at $u = 1-\delta$, $u = 1$ and linear in the interval. It then clearly follows from (2.8) and (2.9) that

$$f(u) \geqq g(u)+\varepsilon/3 \quad (0 \leqq u \leqq 1). \tag{2.10}$$

But $f(u)$ is continuous for $0 \leqq u \leqq 1$. Thus, by the Weierstrass approximation theorem ([12], p. 414) there is a polynomial $H(u)$ such that

$$|H(u)-f(u)| \leqq \varepsilon/3 \qquad (0 \leqq u \leqq 1). \tag{2.11}$$

It now follows from (2.9), (2.10) and (2.11) that

$$H(u) \geqq g(u) \qquad (0 \leqq u \leqq 1),$$
$$H(1) \leqq g(1)+\varepsilon.$$

We can obviously determine $h(u)$ in a similar way.

The next two results are parallel Abelian theorems for the summability methods (L, α), $\alpha > -1$ and $(L, -1)$. The second result introduces a summability method (l), not altogether new (see e.g. [5], Theorem 6.1), which is naturally associated with the method $(L, -1)$.

Lemma 4. *For $s(x)$ assumed to be as at the outset, $s(x) \to S$ $(C, 1)$ implies $s(x) \to S$ (L, α) for every $\alpha > -1$, in the notation* *of* (1.7′), *the transform* $L(t, \alpha)$ *of* $s(x)$ *existing as the non-absolutely convergent integral*

$$L(t, \alpha) = \frac{t^{\alpha+1}}{\Gamma(\alpha+1)} \int_0^\infty e^{-tx} x^\alpha s(x) dx, \qquad t > 0. \text{ [3]}$$

Lemma 4, though perhaps not stated explicitly anywhere else, is easily proved as under. If we write

$$\psi(x) = \frac{e^{-x} x^\alpha}{\Gamma(\alpha+1)}, \quad s_1(x) = \int_0^x s(u) du \qquad (x > 0)$$

then, on the hypothesis $s(x) \to S(C, 1)$ or $s_1(x)/x \to S$ $(x \to \infty)$, the integral on the left side of the following relation exists for all $t > 0$ and the relation itself may be got by a partial integration:

$$-t^2 \int_0^\infty \psi'(tx) s_1(x) dx = t \int_0^\infty \psi(tx) s(x) dx \qquad (t > 0),$$

The integral on the right side of the above relation is the transform $L(t, \alpha)$ of $s(x)$ and so the relation can be written in the form

$$L(t, \alpha) = -\int_0^\infty \psi'(x) x \left\{ \frac{s_1(x/t)}{x/t} \right\} dx.$$

Since $s_1(x)/x \to S$ $(x \to \infty)$, an appeal to the Lebesgue theorem of dominated convergence now leads us without difficulty to the conclusion $L(t, \alpha) \to S (t \to +0)$.

Lemma 5. *For* $s(x)$ *defined as at the outset, let us write*

$$(2.12) \qquad s(x) \to S(l) \text{ if } \frac{1}{\log u} \int_1^u \frac{s(x)}{x} dx \to S (u \to \infty).$$

Then, in the notation of (2.12) *and* (2.3),

$$s(x) \to S(l) \text{ implies } s(x) \to S(L, -1),$$

the transform $L(t, -1)$ *of* $s(x)$ *in* (2.1) *existing with a non-absolutely convergent integral in the numerator.*

Proof. Assuming that $s(x) \to S(l)$, we prove the existence of the $L(t, -1)$ transform of $s(x)$ as defined by (2.1), through the

[3] In the only application made of Lemma 4 in the sequel (to establish Corollary 5.1 B) $s(x)$ is effectively positive and the convergence of the $L(t, \alpha)$ transform is necessarily absolute convergence as stipulated in our definition of (L, α) summability.

following relation where the left-hand integral exists because the condition in (2.12) is assumed and the right-hand integral is derived therefrom by an integration by parts.

$$t\int_1^\infty e^{-tx}dx\int_1^x \frac{s(y)}{y}\,dy = \int_1^\infty e^{-tx}\,\frac{s(x)}{x}\,dx \equiv I(t), \text{ say } (t>0),$$

i.e.

$$\begin{aligned} I(t) &= t\left[\int_1^{x_0}+\int_{x_0}^\infty\right]e^{-tx}dx\int_1^x \frac{s(y)}{y}\,dy \\ &= J_1+J_2 \text{ (say)}, \end{aligned}$$

where we choose $x_0 > 1$ (corresponding to any given $\varepsilon > 0$) so that

$$(S-\varepsilon)\log x < \int_1^x \frac{s(y)}{y}\,dy < (S+\varepsilon)\log x \quad \text{for} \quad x > x_0,$$

by appealing to the condition in (2.12) assumed by us. Thus

$$(2.13) \qquad I(t)\begin{Bmatrix} < J_1+(S+\varepsilon)t\int_{x_0}^\infty e^{-tx}\log x dx \\ > J_1+(S-\varepsilon)t\int_{x_0}^\infty e^{-tx}\log x dx \end{Bmatrix}.$$

Integrating by parts the infinite integral on the right side of (2.13), we get

$$\begin{aligned} t\int_{x_0}^\infty e^{-tx}\log x dx &= e^{-tx_0}\log x_0+\int_{x_0}^\infty e^{-tx}\,\frac{dx}{x} \\ &= C+\int_1^\infty e^{-tx}\,\frac{dx}{x} \end{aligned}$$

where

$$C = e^{-tx_0}\log x_0-\int_1^{x_0} e^{-tx}\,\frac{dx}{x}.$$

Hence (2.13) gives us:

$$(2.14) \qquad I(t)\begin{Bmatrix} < J_1+(S+\varepsilon)C+(S+\varepsilon)\int_1^\infty e^{-tx}\,\dfrac{dx}{x} \\ > J_1+(S-\varepsilon)C+(S-\varepsilon)\int_1^\infty e^{-tx}\,\dfrac{dx}{x} \end{Bmatrix}.$$

In (2.14), J_1 and C tend to finite limits as $t \to +0$, so that, as we wished to prove,

$$\int_1^\infty e^{-tx}\frac{s(x)}{x}\,dx \sim S\int_1^\infty e^{-tx}\frac{dx}{x}\;(t\to+0).$$

The final lemma which follows is a general result due to Rajagopal ([8], Theorem 4).

LEMMA 6. *Let* $s(x)$ *be as stated at the outset and furthermore bounded in* $(0,\infty)$. *Let* $\psi(x)$ *be non-negative and bounded above for* $x>0$, *and let*

$$\int_0^\infty \psi(x)dx = 1.$$

Then the hypothesis

$$\bar{C}(1) \equiv \limsup_{u\to\infty} \frac{1}{u}\int_0^u s(x)dx = \limsup_{x\to\infty} s(x) \equiv \bar{S}$$

implies the conclusion

$$\limsup_{t\to+0} t\int_0^\infty \psi(tx)s(x)dx = \bar{S}.$$

In particular, taking successively

$$\psi(x) = \frac{e^{-x}x^\alpha}{\Gamma(\alpha+1)}, \qquad \alpha \geqq 0,$$

$$\psi(x) = k(1-x)^{k-1},\ k\geqq 1\ (0\leqq x\leqq 1),\ \psi(x)=0 \quad (x>1),$$

we see that

$$\bar{C}(1) = \bar{S} \text{ implies } \bar{L}(\alpha) = \bar{C}(k) = \bar{S},\ \alpha\geqq 0,\ k\geqq 1,$$

in the notation of (1.4) *and* (1.5).

