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by

A. S. Troelstra

1. Introduction

Various investigations in the domain of intuitionistic mathematics constitute our point of departure; in the first place, an unpublished manuscript of a lecture, held by J. J. de Iongh in December 1956 in Amsterdam, concerning notions of finiteness of different constructive content ([4]).

Further important sources were the treatment of relations between cardinalities by L. E. J. Brouwer in [1] and the investigations of A. Heyting on the countability predicates ([2]).

We are trying to treat and to extend these researches from a common point of view.

In the second paragraph general relations of a certain kind are introduced; in the next paragraph these relations are used to define various notions of finiteness and infiniteness, relations between cardinalities, and countability predicates.

In the last paragraph some notions of finiteness, not defined by means of the relations of § 2, are discussed.

I am indebted to prof. J. J. de Iongh for his permission to make use of his results, and to prof. dr. A. Heyting for his valuable help and criticism.

2. *P*-*Q*- and *P*-*Q*-*T*-relations

A relation is a binary predicate; relations will be denoted by capitals: R, P, Q, T, ... Species ([3], 3.2.1) will be denoted by lower case letters: $a, b, c, \ldots$.

Inclusion, product and relative product of two relations R_1, R_2 will be written as $R_1 \subset R_2$, $R_1 \cap R_2$, R_1R_2 respectively. The relative product binds stronger than the product does. We remark the following general laws:

$$(1)\qquad \begin{aligned} &R_1(R_2 \cap R_3) \subset R_1R_2 \cap R_1R_3 \\ &(R_1 \cap R_2)R_3 \subset R_1R_3 \cap R_2R_3 \\ &(R_1R_2)R_3 = R_1(R_2R_3) \end{aligned}$$

The identity relation will be designated by I. We have a general rule:

$$R_1 \subset R_2 \Rightarrow R_1 S \subset R_2 S \text{ and } SR_1 \subset SR_2.$$

As a consequence we have for example:

$$I \subset R_1,\ R_3 \Rightarrow R_2 \subset R_1 R_2 R_3.$$
$$I \subset R_2 \Rightarrow R_1 R_3 \subset R_1 R_2 R_3.$$

For predicates in general we use sometimes set-theoretic notions for sake of convenience, so for example:

$$(x)(P_1(x) \to P_2(x)) \Leftrightarrow P_2 \subset P_1.$$

Further we use $a \in P$, $a \notin P$, $P_1 \cap P_2$, $P_1 \cup P_2$ etc.

A function φ acting on an argument x is always denoted with parentheses: $\varphi(x)$. We define

$$\varphi a = \varphi[a] = \{y : (Ex)\ (y = \varphi(x)\ \&\ x \in a)\}.$$

If φ is explicitly stated to be bi-unique in the context, φ^{-1} denotes the inverse mapping; in all other cases we define

$$\varphi^{-1}(x) = \{y : \varphi y = x\}.$$

Further we use

$$\varphi^{-1} a = \varphi^{-1}[a] = \{y : \varphi y \in a\}.$$

Definition 2.1: A mapping φ will be called bi-unique if we have:

$$\varphi(x) = \varphi(y) \to x = y.$$

Remark 2.1: A weaker notion of bi-uniqueness is given by the condition:

$$x \neq y \to \varphi(x) \neq \varphi(y).$$

If the domain of φ is a, and the equality in a is stable (that is, $\neg\neg(x = y) \to x = y$ for all $x, y \in a$), then this weaker notion is equivalent to the stronger one.

Definition 2.2: A species b is called detachable with respect to a, or detachable in a, if $b \subset a$, and

$$(x)(x \in a \to x \in b \vee x \notin b)$$

is valid (see [3], 3.2.4 def. 2).

Remark 2.2: This notion can be weakened in many ways; one of the most simple is:

$$(x)(x \in a \to \neg\, x \in b \vee \neg\neg\, x \in b).$$

However, if a is weakly detachable in this sense with respect to b, and b with respect to c, we cannot be sure that a is weakly detachable with respect to c; so this is not a useful notion.

We define the following relations between species:

DEFINITION 2.3:

aPb: $b \subset a$.

aP_sb: b is detachable with respect to a.

aQb: b is the image of a by some mapping.

aQ_sb: b is the image of a under a bi-unique mapping.

aTb: a is congruent with b, that is, we have:
$x \in a \rightarrow \neg\neg(x \in b)$ & $x \in b \rightarrow \neg\neg(x \in a)$. ([3], 3.2.4, def. 1)

aT_sb: aTb and $a \subset b$ (this is Brouwers notion of halfidentity introduced in [1]).

P, P_s, Q, Q_s, T, T_s will be called the basic P-Q-T-relations, P, P_s, Q, Q_s, will be called the basic P-Q-relations.

DEFINITION 2.4: A P-Q-relation is a continued relative product of basic P-Q-relations; a P-Q-T-relation is a continued relative product of basic P-Q-T-relations.

We want to investigate the P-Q-relations first. For these relations we introduce a stronger notion of inclusion and identity.

