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On a class of star-like functions [1]

by

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1. Introduction

A function $f(z) = z+\sum_{n=2}^{\infty} a_n z^n$, regular and univalent in $|z| < 1$, is said to be star-like if $\text{Re}\{zf'(z)/f(z)\} \geqq 0$ in $|z| < 1$. Various sub classes of the class of star-like functions have been considered by different authors. The class S_α of star-like functions of order $\alpha (0 \leqq \alpha \leqq 1)$ i.e. the class of star-like functions $f(z)$ for which $\text{Re}\{zf'(z)/f(z)\} \geqq \alpha$, was first introduced by Robertson [1] and has been studied among others by Merkes [5] and Schild [4]. The class $S_{\frac{1}{2}}$ is particularly important. Marx [2] and Strohäcker [6] proved that if $f(z)$ maps $|z| < 1$ on to a convex domain, then $f(z) \in S_{\frac{1}{2}}$. Somewhat later Gabriel [3] showed that the functions of the class $S_{\frac{1}{2}}$ played an important role in the solution of certain differential equations. In the present paper we investigate the class $\bar{S}$ of star-like functions $f(z)$ such that $\text{Re}\{zf'(z)/f(z)\}$ lies in the circle with centre at $(1, 0)$ and radius unity i.e. the class of functions $f(z)$ such that $|zf'(z)/f(z)-1| \leqq 1$ in $|z| < 1$.

2. A representation formula

THEOREM 1: *$f(z) \in \bar{S}$ if and only if*

$$f(z) = z \exp\left\{\int_0^z \phi(t)dt\right\}, \tag{2.1}$$

where $\phi(z)$ is regular and $|\phi(z)| \leqq 1$ in $|z| < 1$.

PROOF. Suppose that $f(z) \in \bar{S}$. Then $|zf'(z)/f(z)-1| \leqq 1$. Also $\{zf'(z)/f(z)-1\} = 0$ when z is 0. Therefore, by Schwarz' lemma it follows that

$$\frac{zf'(z)}{f(z)} - 1 = z\phi(z), \tag{2.2}$$

where $\phi(z)$ is regular and $|\phi(z)| \leqq 1$ in $|z| \leqq 1$.

[1] The author wishes to express his thanks to Prof. V. Singh for helpful guidance.

From (2.2), by integration, it easily follows that $f(z)$ has the representation (2.1).

Conversely, if $f(z)$ has the representation (2.1), then by differentiation etc., we find that $|zf'(z)/f(z)-1| \leqq 1$ and therefore $f(z) \in \bar{S}$.

3. Distortion theorems

THEOREM 2. *For all $f(z) \in \bar{S}$, we have*

$$|z|\, e^{-|z|} \leqq |f(z)| \leqq |z|\, e^{|z|}, \tag{3.1}$$

and

$$(1-|z|)\, e^{-|z|} \leqq |f'(z)| \leqq (1+|z|)e^{|z|}. \tag{3.2}$$

PROOF: Since any function $f(z) \in \bar{S}$ has the representation (2.1), where $\phi(z)$ is regular and $|\phi(z) \leqq 1$ in $|z| < 1$, (3.1) follows immediately.

Inequalities (3.1) are sharp. Equality being attained for the function $f(z) = ze^z$.

Again differentiating (2.1) we get

$$f'(z) = [1+z\phi(z)] \exp\left\{\int_0^z \phi(t)dt\right\}. \tag{3.3}$$

From the above representation of $f'(z)$, (3.2) at once follows. The function $f(z) = ze^z$ shows that (3.2) is sharp.

From (3.2) we have the following:

COROLLARY 1: The conformal map of $|z| < 1$ as yielded by functions of class $\bar{S}$ contains all the points of circle $|w| < 1/e$.

4. Coefficient regions

THEOREM 3: *If $f(z) = z+\sum_{n=2}^{\infty} a_n z^n \in \bar{S}$, then*

$$|an| \leqq \frac{1}{(n-1)}. \tag{4.1}$$

PROOF. The proof is based on a method of Clunie. If $f(z) \in \bar{S}$, then we know that

$$z f'(z)-f(z) = f(z) \cdot z\phi(z). \tag{4.2}$$

If we put $\psi(z) = z\phi(z) = \sum_{n=1}^{\infty} b_n z^n$, then $|\psi(z)| < 1$ in $|z| < 1$, and (4.2) may be written as

$$(4.3) \qquad \sum_{n=2}^{\infty}(n-1)a_n z^n = [z+\sum_{n=2}^{\infty}a_n z^n]\sum_{n=1}^{\infty}b_n z^n.$$