NOTE. The following familiar notation, employed in the sequel may be explained at this point. If two methods of summability (P) and (Q) say, for a function $s(x)$ are such that $s(x)\to S$ (P) implies $s(x)\to S$ (Q), following Hardy ([2], p. 66) we say that the method (P) is included by the method (Q) and write:

$$(P)\subseteqq(Q).$$

If the methods (P) and (Q) are such that $(P)\subseteqq(Q)$ and $(Q)\subseteqq(P)$, then following Hardy again, we say that the methods are equivalent.

3. On summabilities $(L,-1)$ and (L,α), $\alpha>-1$

The first theorem given below extends the scale of summability methods for $s(x)$ consisting of the methods (L,α) for all values of $\alpha > -1$. That the methods (L, α) form a scale is a result due to Jakimovski ([5], Theorem 3.1) stated, for $s(x)$ bounded in $(0, \infty)$, by Rubel ([11], Theorem 2.2).

THEOREM 3.1. *Let $s(x)$ be bounded and integrable in each finite positive interval of x. Then, in the notation of* (1.4) *and* (2.2), *we have, for $\beta > \alpha > -1$,*

$$\bar{L}(-1) \leqq \bar{L}(\alpha) \leqq \bar{L}(\beta),$$

$$\underline{L}(-1) \geqq \underline{L}(\alpha) \geqq \underline{L}(\beta).$$

Here $L(t, \beta)$ is defined as in (1.1) and supposed to exist as an absolutely convergent integral for $t > 0$, $\bar{L}(\beta)$ and $\underline{L}(\beta)$ being defined according to (1.4). Theorem 3.1 is the assertion that by this supposition, we ensure the existence, firstly, of $L(t, \alpha)$ and secondly, of $L(t, -1)$ defined by (2.1), as absolutely convergent integrals for $t > 0$, the associated limits $\bar{L}(\alpha)$, $\underline{L}(\alpha)$, $\bar{L}(-1)$, $\underline{L}(-1)$ of which the two last are defined by (2.2), satisfying the inequalities of Theorem 3.1 (without any condition that some or all of them are finite).

PROOF. The part of Theorem 3.1 which asserts

$$\underline{L}(\beta) \leqq \underline{L}(\alpha) \leqq \bar{L}(\alpha) \leqq \bar{L}(\beta), \qquad \beta > \alpha > -1,$$

is proved by Rubel (*loc. cit.*) for $s(x)$ bounded in $(0, \infty)$ and implicit, without this restriction on $s(x)$, in a result given by Jakimovski (*loc. cit.*). The proof which follows is that of the assertion

$$\underline{L}(\alpha) \leqq \underline{L}(-1) \leqq \bar{L}(-1) \leqq \bar{L}(\alpha), \qquad \alpha > -1. \tag{3.1}$$

We start with the proof of the existence of the $L(t, -1)$ transform of $s(x)$ defined according to (2.1), or more particularly, with the proof of the existence of

$$I(t) = \int_1^\infty e^{-tx} \frac{s(x)}{x}\, dx,$$

as an absolutely convergent integral, on the hypothesis of the existence of the $L(t, \alpha)$ transform in (1.1) as an absolutely convergent integral for $t > 0$. Now, we may clearly suppose, without loss of generality, that

$$s(x) = 0 \qquad (0 \leqq x < 1),$$

and hence, formally for the present,

$$(3.2)\quad I(t) = \frac{1}{\Gamma(\alpha+1)}\int_1^\infty e^{-tx}x^\alpha s(x)dx\int_0^\infty e^{-xy}y^\alpha dy \qquad (t>0)$$

$$(3.3)\qquad = \frac{1}{\Gamma(\alpha+1)}\int_0^\infty y^\alpha dy\int_1^\infty e^{-(t+y)x}x^\alpha s(x)dx.$$

The repeated integral in (3.3) converges absolutely for all $t > 0$, since, after replacement of $s(x)$ by $|s(x)|$ it is obviously less than

$$\frac{1}{\Gamma(\alpha+1)}\int_0^\infty y^\alpha e^{-y}dy\int_1^\infty e^{-tx}x^\alpha|s(x)|dx$$

where, by hypothesis, the inner integral converges for all $t > 0$. Hence, by Fubini's theorem, we may pass from (3.3) to (3.2) and prove that $I(t)$ exists as an absolutely convergent integral for $t > 0$.

To prove the theorem in the case of finite $\bar{L}(\alpha)$ and $\underline{L}(\alpha)$, we have to show that $t_0 > 0$ can be found, corresponding to an arbitrary small $\varepsilon > 0$, so as to make the hypothesis

$$(3.4)\quad \frac{t^{\alpha+1}}{\Gamma(\alpha+1)}\int_1^\infty e^{-tx}x^\alpha s(x)dx \begin{cases} < \bar{L}(\alpha)+\varepsilon \\ > \underline{L}(\alpha)-\varepsilon \end{cases} (0 < t < 2t_0)$$

imply (3.1) which can be written, in the notation of (2.1) and (3.2):

$$(3.5)\quad \lim_{t\to+0}\begin{matrix}\sup\\ \inf\end{matrix}\ L(t, -1) = \lim_{t\to+0}\begin{matrix}\sup\\ \inf\end{matrix}\ \frac{I(t)}{\log 1/t}\begin{Bmatrix} \leqq \bar{L}(\alpha) \\ \geqq \underline{L}(\alpha) \end{Bmatrix}.$$

Now the formula for $I(t)$ in (3.3) gives us:

$$(3.6)\quad \begin{cases} I(t) = \displaystyle\int_0^{t_0}\frac{y^\alpha dy}{\Gamma(\alpha+1)}\int_1^\infty e^{-(t+y)x}x^\alpha s(x)dx + \\ \qquad + \displaystyle\int_{t_0}^\infty \frac{y^\alpha dy}{\Gamma(\alpha+1)}\int_1^\infty e^{-(t+y)x}x^\alpha s(x)dx \\ \quad = I_1+I_2 \text{ (say)}, \end{cases}$$

where I_1 can be written, for $0 < t < t_0$:

$$(3.7)\quad \begin{cases} I_1 = \left\{\displaystyle\int_0^t + \int_t^{t_0}\right\}\frac{y^\alpha dy}{\Gamma(\alpha+1)}\int_1^\infty e^{-(t+y)x}x^\alpha s(x)dx \\ \quad = I_{11}+I_{12} \text{ (say)}, \end{cases}$$

whence we get, using (3.4)

$$I_{11} < \int_0^t y^\alpha dy \frac{\bar{L}(\alpha)+\varepsilon}{(t+y)^{\alpha+1}} < \frac{\bar{L}(\alpha)+\varepsilon}{t^{\alpha+1}} \int_0^t y^\alpha dy = \frac{\bar{L}(\alpha)+\varepsilon}{\alpha+1},$$

$$I_{12} < \int_t^{t_0} y^\alpha dy \frac{\bar{L}(\alpha)+\varepsilon}{(t+y)^{\alpha+1}} < \{\bar{L}(\alpha)+\varepsilon\} \int_t^{t_0} \frac{dy}{y} = \{\bar{L}(\alpha)+\varepsilon\} \log \frac{t_0}{t}.$$