DEFINITION 2.5: Suppose R is a P-Q-relation. If we have a_0Ra_n, and this is testified by the sequence: $b_1 \subset a_0$, $\varphi_1 b_1 = a_1$, $b_2 \subset a_1$, $\varphi_2 b_2 = a_2, \ldots, \varphi_n b_n = a_n$ (possibly $b_1 = a_0$, and/or φ_n is the identity), then $\bigcup_{i=0}^{n} a_i$ is called a substratum of the assertion a_0Ra_n.

DEFINITION 2.6: For P-Q-relations R_1, R_2 we define $R_1 \subset_s R_2$ (substratum-inclusive) by:

$(a)(b)(c)(\mathrm{E}d)(aR_1b$ with a substratum $c \rightarrow aR_2b$ with a substratum $d \subset c)$.

R_1, R_2 are called substratum-identical ($R_1 =_s R_2$) if $R_1 \subset_s R_2$ and $R_2 \subset_s R_1$.

REMARK 2.3: A species is defined as a property of mathematical objects, which themselves have been or could have been defined earlier. So in a natural way a hierarchy of types for species is introduced ([3], 3.2.3) which can be extended to all types corresponding to constructive ordinals.

Predicates can be considered as properties, and are to be considered as completely defined only if a certain class (class used as synonymous with species) of species is given, on the elements of which the meaning of the predicates is defined. (So a predicate also ought to have a type.) Some predicates, for example equality between species, admit a "systematic ambiguity"; for species of every type equality between them can be defined in the same manner. The P-Q-T-relations also admit this systematic ambiguity as regards their definition.

The validity of P-Q-T-relations between certain given species depends on the class of species to which we suppose the meaning of the relation to be restricted in a given context, since the definition of a relative product requires existential quantification.

As a consequence, equality between two relations of this kind also depends on the presupposed class of species.

A substratum-identity between two relations holds on every class of species s with the property:

$$a \in s \;\&\; b \subset a \rightarrow b \in s.$$

Equality in general holds between two relations on every class of species with adequate properties (depending on the proof of the equality). E.g. in lemma 2.4 a class of species with the properties

$$a \in s \;\&\; b \subset a \rightarrow b \in s,$$
$$a \in s \rightarrow \{x : x \subset a\} \in s$$

is certainly adequate.

Remark 2.4: We have: $I \subset P_s \subset P$; $I \subset Q_s \subset Q$; $I \subset T_s \subset T$; $QQ_s = Q$; $PP_s = P$; $TT_s = T$.

Notation 2.1: R_P, R'_P, ... resp. R_Q, R'_Q, ... will be used to denote either P or P_s, resp. Q or Q_s (e.g., aR_Qb can denote either aQb or aQ_sb).

We now proceed with four lemmas.

Lemma 2.1: $R'_P \subset R_P \Rightarrow R_PR_QR'_P =_s R_PR_Q$.
$R_Q \subset R'_Q \Rightarrow R_QR_PR'_Q =_s R_PR'_Q$.

Proof: De Iongh has already demonstrated in [1] $PQP = PQ$; in fact the proof shows that we have $PQP =_s PQ$. We generalize his argument. Suppose $a_1 \subset a$, $\varphi a_1 = b_1$, $b \subset b_1$. Then we have $\varphi^{-1}b \subset a_1 \subset a$.

If a_1 is detachable with respect to a, b with respect to b_1, then also $\varphi^{-1}b$ is detachable with respect to a, since:

$$\begin{aligned} x \in a &\to x \in a_1 \vee x \notin a_1 \\ x \in a_1 &\to \varphi(x) \in b_1 \\ \varphi(x) \in b_1 &\to \varphi(x) \in b \vee \varphi(x) \notin b \\ &\to x \in \varphi^{-1}b \vee x \notin \varphi^{-1}b \\ x \notin a_1 &\to x \in \varphi^{-1}b. \end{aligned}$$

So $\varphi^{-1}b$ is detachable with respect to a, and the first part of the lemma is established. The second line is proved in the following way:

$$R_P R'_Q \subset_s R_Q R_P R'_Q \subset_s R_P R_Q R_P R'_Q =_s R_P R_Q R'_Q =_s R_P R'_Q.$$

LEMMA 2.2: $P_s Q \subset_s QP$.

PROOF: Suppose aP_sQd, demonstrated by $a_1 \subset a$, $\varphi a_1 = d$. Define φ' by:

$$\begin{aligned} x \in a_1 &\to \varphi'(x) = \varphi(x) \\ x \notin a_1 &\to \varphi'(x) = x. \end{aligned}$$

Then we can write d as

$$\{y : (Ex)(x \in a_1 \;\&\; y = \varphi(x))\}.$$

So we have $aQPd$ with substratum $a \cup \varphi a_1$.

LEMMA 2.3: $P_s R_Q = R_Q P_s$.

PROOF: Suppose aP_sQb; $a_1 \subset a$, $\varphi a_1 = b$.

Define φ' by:

$$\begin{aligned} x \in a_1 &\to \varphi'(x) = \varphi(x) \\ x \notin a_1 &\to \varphi'(x) = \{x, \varphi a_1\}. \end{aligned}$$

Then we have:

$$b = \{y : (Ex)(x \in a_1 \;\&\; \varphi(x) = y)\},\ b \subset \varphi' a.$$

If φ is one-to-one then so is φ'.