Comparing the coefficients of z^n on both the sides of (4.3) we get

$$(n-1)\,a_n = b_1 a_{n-1}+b_2 a_{n-2}+b_3 a_{n-3}+\ldots+b_{n-1},\ n \geqq 2.$$

This shows that the coefficient a_n on the left side of (4.3) depends only on the coefficients $a_2, a_3, \ldots, a_{n-1}$ on the right hand side of (4.3). Hence for $n \geqq 2$, we may write

$$(4.4) \qquad \sum_{k=2}^{n}(k-1)a_k z^k+\sum_{k=n+1}^{\infty}c_k z^k = [z+\sum_{k=2}^{n-1}a_k z^k]\psi(z)$$

say. Squaring the moduli of both sides of (4.4) and integrating round $|z| = r < 1$, we get, using the fact that $|\psi(z)| < 1$ in $|z| < 1$,

$$\sum_{k=2}^{n}(k-1)^2|a_k|^2 r^{2k}+\sum|c_k|^2 r^{2k} < 1+\sum_{k=2}^{n-1}|a_k|^2$$

making $r \to 1$, we find that

$$\sum_{2}^{n}(k-1)^2\,|a_k|^2 \leqq 1+\sum_{2}^{n-1}|a_k|^2$$

or

$$(n-1)^2\,|a_n|^2 \leqq 1-\sum_{2}^{n-1}(k-2)k\,|a_k|^2.$$

It, therefore follows that for all $n \geqq 2$

$$|a_n| \leqq \frac{1}{(n-1)}.$$

Equality is attained for $f(z) = ze^{z^{n-1}/n-1}$, a function of $\bar{S}$ for which $a_2 = a_3 = \ldots a_{n-1} = 0$.

5. Radius of convexity

THEOREM 4. Each function $f(z) \in \bar{S}$ maps

$$(5.1) \qquad |z| \leqq \frac{\sqrt{13}-3}{2}$$

onto a convex domain.

PROOF. We know that if $f(z) \in \bar{S}$, then

$$\frac{zf'(z)}{f(z)} = 1+z\phi(z).$$

Logarithmic differentiation of the above equation yields

$$\frac{zf''(z)}{f'(z)} = \frac{zf'(z)}{f(z)} - 1 + \frac{z\phi(z) + z^2\phi'(z)}{1 + z\phi(z)}$$

$$= z\phi(z) + \frac{z\phi(z) + z^2\phi'(z)}{1+z\phi(z)}.$$

Now making use of the fact that $|\phi'(z)| \leqq 1-|\phi(z)|^2/1-|z|^2$, we get

$$\text{(5.2)} \qquad \left|\frac{zf''(z)}{f'(z)}\right| \leqq \frac{|z|[2|\phi|-2|\phi|\ |z|^2+|z| - |z| \cdot |z\phi|^2]}{(1-|z\phi|)(1-|z|^2)}.$$

$f(z)$ will map $|z| \leqq r$ onto a convex domain if the right hand side of (5.2) is less than or equal to 1 for $|z| \leqq r$, i.e. if

$$\text{(5.3)} \quad (1-|z\phi|)(1-|z|^2) \geqq |z|[2|\phi|-2|\phi| \cdot |z|^2+|z| - |z| \cdot |z\phi|^2].$$

Putting $|z| = p$, $|\phi| = q$ and $pq = x$, where $0 \leqq p \leqq 1$, $0 \leqq q \leqq 1$, $0 \leqq x \leqq p$, (5.3) may be written as

$$\text{(5.4)} \qquad p^2x^2-3(1-p^2)x+(1-2p^2) \geqq 0.$$

We consider $f(x) = p^2x^2-3\ (1-p^2)x+(1-2p^2)$.
Clearly $f(x)$ decreases for $0 \leqq x \leqq p$, if

$$2p^2x-3(1-p^2) < 0$$

or

$$2p^2 \cdot p-3(1-p^2) < 0$$

or

$$\text{(5.5)} \qquad p < \tfrac{1}{2}[(5-2\sqrt{6})^{\frac{1}{3}}+(5+2\sqrt{6})^{\frac{1}{3}}-1]$$

The value of p as given by (5.5) is easily seen to be less than $\frac{13}{16}$.

The least value of $f(x)$ is $f(p)$, when p is given by (5.5). Therefore, (5.4) will hold if

$$f(p) \geqq 0$$

or

$$p^4+3p^3-2p^2-3p+1 \geqq 0$$

or

$$(1-p^2)(p^2+3p-1) \leqq 0$$

or

$$p^2+3p-1 \leqq 0$$

or

$$p \leqq \frac{\sqrt{13}-3}{2}. \tag{5.6}$$

Clearly the value of p as given by (5.6) is less than that given by (5.5). Hence every function $f(z) \in \bar{S}$ maps $|z| \leqq \sqrt{13}-3/2$ onto a convex domain.

The function $f_0(z) = ze^z$ shows that this bound for the radius of convexity cannot be improved.

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