Using the above upper estimates for I_{11} and I_{12} in (3.7), we see that, for $0 < t < t_0$,

$$(3.8) \qquad I_1 < \{\bar{L}(\alpha)+\varepsilon\} \left(\frac{1}{\alpha+1} + \log t_0\right) + \{\bar{L}(\alpha)+\varepsilon\} \log 1/t.$$

Next we find that, in (3.6),

$$(3.9) \quad \begin{cases} |I_2| \leqq \displaystyle\int_{t_0}^\infty \frac{y^\alpha e^{-y/2}}{\Gamma(\alpha+1)} dy \int_1^\infty e^{-(t+y/2)x} x^\alpha |s(x)| dx \quad (t+y/2 > t_0/2) \\ \quad < \displaystyle\int_{t_0}^\infty y^\alpha e^{-y/2} dy \frac{K}{(t+y/2)^{\alpha+1}} \text{ if } t+y/2 > t_0/2 \geqq T_0 \text{ (say)}. \end{cases}$$

To obtain (3.9) we use the fact that the existence of the $L(t, \alpha)$ transform of (1.1), as an absolutely convergent integral for $t > 0$, implies (as in [13], p. 183, Corollary 1c, and [5], Lemma 2.1)

$$\limsup_{t\to\infty} \frac{t^{\alpha+1}}{\Gamma(\alpha+1)} \int_0^\infty e^{-tx} x^\alpha |s(x)| dx \leqq \limsup_{x\to+0} |s(x)| = 0$$

since $s(x) = 0$ $(0 \leqq x < 1)$ by supposition. Hence

$$\frac{T^{\alpha+1}}{\Gamma(\alpha+1)} \int_1^\infty e^{-Tx} x^\alpha |s(x)| dx < K \text{ for } T \geqq T_0 > 0.$$

From (3.9), we have

$$(3.10) \qquad |I_2| < K \int_{t_0}^\infty \frac{y^\alpha e^{-y/2} dy}{(y/2)^{\alpha+1}} = K\, 2^{\alpha+1} \int_{t_0}^\infty \frac{e^{-y/2}}{y} dy.$$

Employing (3.8) and (3.10) in (3.6), then dividing both sides of (3.6) by $\log 1/t$ and letting $t \to +0$, we get

$$\limsup_{t\to+0} \frac{I(t)}{\log 1/t} \leqq \bar{L}(\alpha)+\varepsilon$$

which is the first inequality of (3.5), ε being arbitrary. The second inequality of (3.5) may be deduced from the first, by considering the transforms $L(t, \alpha)$ and $L(t, -1)$ of the function $-s(x)$ instead of the function $s(x)$.

To complete the proof of (3.1), we have to consider the further cases:

(a) $\bar{L}(\alpha) = \infty$, $\underline{L}(\alpha) = -\infty$,
(b) $|\bar{L}(\alpha)| < \infty$, $\underline{L}(\alpha) = -\infty$,
(c) $|\underline{L}(\alpha)| < \infty$, $\bar{L}(\alpha) = \infty$,
(d) $\bar{L}(\alpha) = \underline{L}(\alpha) = \infty$,
(e) $\bar{L}(\alpha) = \underline{L}(\alpha) = -\infty$.

Obviously (a), (b), (c) require no proof after the preceding treatment of the case $|\bar{L}(\alpha)| < \infty$, $|\underline{L}(\alpha)| < \infty$. Finally (d) is proved as follows and (e) similarly. In the notation of (3.6) and (1.1)

$$I_1 = \int_0^{t_0} \frac{y^\alpha dy}{\Gamma(\alpha+1)} \int_1^\infty e^{-(t+y)x} x^\alpha s(x) dx$$
$$= \int_0^{t_0} \frac{y^\alpha dy}{(t+y)^{\alpha+1}} L(t+y, \alpha),$$

where now $t_0 > 0$ is chosen so that $L(t, \alpha) > G$ (given arbitrarily large positive number) for $0 < t < 2t_0$, using our present hypothesis, $L(t, \alpha) \to \infty$ as $t \to +0$. Therefore, for $0 < t < t_0$,

$$I_1 > G \int_t^{t_0} \frac{y^\alpha dy}{(2y)^{\alpha+1}}$$
$$= \frac{G}{2^{\alpha+1}} \log \frac{t_0}{t}.$$

An upper estimate for $|I_2|$ as in (3.10) is still valid and, in conjunction with the above lower estimate for I_1, leads to the required result, expressible as follows in the notation of (3.5) and (3.6):

$$\lim_{t\to+0} L(t, -1) = \lim_{t\to+0} \frac{I(t)}{\log 1/t} = \lim_{t\to+0} \frac{I_1+I_2}{\log 1/t} = \infty.$$

Thus (3.1) is proved whether its extreme members are both finite, or both infinite, or finite one at a time.

Corollary 3.1. *In the notation recalled at the end of Section* 2,

$$(L, \beta) \subseteqq (L, \alpha) \subseteqq (L, -1), \qquad \beta > \alpha \geqq -1.$$

This corollary follows also from certain general considerations which may be of interest in themselves [10].

The next theorem leads on to a Tauberian counterpart of the Abelian result in Lemma 5. It is an adaptation to summability $(L, -1)$ of a technique of Karamata's ([2], p. 157, Theorem 100) which has become classical. Clearly it is one of a class of results, such as Theorems 3.2, 4.1 of the sequel, in which the hypothesis of 'boundedness below of $s(x)$' may be changed to that of 'boundedness on one side of $s(x)$', since in case $s(x)$ in the theorems is bounded above, we may replace $s(x)$ by $s^*(x) \equiv -s(x)$ which is bounded below.

THEOREM 3.2. *Let $s(x)$ be bounded and integrable in each finite positive interval of x. Let $g(u)$ be bounded and integrable in the interval $0 \leqq u \leqq 1$, and be furthermore continuous on the left at $u = 1$. Suppose that*

(i) *$s(x)$ is bounded below in $(0, \infty)$,*

(ii) *$s(x) \to S$ $(L, -1)$ in the sense of (2.3).*

Then the integral in (3.11) below is convergent absolutely for $t > 0$ and

$$\lim_{t\to+0} \frac{1}{\log 1/t} \int_1^\infty e^{-tx} g(e^{-tx}) \frac{s(x)}{x}\, dx = Sg(1). \tag{3.11}$$

PROOF. The convergence of the integral of (3.11) for $t > 0$ follows from that of the integral which is the $L(t, -1)$ transform of $s(x)$ according to (2.1), the convergence of these two integrals being in effect absolute convergence on account of hypothesis (i) being assumable (without loss of generality) as $s(x) > 0$ in $(0, \infty)$. For, in case $s(x) > -K$ in $(0, \infty)$, we may change $s(x)$ to $s(x)+K$ in both hypothesis (ii) and the conclusion (3.11) in view of the relation

$$\lim_{t\to+0} \frac{1}{\log 1/t} \int_1^\infty e^{-tx} g(e^{-tx}) \frac{dx}{x} = g(1) \tag{3.12}$$

which is easily proved in the form

$$\frac{1}{\log 1/t} \int_t^\infty e^{-x} g(e^{-x}) \frac{dx}{x} \to g(1) \qquad (t \to +0). \tag{3.13}$$