LEMMA 2.4: $PQ \subset QPQ_s$.

PROOF: Suppose $aPQb$, $a_1 \subset a$, $\varphi a_1 = b$. (φ only defined on a). Now we define successively:

$$x \in a \to \psi(x) = \varphi^{-1}[\varphi(x)]$$

with $\varphi^{-1}[\varphi(x)] = \{y : y \in a \;\&\; \varphi(y) = \varphi(x)\}$.

$\psi a = a_2$.

Since φ is defined on a_1 only, we can proceed:

$$a_3 = \{\psi(x) : (Ey)(y \in \psi(x))\}$$

and define φ' by: $\varphi'(\psi(x)) = \varphi(x)$.

So $\varphi' a_3 = b$. We have to show φ' is a one-to-one mapping:

$$\varphi'(\psi(x)) = \varphi'(\psi(y)) \Rightarrow \varphi(x) = \varphi(y) \Rightarrow$$
$$\varphi^{-1}[\varphi(x)] = \varphi^{-1}[\varphi(y)] \Rightarrow \psi(x) = \psi(y).$$

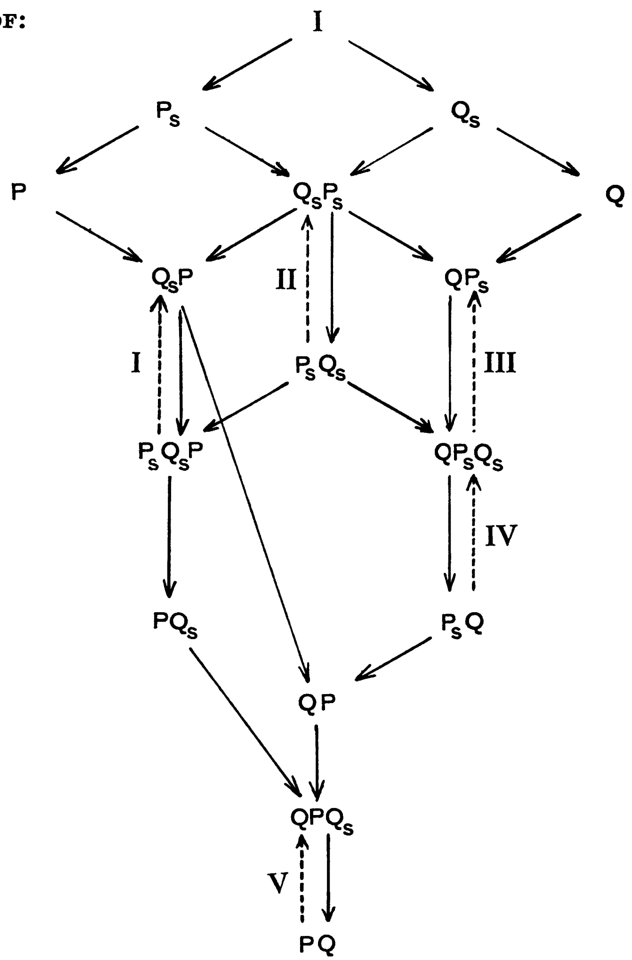
These lemmas may be combined in the following theorem:

THEOREM 2.5: *a) Scheme I shows all possible P-Q-relations with respect to substrate-identity.*

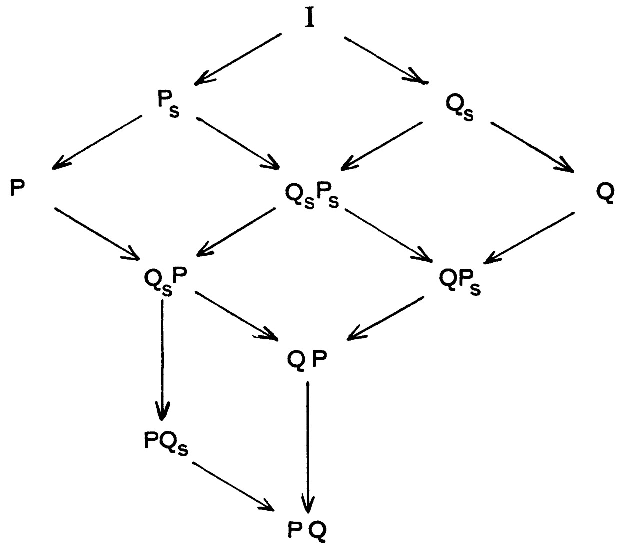
b) Scheme II shows all possible P-Q-relations.

(The dashed arrows in scheme I are inclusions which have not been proved with respect to substratum-inclusion).

PROOF:



Scheme I



Scheme II

a) Scheme I is the result of the application of the lemmas 2.1 and 2.2. After application of lemma 2.1 all one- and two-factor relative products remain, and further QP_sQ_s, P_sQ_sP, QPQ_s, P_sQP, QP_sQ_sP, P_sQPQ_s.

Lemma 2.2 implies:

$$QP \subset_s P_sQP \subset_s QPP =_s QP; \quad QP \subset_s QP_sQ_sP \subset_s QQPP =_s QP$$

and $QPQ_s \subset_s P_sQPQ_s \subset_s QPPQ_s =_s QPQ_s$.

b) Application of lemma 2.3 results in:

$$Q_sP_s = P_sQ_s, \ P_sQ = QP_s = QP_sQ_s \text{ and } P_sQ_sP = Q_sP_sP = Q_sP.$$

Lemma 2.4 implies $PQ = QPQ_s$.