Because, we can choose $\delta > 0$ corresponding to an arbitrary $\varepsilon > 0$, so that, for $0 < t \leqq x \leqq \delta$,

$$g(e^{-x}) = g(1)+\eta(x), \text{ where } |\eta(x)| < \varepsilon;$$

and consequently, for $0 < t < \delta$,

$$
(3.14)\quad \begin{cases} \dfrac{1}{\log 1/t}\displaystyle\int_t^\infty e^{-x}g(e^{-x})\,\frac{dx}{x} \\ \qquad < \dfrac{g(1)}{\log 1/t}\displaystyle\int_t^\delta \frac{e^{-x}}{x}\,dx + \frac{1}{\log 1/t}\int_\delta^\infty e^{-x}g(e^{-x})\,\frac{dx}{x} + \\ \qquad\qquad + \dfrac{\varepsilon}{\log 1/t}\displaystyle\int_t^\delta e^{-x}\,\frac{dx}{x}, \\ \qquad > \dfrac{g(1)}{\log 1/t}\displaystyle\int_t^\delta \frac{e^{-x}}{x}\,dx + \frac{1}{\log 1/t}\int_\delta^\infty e^{-x}g(e^{-x})\,\frac{dx}{x} - \\ \qquad\qquad - \dfrac{\varepsilon}{\log 1/t}\displaystyle\int_t^\delta e^{-x}\,\frac{dx}{x}. \end{cases}
$$

The first term on the right side of (3.14) tends to $g(1)$ as $t \to +0$, since

$$
(3.15)\quad \log 1/t - \int_t^\infty e^{-x}\,\frac{dx}{x} \to \text{Euler's constant} \quad (t \to +0).
$$

The second term on the right side of (3.14) tends to 0 as $t \to +0$, since $g(e^{-x})$ is bounded for $\delta \leqq x < \infty$. And finally the absolute value of each of the last terms on the right side of (3.14) is less than

$$
\frac{\varepsilon}{\log 1/t}\int_t^\infty e^{-x}\,\frac{dx}{x} \to \varepsilon \qquad (t \to +0)
$$

by (3.15) again. Hence (3.14) readily gives us (3.13), or equivalently (3.12).

We proceed to prove (3.11), supposing that $s(x) > 0$ in $(0, \infty)$. By definition, hypothesis (ii) is that

$$
\int_1^\infty e^{-tx}\,\frac{s(x)}{x}\,dx \sim S\int_1^\infty e^{-tx}\,\frac{dx}{x} \qquad (t \to +0)
$$

and yields the following relation when we replace t by $(n+1)t$, $n = 0, 1, 2, \ldots$:

$$
(3.16)\qquad \int_1^\infty e^{-tx}e^{-ntx}\,\frac{s(x)}{x}\,dx \sim S\int_1^\infty e^{-tx}e^{-ntx}\,\frac{dx}{x}.
$$

From (3.16) we see that, for any polynomial $G(u)$ in $u = e^{-tx}$ $(t > 0,\ 0 \leqq x < \infty)$, we have the relation:

$$
(3.17)\qquad \int_1^\infty e^{-tx}G(e^{-tx})\,\frac{s(x)}{x}\,dx \sim S\int_1^\infty e^{-tx}G(e^{-tx})\,\frac{dx}{x}.
$$

In (3.17), we may take $G(u)$ to be successively the polynomials $H(u)$, $h(u)$ associated with $g(u)$ as in Lemma 3. Recalling the relation between $g(u)$ and $H(u)$ in Lemma 3, along with our supposition $s(x) > 0$, we get from (3.17) with $G(u) = H(u)$

$$\text{(3.18)} \quad \begin{cases} \displaystyle\int_1^\infty e^{-tx} g(e^{-tx}) \frac{s(x)}{x}\, dx \leqq \int_1^\infty e^{-tx} H(e^{-tx}) \frac{s(x)}{x}\, dx \\ \displaystyle\qquad\qquad\qquad\qquad \sim S \int_1^\infty e^{-tx} H(e^{-tx}) \frac{dx}{x}. \end{cases}$$

Now we have, by Lemma 3,

$$H(1) < g(1) + \varepsilon$$

and, by continuity of $H(u)$ and $g(u)$ to the left of $u = 1$, there is a $\delta > 0$ such that

$$\text{(3.19)} \qquad H(u) < g(u) + \varepsilon \qquad (1-\delta \leqq u \leqq 1).$$

As $t \to +0$ finally, we may suppose that $e^{-t} \leqq 1-\delta$. For any fixed $t > 0$ subject to this condition, we get by using (3.19) in the final integral of (3.18):

$$\begin{aligned} \int_1^\infty e^{-tx} H(e^{-tx}) \frac{dx}{x} &= \left\{ \int_{e^{-tx}=e^{-t}}^{e^{-tx}=1-\delta} + \int_{e^{-tx}=1-\delta}^{e^{-tx}=0 (x=\infty)} \right\} e^{-tx} H(e^{-tx}) \frac{dx}{x} \\ &< \int_{e^{-tx}=e^{-t}}^{e^{-tx}=1-\delta} e^{-tx} g(e^{-tx}) \frac{dx}{x} + \varepsilon \int_{e^{-tx}=e^{-t}}^{e^{-tx}=1-\delta} e^{-tx} \frac{dx}{x} + \\ &\quad + \int_{e^{-tx}=1-\delta}^{x=\infty} e^{-tx} H(e^{-tx}) \frac{dx}{x}. \end{aligned}$$

Hence

$$\text{(3.20)} \quad \begin{aligned} \int_1^\infty e^{-tx} H(e^{-tx}) \frac{dx}{x} &< \int_1^\infty e^{-tx} g(e^{-tx}) \frac{dx}{x} + \varepsilon \int_{e^{-tx}=e^{-t}}^{e^{-tx}=1-\delta} e^{-tx} \frac{dx}{x} + \\ &\quad + \int_{e^{-tx}=1-\delta}^{x=\infty} e^{-tx} H(e^{-tx}) \frac{dx}{x} - \int_{e^{-tx}=1-\delta}^{x=\infty} e^{-tx} g(e^{-tx}) \frac{dx}{x}. \end{aligned}$$

Since H, g are bounded, it is easily seen that, for any fixed $\delta > 0$, the third and the fourth integrals on the right side of (3.20) are each o $(\log 1/t)$ as $t \to +0$; also the second integral is less than

$$\varepsilon \int_1^\infty e^{-tx} \frac{dx}{x} \sim \varepsilon \log 1/t \qquad (t \to +0).$$

Therefore, using (3.20) in (3.18), then dividing both sides of (3.18) by $\log 1/t$ and letting $t \to +0$ we obtain

$$(3.21) \qquad \limsup_{t\to+0} \frac{1}{\log 1/t}\int_1^\infty e^{-tx}g(e^{-tx})\,\frac{s(x)}{x}\,dx \leqq Sg(1).$$

Similarly,

$$(3.22) \qquad \liminf_{t\to+0} \frac{1}{\log 1/t}\int_1^\infty e^{-tx}g(e^{-tx})\,\frac{s(x)}{x}\,dx \geqq Sg(1),$$

and (3.11), which we require, follows from (3.21) and (3.22) together.