We are able to show, by means of some examples, that the inclusions, denoted by dashed arrows and numbered I, . . ., V in scheme I cannot be strengthened to substratum-inclusions.

To describe these and other counterexamples we introduce a standard problem for which no solution is known.

NOTATION 2.2: $\{1, 2, \ldots, n\} = \bar{n}$, $\emptyset = \bar{0}$. The species of natural numbers will be denoted by $\overline{\omega}$, the species of real numbers by Ω.

We introduce the predicate $\Pi^k_n(m)$, k, n, $m \in \overline{\omega}$. $\Pi^k_n(m) \Leftrightarrow m$ is the number of the last decimal of the n^{th} sequence of ten consecutive sevens in the decimal notation for π^k. Further we define "floating" numbers r_k in the following manner:

$$r_k = \sum_{i=1}^{\infty} a_{i,k}\, 10^{-i} \text{ with } a_{i,k} = 1 \Leftrightarrow \Pi_1^k(i)$$
$$= 0 \text{ if otherwise.}$$

Example 2.1: Refutation of II and III in scheme I.

These inclusions can be disproved simultaneously by refuting $P_sQ_s \subset_s QP_s$. We consider:

$$a = \{n : n = 1 \vee (n = 2 \;\&\; (Em)\Pi_1^1(m))\}.$$

A subspecies a_1 of a is defined by:

$$a_1 = \{n : n \in a \;\&\; n = 2\}.$$

a_1 is detachable with respect to a. We define a mapping φ on a_1:

$$2 \in a_1 \Rightarrow \varphi(2) = 1+r_1.$$

We put $\varphi a_1 = b$.

φ is a bi-unique mapping from a_1 onto b. So we have aP_sQ_sb with a substratum $d_1 \subset d$. If aQP_sb is demonstrated by $\varphi' a = b_1$, b_1P_sb, $d_1 = a \cup b_1 \subset d$, then we have to be sure that $\varphi'(1)$ is well defined. As long as it is unknown if $(Em)\Pi_1^1(m)$ holds, the only element of $a \cup b$ of which it is certain that it is contained in d, is 1, so $\varphi'(1) = 1$.

Necessarily, $2 \in a_1$ implies $\varphi'(2) = 1+r_1$. Therefore it cannot be proved that b is detachable with respect to $b_1 = \varphi' a$.

Example 2.2: Refutation of inclusion IV in scheme I. Take $a = \overline{\omega}$, $a_1 = \{n : (Em)\Pi_m^1(n)\}$. a_1 is detachable with respect to a. We define a mapping φ by:

$$\left.\begin{array}{l} 2n \in a_1 \Rightarrow \varphi(2n) = n+r_{2n} \\ 2n+1 \in a_1 \Rightarrow \varphi(2n+1) = n+r_{2n+1} \end{array}\right\} \varphi a_1 = b.$$

The substratum is equal to $a \cup b$.

If we want to show aQP_sQ_sb, then we must construct b_1, b_2, φ', φ'' such that $\varphi' a = b_1$, b_2 detachable with respect to b_1, $\varphi'' b_2 = b$; φ'' is a bi-unique mapping.

If we do not know a solution to our standard problem, $2n$, $2n+1 \in a_1$ is possible, while it is unknown whether $n+r_{2n}$ and $n+r_{2n+1}$ differ or not. So φ'' can only be the identity.

There remains to be shown that aQP_sb cannot be proved. We have:

$$\varphi'(1) = n \vee \varphi'(1) = n+r_{2n} \vee \varphi'(1) = n+r_{2n+1}.$$

As long as our problem is completely unsolved, $\varphi'(1) = n$ is the only possibility. Since it is always possible that $n+r_{2n}$ or

$n+r_{2n+1}$ belongs to b, it is impossible to show that b is a detachable subspecies of b_1.

Example 2.3: Refutation of inclusion I in scheme I.

Take $a = [0, 1] \cup [2, 3]$, $a_1 = [0, 1] \subset a$. a_1 is detachable with respect to a.

If we take $\varphi(x) = 3x$, then $\varphi a_1 = [0, 3] = b$. So we have aP_sQ_sb, with a substratum $[0, 3]$. If we wanted to prove aQ_sPb by aQ_sb_1, b_1Pb, $a \cup b_1 \subset [0, 3]$, b_1 has to be $[0, 3] = b$. But it is impossible to prove aQ_sb, since the only detachable subspecies of $[0, 3]$ are $\emptyset$ and $[0, 3]$, as a consequence of the fan theorem ([3], 3.4.3 theorem 2).