4. Tauberian theorems for summability $(L, -1)$

The theorems of this section may be viewed as successive consequences of Theorem 3.2.

THEOREM 4.1. *Let $s(x)$ be bounded and integrable in each finite positive interval of x. Suppose that*

(i) *$s(x)$ is bounded below in $(0, \infty)$,*

(ii) *$s(x) \to S$ $(L, -1)$ in the sense of* (2.3). *Then, in the sense of* (2.12)

$$s(x) \to S(l).$$

PROOF. Theorem 4.1 is the particular case of Theorem 3.2 with $g(x)$ defined thus:

$$g(x) = \begin{cases} 0 & \text{for } \; 0 \leqq x \leqq e^{-1} \\ x^{-1} & \text{for } \; e^{-1} \leqq x < \infty. \end{cases}$$

The following corollary to Theorem 4.1 gives the essential content of a theorem proved by Jakimovski ([5], Theorem 6.1) independently of Theorem 4.1, by means of his $M(\alpha, \beta)$ transform and Frullani's integral. In the proof given below, only Lemma 1 and Lemma 2(a) are required in addition to Theorem 4.1.

COROLLARY 4.1 (Jakimovski). *If $s(x)$ is defined as in Theorem 4.1, then the hypotheses*

(i) $\liminf_{u\to\infty} \operatorname{lower\,bd}_{u<u'<\lambda u} \{s(u')-s(u)\} = -w(\lambda) > -\infty$ *for a* $\lambda > 1$,

(ii) *$s(x) \to S$ (L, α), $\alpha > -1$, in the notation of* (1.7′), *together imply in the notation of* (2.12):

$$s(x) \to S(l).$$

Proof. In Lemma 2(a), let $\Lambda(x) = x$. Then hypothesis (i) of our corollary yields:

$$(4.1) \quad s(v)-s(u) > -a\,(\log v-\log u)-b \quad \text{for} \quad v > u > 0.$$

Next, in (2.4), let

$$(4.2) \qquad c(y, x) = \frac{e^{-x/y}x^{\alpha}}{y^{\alpha+1}\Gamma(\alpha+1)}, \ \alpha > -1, \ x > x_0 = 0.$$

Then the transform $\tau(y)$ of (2.4) is the same as the transform $L(1/y, \alpha)$ as defined by (1.1). Also, the choice of $c(y, x)$ in (4.2), together with the choice $\Phi(x) = \log x$ satisfies the condition (i) of Lemma 1, as we can easily verify [4]. Furthermore, in consequence of hypothesis (i) of our corollary, with its implication (4.1), $s(x)$ satisfies the condition (ii) of Lemma 1 with $\Phi(x) = \log x$ again. Hence, firstly, hypothesis (ii) of our corollary, or even the broader hypothesis that $s(x) = O(1)\ (L, \alpha)$ in the sense of (1.8′), implies $s(x) = O(1)$, $x \to \infty$, by Lemma 1. And, secondly, hypothesis (ii) of our corollary implies $s(x) \to S(L, -1)$ by Corollary 3.1. From the two implications last stated, it follows by Theorem 4.1 that $s(x) \to S(l)$ as we wished to prove.

The next theorem is the analogue for summability $(L, -1)$ of Jakimovski's theorem for summability (L, α), $\alpha > -1$ ([5], Theorem 5.6 with $M = 1$, $\beta = 0$, $c = 0$).

[4]

$$\int_0^M \frac{e^{-x/y}x^{\alpha}}{y^{\alpha+1}\Gamma(\alpha+1)}\,dx = \frac{1}{\Gamma(\alpha+1)}\int_0^{M/y} e^{-u}u^{\alpha}\,du < \frac{1}{\Gamma(\alpha+1)}\,(M/y)^{\alpha+1} = o(1)$$

if $\log y/M \to \infty$ or $y/M \to \infty$ and $\alpha > -1$;

$$\int_N^{\infty} \frac{e^{-x/y}x^{\alpha}}{y^{\alpha+1}\Gamma(\alpha+1)}\,dx = \frac{1}{\Gamma(\alpha+1)}\int_{N/y}^{\infty} e^{-u}u^{\alpha}\,du = o(1)$$

if $\log N/y \to \infty$ or $N/y \to \infty$, since $\alpha > -1$ and $\int_0^{\infty} e^{-u}u^{\alpha}\,du < \infty$; and using the fact $\log X \leqq X-1$ $(X > 0)$

$$\begin{aligned}\int_N^{\infty} \frac{e^{-x/y}x^{\alpha}}{y^{\alpha+1}\Gamma(\alpha+1)}\log\frac{x}{N}\,dx &< \int_N^{\infty} \frac{e^{-x/y}x^{\alpha}}{y^{\alpha+1}\Gamma(\alpha+1)}\cdot\frac{x}{N}\cdot dx \\ &= \frac{1}{\Gamma(\alpha+1)}\cdot\frac{y}{N}\int_{N/y}^{\infty} e^{-u}u^{\alpha}\,du \\ &< \frac{1}{\Gamma(\alpha+1)}\int_{N/y}^{\infty} e^{-u}u^{\alpha}\,du = o(1)\end{aligned}$$

as we can assume y to be less than N if $\log N/y \to \infty$ or $N/y \to \infty$, and again, since $\int_0^{\infty} e^{-u}u^{\alpha}\,du < \infty$, $\alpha > -1$.

THEOREM 4.2. *If $s(x)$ is bounded and integrable in each finite positive interval of x, then the hypotheses*

(i) $\liminf_{u\to\infty} \operatorname{lower\ bd}_{u<u'<u^\lambda} \{s(u')-s(u)\} = -w(\lambda) \uparrow 0 \ (\lambda \to 1+0)$,

(ii) $s(x) \to S \ (L, -1)$ *in the sense of* (2.3), *together imply*

$$s(x) \to S \ (x \to \infty).$$

A result intermediate between Theorems 4.1 and 4.2 is the following Theorem 4.2′ which invites comparison with Corollary 4.1 and which, in fact, may be proved like that corollary.

THEOREM 4.2′. *If, in Theorem* 4.2, *hypothesis* (i) *alone is changed to*

(i′) $\liminf_{u\to\infty} \operatorname{lower\ bd}_{u<u'<u^\lambda} \{s(u')-s(u)\} = -w(\lambda) > -\infty$ *for a* $\lambda > 1$,

then the conclusion will be changed to the following in the sense of (2.12):

$$s(x) \to S(l).$$

PROOF OF THEOREM 4.2′. Lemma (2a) with $\Lambda(x) = \log x$ shows that hypothesis (i′) implies:

(4.3) $s(v)-s(u) > -a\,(\log\log v - \log\log u) - b,\ v > u > x_0 > 0.$

Also, the transform $\tau(y)$ of $s(x)$ in (2.4), with

$$c(y, x) = \frac{e^{-x/y}/x}{\log y},\quad x \geqq 1, \tag{4.4}$$

is, to recall (2.1), asymptotically equal (as $y \to \infty$) to the transform $L(1/y, -1)$ of $s(x)$. Then, firstly, the condition (i) of Lemma 1 holds for the function-to-function transform $\tau(y)$ with the choice of $c(y, x)$ in (4.4) and the choice $\Phi(x) = \log\log x$ by an argument essentially the same as that used by Rangachari [9] in the corresponding case of the sequence-to-function transform $\sum c_n(x)s_n$ [5]. And, secondly, the condition (ii) of Lemma 1, with