Example 2.4: Refutation of inclusion V in scheme I. Take $a = \overline{2}$. We define $a_1 \subset a$:

$$a_1 = \{x : (x=1 \;\&\; (Em)\{(En)\Pi_1^{2m}(n) \;\&\; (k)(k<m \rightarrow \neg\, (En)\Pi_1^{2k}(n))\}) \vee (x=2 \;\&\; (Em)\{(En)\Pi_1^{2m+1}(n) \;\&\; (k)(k<m \rightarrow \neg\, (En)\Pi_1^{2k+1}(n))\})\}.$$

We define a mapping φ $(\varphi a_1 = b)$ in the following way:

$$1 \in a_1 \;\&\; (En)\Pi_1^{2d}(n) \;\&\; (k)(k < d \rightarrow \neg\, (En)\Pi_1^{2k}(n)) \Rightarrow \varphi(1) = d.$$
$$2 \in a_1 \;\&\; (En)\Pi_1^{2e+1}(n) \& (k)(k < e \rightarrow \neg\, (En)\Pi_1^{2k+1}(n)) \Rightarrow \varphi(2) = e.$$

The substratum of $aPQb$ is equal to $a \cup b$; only natural numbers occur as elements of $a \cup b$. Suppose we were able to prove $aQPQ_sb$, without having a solution to our standard problem, by exhibiting b_1, b_2 such that aQb_1, b_1Pb_2, b_2Q_sb with $a \cup b_1 \cup b \subset a \cup b$, or equivalently, $b_1 \subset a \cup b$.

If $\varphi' a = b_1$, the only possibilities are: $b_1 = \{1, 2\}$, $\{1\}$, or $\{2\}$. As it is always possible that b contains two elements, the only possibility is $\{1, 2\} = b_1$. So there remains to show the impossibility of proving aPQ_sb.

If aPQ_sb was demonstrated by aPb_1, $\varphi'' b_1 = b$, b_1 is a species of the following type:

$$b_1 = \{x : (x = 1 \;\&\; F_1) \vee (x = 2 \;\&\; F_2)\}.$$

If $1 \in a_1$ is known, we have no guarantee that $(2 \in a_1 \vee 2 \notin a_1)$ is known. Therefore, the definition has to take the following form:

$$b_1 = \{x : (x = 1 \;\&\; 1 \in a_1) \vee (x = 2 \;\&\; 2 \in a_1 \;\&\; ((1 \in a_1 \;\&\; 2 \in a_1) \rightarrow c \neq d))\}.$$

This definition does not agree with the symmetry of the problem; hence, if we only know $2 \in a_1$, we cannot be sure that $2 \in b_1$, so we do not know if φb_1 is empty or not, while b cannot be empty.

We proceed with the treatment of the system of P-Q-T-relations. We do not intend to give a complete system of reductions, but instead we restrict ourselves to the derivation of the most important reductions in the system which guarantee its finiteness.

In the next theorem we restrict ourselves to combinations with T_s only.

THEOREM 2.6: $a)$ $T_sP = PT_s = PT = TP$
$b)$ $PR_QT_s = PR_QT$
$c)$ $T_sR \subset RT_s$, if R is a P-Q-relation.

PROOF: $a)$ aTa', $a'Pb \Rightarrow (a \cap b)Tb$, $aP(a \cap b)$; hence $TP \subset PT_s \subset PT$.

$b)$ aPa', $\varphi a' = b'$, $b'Tb \Rightarrow (b' \cap b)T_sb$, $\varphi^{-1}[b' \cap b] \subset a' \subset a$, so $PR_QT \subset PR_QT_s \subset PR_QT$.

$c)$ Analogous to $a)$, aT_sa', $a'P_sb \Rightarrow aP_s(a \cap b)$, $T_sP_s \subset P_sT_s$.

Now we suppose aT_sa', $\varphi a' = b$; then $\varphi a = b'$, $b'T_sb$. So $T_sR_Q \subset R_QT_s$. After this, $c)$ can be proved by induction for every P-Q-relation R.

REMARK 2.5: In the proof of theorem 2.6$c)$ we used $aTb \rightarrow \varphi aT\varphi b$, where φ is not necessarily defined on $a \cup b$. The proof of this rule is as follows:

$$y \in \varphi a \rightarrow (Ex)(x \in a \;\&\; \varphi(x) = y); \text{ so } \neg\neg\, x \in b.$$
$$x \in b \rightarrow \varphi(x) = y \in \varphi b, \text{ so } \neg\neg\, x \in b \rightarrow \neg\neg\, y \in \varphi b, \text{ q.e.d.}$$

COROLLARY 2.6.1.: $T_sRT_s = RT_s$ if R is any P-Q-relation, since $RT_s \subset T_sRT_s \subset RT_sT_s = RT_s$. This implies that the number of relative products of P, P_s, Q, Q_s and T_s is finite.

THEOREM 2.7: $TRT = TRT_s$ *if R is any P-Q-relation.*

PROOF: Since $TP = PT$, we may restrict ourselves to the possibilities $R = R_Q$, P_s, P_sR_Q.

$a)$ Suppose aTR_QTb, i.e. aTa', $\varphi a' = b'$, $b'Tb \Rightarrow (b' \cap b)T_sb$, $\varphi^{-1}[b' \cap b]T_sa$. So $TR_QT = TR_QT_s$.