[5]

$$\int_1^M \frac{e^{-x/y}}{x\cdot\log y}\,dx < \frac{1}{\log y}\int_1^M \frac{dx}{x} = \frac{\log M}{\log y} = o(1)$$

if $\log y/\log M \to \infty$;

$$\int_N^\infty \frac{e^{-x/y}}{x\cdot\log y}\,dx < \frac{1}{\log y}\cdot\frac{1}{N}\int_N^\infty e^{-x/y}\,dx = \frac{1}{\log y}\cdot\frac{y}{N}e^{-N/y} = o(1)$$

as we may assume y to be less than N, if $\log N/\log y \to \infty$; and lastly, using the fact that $\log X \leqq X-1$ $(X > 0)$ (*Continued on page* 186)

$\Phi(x) = \log\log x$ is realized in (4.3). Hence, by Lemma 1, $s(x) = O(1)$, $x \to \infty$, in Theorem 4.2′. Since also $s(x) \to S(L, -1)$, it follows by Theorem 4.1 that $s(x) \to S(l)$ as required.

PROOF OF THEOREM 4.2. Hypothesis (i) of Theorem 4.2 implies hypothesis (i′) of Theorem 4.2′. Consequently, by Theorem 4.2′, $s(x) \to S(l)$ in Theorem 4.2; and $s(x) \to S(l)$ leads to the conclusion $s(x) \to S$ by Lemma 2(b), since hypothesis (i) of Theorem 4.2 is just the supposition $w(\lambda) \uparrow 0$ $(\lambda \downarrow 1+0)$ of Lemma 2(b) with $\Lambda(x) = \log x$.

The easy deduction from Theorem 4.2 given below as a corollary is the precise analogue for a function $s(x)$ of a theorem by Rangachari for a sequence s_n ([9], Theorem I (L)).

COROLLARY 4.2. *Let $a(x)$ be bounded and integrable in every finite positive interval of x; and let*

$$s(x) = \int_0^x a(u)du \qquad (x > 0).$$

Then the two suppositions

$$\text{(i)} \qquad a(x) = O_L\left(\frac{1}{x\log x}\right) \qquad (x \to \infty),$$

$$\text{(ii)} \qquad s(x) \to S\ (L, -1)$$

together imply

$$s(x) \to S \qquad (x \to \infty).$$

[5] (*Continuation of page* 185)

$$\begin{aligned}
\int_N^\infty \frac{e^{-x/y}}{x\cdot\log y}\log\frac{\log x}{\log N}\,dx &< \frac{1}{\log y}\int_N^\infty \frac{e^{-x/y}}{x}\left(\frac{\log x}{\log N}-1\right)dx \\
&< \frac{1}{\log y\cdot\log N}\int_N^\infty \frac{e^{-x/y}}{x}\log\frac{x}{N}\,dx \\
&= \frac{1}{\log y\cdot\log N}\int_0^\infty \frac{e^{-(x+N)/y}}{x+N}\log(1+x/N)dx \\
&< \frac{1}{\log y\cdot\log N}\int_0^\infty \frac{e^{-(x+N)/y}}{x+N}\,\frac{x}{N}\,dx \\
&< \frac{1}{\log y\cdot\log N}\cdot\frac{1}{N^2}\int_0^\infty x\,e^{-(x+N)/y}\,dx \\
&= \frac{1}{\log y\cdot\log N}\cdot\frac{y^2}{N^2}\,e^{-N/y} \\
&< \frac{1}{\log y\cdot\log N} = o(1)
\end{aligned}$$

if $\log N/\log y \to \infty$ in which case we may assume y to be less than N.

5. Inclusion and oscillation theorems for the methods of summability in Section 1

In Theorem 5.1 which follows, $s(x)$ is not necessarily bounded in $(0, \infty)$, but $\bar{P}(\theta)$, $\underline{P}(\theta)$, $\bar{C}(1)$, and $\underline{C}(1)$, defined according to (1.5) and (1.6), are finite. An example of such a function $s(x)$ is $s(x) \equiv x \sin x$, which is considered in the addendum.

THEOREM 5.1. *Let $s(x)$ be bounded and integrable in each finite positive interval of x. Then, either of the two assumptions, $s(x) = O(1)$ (P, θ) and $s(x) = O(1)(C, 1)$ in the sense of (1.8′), implies the other. Furthermore we have, with either assumption,*

$$\text{(5.1)} \quad \frac{\underline{C}(1)-\theta\bar{C}(1)}{1-\theta} \leqq \underline{P}(\theta) \leqq \underline{C}(1) \leqq \bar{C}(1) \leqq \bar{P}(\theta) \leqq \frac{\bar{C}(1)-\theta\underline{C}(1)}{1-\theta}.$$

PROOF. Suppose first that $\bar{P}(\theta)$, $\underline{P}(\theta)$ are finite for a θ such that $0 < \theta < 1$. Then, from definition (1.6), we have, for any given positive ε,

$$P(u, \theta) \equiv \frac{1}{u-\theta u}\int_{\theta u}^{u} s(x)dx < \bar{P}(\theta)+\varepsilon,\ u > u_0 = u_0(\varepsilon).$$

In the notation of (1.2), the above inequality is

$$\text{(5.2)} \qquad s_1(u)-s_1(\theta u) < (1-\theta)u[\bar{P}(\theta)+\varepsilon], \qquad u > u_0.$$

$s_1(u)$ being an indefinite integral is bounded in any finite range of u; hence there is a K (depending on u_0, and thus on ε, but fixed once u_0 has been chosen) such that

$$\text{(5.3)} \qquad |s_1(u)| \leqq K \qquad (u \leqq u_0).$$

For any $u > u_0$ let integer $n = n(u)$ be such that $u\theta^n \leqq u_0 < u\theta^{n-1}$. Then we write

$$s_1(u) = \sum_{i=0}^{n-1} \{s_1(u\theta^i)-s_1(u\theta^{i+1})\}+s_1(u\theta^n),$$

and get from (5.2) and (5.3):

$$\begin{aligned} s_1(u) &\leqq (1-\theta)u[\bar{P}(\theta)+\varepsilon](\sum_{i=0}^{n-1}\theta^i)+K \\ &= u[\bar{P}(\theta)+\varepsilon]-u\theta^n[\bar{P}(\theta)+\varepsilon]+K. \end{aligned}$$

Since $\bar{P}(\theta)$ is finite and $u\theta^n$ is bounded, being such that $u_0\theta < u\theta^n \leqq u_0$, we get

$$s_1(u) < u[\bar{P}(\theta)+\varepsilon]+K'+K, \qquad u > u_0,$$

where K' is a constant. For sufficiently large u, $K'+K < \varepsilon u$ and thus

$$C(u, 1) \equiv \frac{s_1(u)}{u} < \bar{P}(\theta)+2\varepsilon.$$

Letting $u \to \infty$, we find that $\bar{C}(1)$ is finite and satisfies the inequality $\bar{C}(1) \leqq \bar{P}(\theta)$. This result along with the corresponding one for lower limits (similarly proved), gives us

$$\underline{P}(\theta) \leqq \underline{C}(1) \leqq \bar{C}(1) \leqq \bar{P}(\theta). \tag{5.4}$$

Now we have only to use the identity

$$P(u, \theta) = \frac{C(u, 1)-\theta C(\theta u, 1)}{1-\theta},$$

take upper or lower limits of both sides as $u \to \infty$ and get the following inequalities which, together with (5.4), give us the desired conclusions:

$$\frac{\underline{C}(1)-\theta\bar{C}(1)}{1-\theta} \leqq \underline{P}(\theta) \leqq \bar{P}(\theta) \leqq \frac{\bar{C}(1)-\theta\underline{C}(1)}{1-\theta}. \tag{5.5}$$

Suppose next that $\bar{C}(1)$ and $\underline{C}(1)$ are finite and θ is any given number such that $0 < \theta < 1$. Then we prove that $\bar{P}(\theta)$ and $\underline{P}(\theta)$ are finite and satisfy (5.5). The proof of (5.4) is as before.