$b)$ Suppose aTP_sR_QTb, i.e. aTa', $a'P_sa''$, $\varphi a'' = b'$, $b'Tb \Rightarrow (b' \cap b)T_sb$, $\varphi^{-1}[b' \cap b] \subset a'' \Rightarrow aT\big((a'-a'') \cup \varphi^{-1}[b' \cap b]\big)$; $\big((a'-a'') \cup \varphi^{-1}[b' \cap b]\big)P_s\varphi^{-1}[b' \cap b]$, since $x \in (a'-a'') \cup \varphi^{-1}[b' \cap b] \Rightarrow x \in a'-a'' \vee x \in \varphi^{-1}[b' \cap b]$. If $x \in a'-a''$, then $x \notin \varphi^{-1}[b' \cap b]$.

Consequently, $TP_s R_Q T = TP_s R_Q T_s$. In the case that φ is the identical mapping we have $TP_s T = TP_s T_s$.

COROLLARY 2.7.1: There are only finitely many P-Q-T-relations.

REMARK 2.6: We could have introduced a specialization of the relation $aPb : aT_r b$, defined by: $aT_r b \Leftrightarrow bT_s a$. Every relation aTb can be written as $aT_r T_s b$ (because $aT_r(a \cap b)$, $(a \cap b)T_s b$) or as $aT_s T_r b$ (because $aT_s(a \cup b)$, $(a \cup b)T_r b$). Consequently $T_s T_r = T_r T_s$.

DEFINITION 2.7: Suppose R to be a P-Q-T-relation and P a predicate of species; P is said to be invariant with respect to R, if we have:

$$b_1 R b_2 \rightarrow (b_1 \in P \rightarrow b_2 \in P).$$

THEOREM 2.8: R is a P-Q-T-relation, and P a predicate of species. With every P a predicate P_1 is associated by:

$$x \in P_1 \Leftrightarrow (Ey)(y \in P \,\&\, yRx).$$

P_1 is invariant with respect to R for every P, if and only if $RR = R$.

PROOF: Suppose $RR = R$. This implies $x \in P_1 \,\&\, xRy \Rightarrow (Ez)(z \in P \,\&\, zRx)$; $\Rightarrow zRRy$; hence $(Ez)(z \in P \,\&\, zRy)$, so $y \in P_1$.
Conversely, let P_1 be invariant with respect to R for every P. We have to show $(x)(y)(xRRy \rightarrow xRy)$. Take P to be defined by: $x \in P \Leftrightarrow x = x_0$, and suppose $x_0 RR y_0$. We obtain $(Ez)(x_0 Rz \,\&\, zRy_0)$; so $z \in P_1$, $y_0 \in P_1$. Therefore we have $x_0 R y_0$ according to the definition.

COROLLARY 2.8.1: If R is a P-Q-T-relation, composed of basic P-Q-T-relations $R_1, R_2, \ldots$ then P_1 (defined as before) is R-invariant for every P if and only if $RR_i = R$, $i = 1, 2, \ldots$ (or stated otherwise, P_1 is R_i-invariant $(i = 1, 2, \ldots)$ for every P).

3. Applications

We introduce the notion of comparison-relation by the following definition:

DEFINITION 3.1: A comparison-relation is a relation R, such that $I \subset R$.

REMARK 3.1: The most important examples of comparison-relations are transitive (i.e. $RR = R$).

A reflexive and transitive relation R on a species b always induces a half-order relation (partial ordering) on a system of equivalence classes of b. (See [3], 7.3.1, def. 1).

We divide this paragraph into different sections for the applications.

A. Cardinalities.

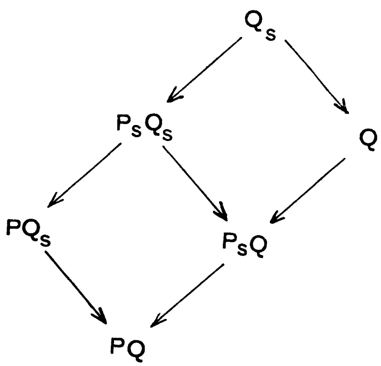
DEFINITION 3.2: The cardinal number of a species a is a predicate $k(a)$; $x \in k(a) \Leftrightarrow aQ_sx$.

To obtain a good analogue to the corresponding classical notions of "$\geqq$" and equality for cardinal numbers we define:

DEFINITION 3.3: A cardinality-relation R is reflexive, transitive, $Q_sR = R$ and $RQ_s = R$.

Of the P-Q-relations of scheme II, Q_s, Q, P_sQ_s, PQ_s, P_sQ, PQ fulfil these requirements.

In [1] Brouwer has considered Q_s (equivalence, German: equivalent), Q (denoted by $\overset{\therefore}{\geqq}$, German: überdeckt), P_sQ_s (in the special case of the notion "zählbar"), PQ_s (denoted by $\geqq$) and PQ (denoted by $\dot{\geqq}$, German: überlagert). Brouwer also considers notions denoted by $\ddot{\geqq}$ (German: superponiert) and $\mathring{\geqq}$ (German: übergeordnet), corresponding to QT_s and PQT_s.



Scheme III

The six P-Q-cardinality-relations are all different; in [2] counterexamples to the inclusions $P_sQ \subset Q$, $PQ \subset P_sQ$, $P_sQ_s \subset Q_s$ and $Q \subset Q_s$ are presented. $PQ \subset PQ_s$ cannot be proved as is shown by example 2.4. We complete the counterexamples by:

Example 3.1: $PQ_s \subset Q$ and $PQ_s \subset P_sQ_s$ are not provable. Take $b \subset \overline{\omega}$, $b = \{n : \neg\, (Em)\Pi_1^n(m)\}$. Q_s is represented by the identity, so $\overline{\omega}PQ_sb$.