Corollary 5.1A. *If* $\bar{C}(1) = \underline{C}(1) = S$ *(finite), then* $\bar{P}(\theta) = \underline{P}(\theta) = S$ *and conversely, i.e. summability* $(C, 1)$ *and summability* (P, θ) *are equivalent.*

The above corollary is due to Dr. B. Kuttner and was kindly communicated by him in a letter to the author.

Corollary 5.1B. *For* $s(x)$ *bounded and integrable in each finite positive interval of* x *and also bounded on one side in* $(0, \infty)$, *summability* (L, α) *for all* $\alpha > -1$, *summability* (C, k) *for all* $k \geqq 1$, *and summability* (P, θ) *for all* θ *such that* $0 < \theta < 1$, *are equivalent.*

Corollary 5.1B follows from Corollary 5.1A taken in conjunction with Lemma 4 and the fact that, for $s(x)$ as in Corollary 5.1B,

(i) $(C, k) \subseteqq (C, 1)$, $k > 1$,

(ii) $(L, \alpha) \subseteqq (C, 1)$, $\alpha > -1$ ([5], Theorem H).

In addition to the particular cases stated as part of Lemma 6, we may have the case

$$\psi(x) = \frac{1}{1-\theta} \qquad (\theta \leqq x \leqq 1)$$
$$= 0 \qquad \text{(otherwise)}$$

which makes

$$t\int_0^\infty \psi(tx)s(x)dx = \frac{1}{u(1-\theta)}\int_{\theta u}^u s(x)dx = P(u, \theta) \qquad (u = 1/t).$$

From Lemma 6 thus augmented [6] we get the following theorem.

THEOREM 5.2. *Let $s(x)$ be bounded and integrable in each finite positive interval of x and also bounded in $(0, \infty)$. Then the hypothesis*

$$\bar{C}(1) \equiv \limsup_{u\to\infty} \frac{1}{u}\int_0^u s(x)dx = \limsup_{x\to\infty} s(x) \equiv \bar{S},$$

implies the conclusion

$$\bar{L}(\alpha) = \bar{C}(k) = \bar{P}(\theta) = \bar{S}$$

for all $\alpha \geqq 0$, $k > 0$, $0 < \theta < 1$, the notation being that of (1.4), (1.5) *and* (1.6).

The final result which follows is in the same class as the preceding.

THEOREM 5.3. (i) *Let $s(x)$ be bounded and integrable in each finite positive interval of x and furthermore either slowly increasing or slowly decreasing. Then*

$$\bar{P}(1) \equiv \lim_{\theta\to 1-0} \bar{P}(\theta), \quad \underline{P}(1) \equiv \lim_{\theta\to 1-0} \underline{P}(\theta)$$

both exist (whether they be finite or not), and

$$\bar{P}(1) = \limsup_{x\to\infty} s(x) = \bar{S}, \quad \underline{P}(1) = \liminf_{x\to\infty} s(x) = \underline{S}.$$

(ii) *In case $s(x)$ is also bounded in $(0, \infty)$ we have*

$$\bar{L}(\infty) = \bar{P}(1) = \bar{S}, \quad \underline{L}(\infty) = \underline{P}(1) = \underline{S},$$

the notation being that of (1.10).

PROOF. (i) We shall prove the required result on the hypothesis that $s(x)$ is slowly increasing, i.e.

$$\limsup_{u\to\infty} \ \underset{u<x<\lambda u}{\text{lower bd}} \{s(x)-s(u)\} = w(\lambda) \downarrow 0 \text{ as } \lambda \downarrow 1+0.$$

[6] This augmentation of Lemma 6 was suggested by Dr. Kuttner. Theorem 5.2 had been originally obtained by combining Lemma 6 and Theorem 5.1.

For $\lambda > 1$, we have the identity:

$$-s(u) = -\frac{1}{(\lambda-1)u}\int_u^{\lambda u} s(x)dx + \frac{1}{(\lambda-1)u}\int_u^{\lambda u}\{s(x)-s(u)\}dx.$$

Taking upper limits as $u \to \infty$ of the two sides of this identity, we get

$$-\underline{S} \leqq -\underline{P}(1/\lambda)+w(\lambda).$$

Hence, recalling that $\underline{P}(1/\lambda) \geqq \underline{S}$ universally, we have

$$-\underline{S}-w(\lambda) \leqq -\underline{P}(1/\lambda) \leqq -\underline{S}$$

whence we obtain when $\lambda \to 1+0$:

$$-\underline{P}(1) = -\underline{S}.$$

We then complete the proof, showing that $\bar{P}(1) = \bar{S}$, by arguments similar to the above applied to the identity:

$$s(u) = \frac{1}{(1-\theta)u}\int_{\theta u}^{u} s(x)dx + \frac{1}{(1-\theta)u}\int_{\theta u}^{u}\{s(u)-s(x)\}dx \quad (0<\theta<1).$$

The preceding argument tacitly assumes that $\underline{S}$ and $\bar{S}$ are finite. In the case of one or both of $\underline{S}$ and $\bar{S}$ being non-finite, the modification to be made in the argument is obvious.

(ii) From a theorem of Rubel ([11], Theorem 3.1) we have now the additional equalities

$$\bar{L}(\infty) = \bar{P}(1),\ \underline{L}(\infty) = \underline{P}(1),$$

which, in conjunction with the equalities proved in part (i), lead to the desired conclusion.

An example of Theorem 5.3 is furnished by the function $s(x) = \sin\log x = \mathscr{I}x^i$ (the imaginary part of x^i) considered by Rubel ([11], pp. 1001—2). For this function

$$\bar{S} = 1,\ \underline{S} = -1,$$

while Rubel has shown that

$$\bar{L}(\alpha) = \frac{|\Gamma(\alpha+1+i)|}{\Gamma(\alpha+1)},\quad \underline{L}(\alpha) = -\frac{|\Gamma(\alpha+1+i)|}{\Gamma(\alpha+1)},$$
$$\bar{L}(\infty) = 1,\quad \underline{L}(\infty) = -1.$$