We do not know if b contains an element or not, so $\overline{\omega}Qb$ cannot be proved.

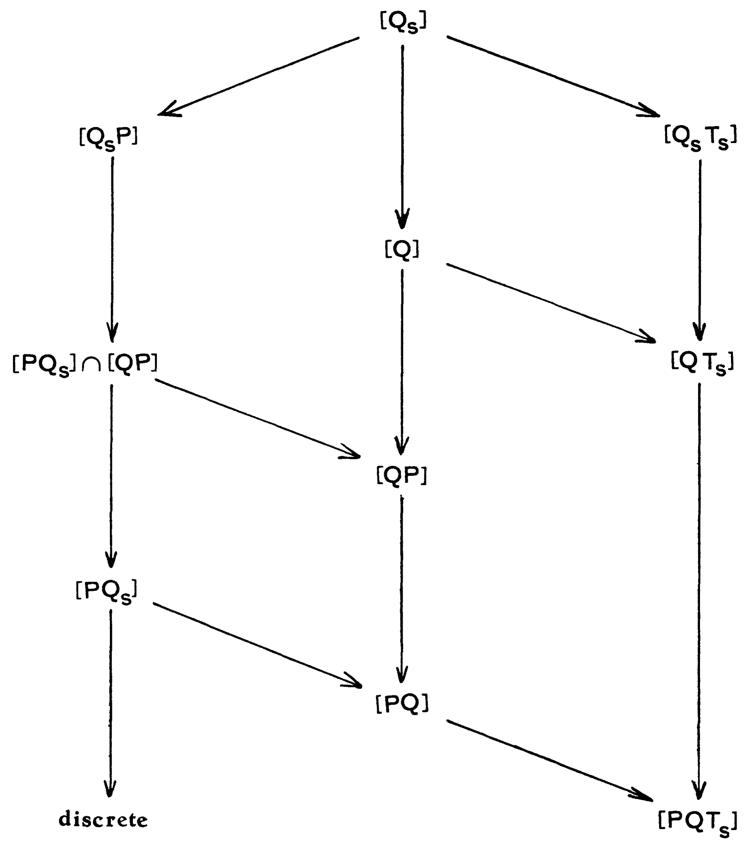
In the same manner $\overline{\omega}P_sQ_sb$ cannot be proved, because this requires the existence of a method which produces the elements of b, if there are any, and no such method is known.

B. P-Q-T-notions of finiteness.

DEFINITION 3.4: Suppose R to be a comparison-relation. The predicate $[R]$ is defined by:

$$x \in [R] \Leftrightarrow (En)(\bar{n}Q_sRx \;\&\; n \in \overline{\omega} \cup \{0\}).$$

Especially, $[Q_s]$ represents the notion of finiteness. If R is a P-Q-(T)-relation, then $[R]$ is a P-Q-(T)-notion of finiteness. Without restriction we may suppose $Q_sR = R$.



Scheme IV

REMARK 3.2: A detachable subspecies of a finite species is a finite species.

We discuss all P-Q-notions of finiteness, but as far as other P-Q-T-notions of finiteness are concerned, we restrict ourselves to the most important ones.

Scheme IV contains all the P-Q-T-notions of finiteness which we want to discuss.

THEOREM 3.1: Scheme IV contains all P-Q-notions of finiteness with their intersections.

PROOF: By using remark 3.3 and supposing R to satisfy $Q_s R = R$, we obtain from scheme II the following notions of finiteness: $[Q_s]$, $[Q]$, $[QP]$, $[PQ]$, $[PQ_s]$ and $[Q_sP]$. Since we have:

$$a \text{ discrete and } a \in [Q] \Rightarrow a \in [Q_s]$$

(a species is called discrete if for every pair x, y of its elements $x = y \vee x \neq y$ holds) only $[PQ_s] \cap [QP]$ does not necessarily coincide with a P-Q-notion of finiteness.

REMARK 3.3: If we require substrate-equivalence instead of equivalence, the set of P-Q-notions of finiteness is only enlarged by $[QPQ_s]$.

REMARK 3.4: $[Q_s]$, $[Q]$, $[Q_sP]$, $[QP]$, $[PQ]$ have already been considered by J. J. de Iongh. The only new notion is $[PQ_s]$ as is shown by example 3.2, 3.3.

Example 3.2: $[PQ_s]$ is not contained in $[QP]$.

Take $a = \overline{2}$, $b = \{x : (x = 1 \;\&\; (Ex)\Pi_1^1(x)) \vee x = 2\}$. We define a mapping φ by:

$$\varphi(2) = 2,\ \varphi(1) = d \text{ if } \Pi_1^1(d);\ \varphi b = c.$$

Therefore aPQ_sc. If we suppose $aQPc$, then $(Ex)\Pi_1^1(x) \vee \neg\, (Ex)\Pi_1^1(x)$ could be decided, and in the first case, d could be determined; but no decision method is known.