On the other hand,

$$P(u, \theta) = \frac{1}{u-\theta u}\int_{\theta u}^{u} s(x)dx = \mathscr{I}\left\{\frac{u^{1+i}-(\theta u)^{1+i}}{u(1-\theta)(1+i)}\right\}$$

$$= \mathscr{I}\left\{\frac{[(\cos\log u + i\sin\log u)-(\theta\cos\log u\theta + i\theta\sin\log u\theta)](1-i)}{2(1-\theta)}\right\}$$

$$= \frac{1}{2(1-\theta)}\{-\cos\log u + \theta\cos\log u\theta + \sin\log u - \theta\sin\log u\theta\}$$

$$= \frac{1}{2(1-\theta)}\{(-1+\theta\cos\log\theta - \theta\sin\log\theta)\cos\log u + (1-\theta\cos\log\theta - \theta\sin\log\theta)\sin\log u\}$$

$$= \frac{U\cos\log u + V\sin\log u}{2(1-\theta)}$$

where

$$U = -1+\theta\cos\log\theta - \theta\sin\log\theta,$$
$$V = 1-\theta\cos\log\theta - \theta\sin\log\theta.$$

Hence

$$\bar{P}(\theta) = \frac{\sqrt{(U^2+V^2)}}{2(1-\theta)}, \quad \underline{P}(\theta) = -\frac{\sqrt{(U^2+V^2)}}{2(1-\theta)}$$
$$\bar{P}(1) = 1, \quad \underline{P}(1) = -1.$$

Addendum

Remarks on the existence of $\bar{P}(1)$ *and* $\underline{P}(1)$. The remarks which follow have been kindly communicated to the author by Dr. L. A. Rubel and Dr. B. Kuttner.

Dr. Rubel suggests that his assertion without proof in [11], Theorem 1.3, viz. that $\bar{P}(1)$ and $\underline{P}(1)$ exist for $s(x)$ bounded in $(0, \infty)$, may be established as follows.

Given u, ξ such that $0 < u$, $\xi < 1$, we can choose a positive integer $n = n(\xi)$ such that $\xi^{n+1} < u < \xi^n$. Then $\xi^n - u \to 0$ as $\xi \to 1-0$. Also it is known ([11], Theorem 1.1) that $\bar{P}(\theta)$ is a continuous function of θ for $0 < \theta < 1$ when $s(x)$ is bounded in $(0, \infty)$. Therefore

$$\bar{P}(u) = \lim_{\xi\to 1-0} \bar{P}(\xi^n).$$

But, by another known result ([11], Theorem 1.2), we have $\bar{P}(\xi^n) \leqq \bar{P}(\xi)$, and so

$$\bar{P}(u) = \lim_{\xi\to 1-0} \bar{P}(\xi^n) \equiv \liminf_{\xi\to 1-0} \bar{P}(\xi^n) \leqq \liminf_{\xi\to 1-0} \bar{P}(\xi).$$

Since u and ξ are independent of each other, we get from the extreme members of the last step:

$$\limsup_{u\to 1-0} \bar{P}(u) \leqq \liminf_{\xi\to 1-0} \bar{P}(\xi),$$

while, by definition,

$$\liminf_{\xi\to 1-0} \bar{P}(\xi) \leqq \limsup_{u\to 1-0} \bar{P}(u).$$

Hence $\bar{P}(1)$ exists and is given by

$$\bar{P}(1) = \liminf_{\theta\to 1-0} \bar{P}(\theta) = \limsup_{\theta\to 1-0} \bar{P}(\theta).$$

Similarly $\underline{P}(1)$ exists.

When $s(x)$ is unbounded in $(0, \infty)$ above as well as below, we may have $\bar{P}(1) = \infty$, $\underline{P}(1) = -\infty$, as Dr. Kuttner shows by considering the function

$$s(x) = x \sin x.$$

For this function

$$P(u, \theta) = \frac{1}{u-\theta u}\int_{\theta u}^{u} x \sin x \, dx \qquad (0 \leqq \theta < 1)$$

$$= \frac{-\cos u + \theta \cos \theta u}{1-\theta} + O\left(\frac{1}{u}\right),$$

on integration by parts. Thus

$$\text{(a)} \qquad \begin{cases} \bar{P}(\theta) = \dfrac{1}{1-\theta} \limsup_{u\to\infty} F(u), \quad \underline{P}(\theta) = \dfrac{1}{1-\theta} \liminf_{u\to\infty} F(u), \\ \text{where} \quad F(u) = -\cos u + \theta \cos \theta u. \end{cases}$$

First, if θ is irrational, then by Kronecker's theorem (e.g. Hardy and Wright [3], p. 380, Theorem 444) we can find arbitrarily large values of u such that θu is arbitrarily near to an even multiple of π, and u to an odd multiple of π. Also, we can find arbitrarily large u such that θu is arbitrarily near to an odd multiple of π, and u to an even multiple of π. Thus from (a),

$$\text{(b)} \qquad \bar{P}(\theta) = \frac{1+\theta}{1-\theta}, \quad \underline{P}(\theta) = -\left(\frac{1+\theta}{1-\theta}\right).$$

Next, if θ is rational, let $\theta = p/q$ (p, q integers with no common factor). Then, in (a),

$$F(u) = -\cos u + \frac{p}{q} \cos \frac{p}{q} u$$

has period $2\pi q$, so that

$$\limsup_{u\to\infty} F(u) = \sup_{0\leqq u\leqq 2\pi q} F(u)$$

(with a corresponding result for lim inf).

Now

(c) $$\int_0^{2\pi q} \{F(u)\}^2 du = \pi q\left(1+\frac{p^2}{q^2}\right),$$

(d) $$\int_0^{2\pi q} F(u) du = 0,$$

and trivially,

$$|F(u)| \leqq 1+\frac{p}{q};$$

so that, from (c), we have

(e) $$\int_0^{2\pi q} |F(u)|\, du \geqq \frac{1}{1+\dfrac{p}{q}}\int_0^{2\pi q}\{F(u)\}^2 du = \frac{\pi q\left(1+\dfrac{p^2}{q^2}\right)}{1+\dfrac{p}{q}}.$$

If we write, as usual,

$$F^+(u) = \max\big(0,\, F(u)\big),\quad F^-(u) = \max\big(0,\, -F(u)\big)$$

it follows, from (d) and (e) that

$$\int_0^{2\pi q} F^+(u)du = \int_0^{2\pi q} F^-(u)du = \tfrac{1}{2}\int_0^{2\pi q}|F(u)|\,du$$

$$\geqq \frac{\pi q\left(1+\dfrac{p^2}{q^2}\right)}{2\left(1+\dfrac{p}{q}\right)}.$$

Hence, clearly

$$\left.\begin{array}{l}\sup F^+(u)\\ \sup F^-(u)\end{array}\right\} \geqq \frac{1}{4}\,\frac{\left(1+\dfrac{p^2}{q^2}\right)}{\left(1+\dfrac{p}{q}\right)} = \frac{1+\theta^2}{4(1+\theta)},$$

and, by (a),

(f) $$\bar{P}(\theta) \geqq \frac{1+\theta^2}{4(1-\theta^2)},\quad \underline{P}(\theta) \leqq -\frac{1+\theta^2}{4(1-\theta^2)}.$$

It is evident, from (b) in the case of irrational θ and from (f) in the case of rational θ, that $\bar{P}(\theta) \to \infty$, and $\underline{P}(\theta) \to -\infty$ as $\theta \to 1-0$, for $s(x) = x\sin x$.

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Added in proof. Recently B. Kwee has given (Proc. Cambridge Philos. Soc. 63 (1967), 401—405) the analogue of Theorem 4.2 for a sequence $\{s_n\}$ and Borwein's sequence-to-function transform of s_n [1] corresponding to the transform $(L, -1)$ of $s(x)$.

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