Example 3.3: $[QP]$ is not contained in $[PQ_s]$ and $[Q]$.

Take $a = \overline{3}$. We define φ by: $\varphi(1) = 1$, $\varphi(2) = 1 + r_1$, $\varphi(3) = 3$; $\varphi a = b$. We define $c \subset b$ by:

$$c = \{x : x = 1 \vee x = 1 + r_1 \vee (x = 3 \;\&\; (Ex)\Pi_1^2(x))\}.$$

It follows from example 2.4 that $[PQ]$ is not contained in $[PQ_s]$ or $[QP]$. Other counterexamples to show the difference between P-Q-notions of finiteness are easily found and have already been given by de Iongh.

De Iongh has given an example of a species from $[Q_sT_s]$, not belonging to $[PQ]$, with unstable equality. Here we present counterexamples within Ω.

Example 3.4: $[QT_s]$ is not contained in $[PQ]$, $[Q_sT_s]$.

r_j, a floating number, has already been defined; if

$r_j = \sum_{i=1}^{\infty} a_{i,j}\, 10^{-i}$, we define:

$$s_j = \sum_{i=1}^{\infty} b_{i,j}\, 10^{-i} \text{ with } b_{i,j} = \begin{cases} a_{i,1} & \text{if } (Ey)(y \leqq i \;\&\; \Pi_1^j(y)), \\ a_{i,2} & \text{if } \neg(Ey)(y \leqq i \;\&\; \Pi_1^j(y)). \end{cases}$$

We introduce a species c:

$$c = \{r_1, r_2\} \cup \{0\} \cup \{s_i\}_{i=4}^{\infty}.$$

For every $i > 3$ we have:

$$\neg\neg(s_i = r_1 \vee s_i = r_2 \vee s_i = 0),$$

but we cannot prove $s_i = r_1 \vee s_i = r_2 \vee s_i = 0$. So $c \in [PQ]$, $c \in [Q_sT_s]$ cannot be proved.

Example 3.5: $[PQT_s]$ is not contained in $[QT_s]$, $[PQ]$.
We modify example 3.4 by defining a species d:

$$d = c \cup \{x: x = 1 \;\&\; (Ey)\Pi_1^3(y)\}.$$

The reasoning is along the same lines as in the preceding example.

If we restrict ourselves to subspecies of the natural numbers, scheme IV is much simplified, the result being scheme V.

$$[Q_s] \rightarrow [Q_sP] \rightarrow [PQ_s] \rightarrow [PQ] \rightarrow [PQT]$$

Scheme V

This is shown with the aid of the following theorem:

THEOREM 3.2: *For subspecies of* $\overline{\omega}$, $[Q_sPT_s] = [Q_sP]$.

PROOF: Suppose $\bar{n}Q_sPT_sa$. Without essential restriction we may suppose: $\bar{n}PT_sa$. For natural numbers we have: $(m)(m \leqq n \vee n < m)$. If $b \subset \bar{n}$, bT_sa, we get: $m \leqq n \rightarrow m \in n$, $m > n \rightarrow m \notin b$, $m \notin b \rightarrow m \notin a$, hence $a \subset \bar{n}$.

Corollary 3.2.1: Subspecies of natural numbers are always discrete; using this together with theorem 3.2 we obtain scheme V from scheme IV.

C. Predicates of countability.

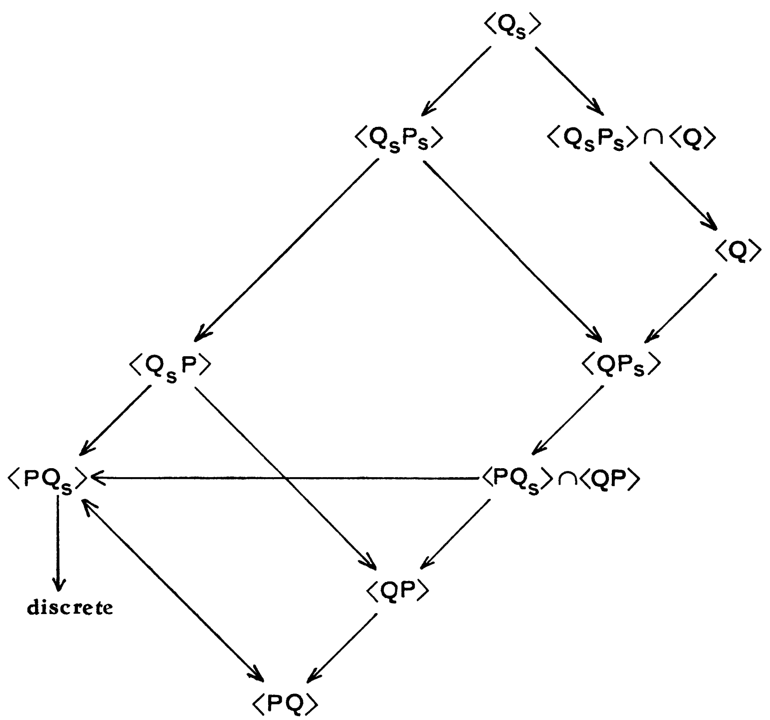
Definition 3.5: If R is a comparison-relation, $\langle R\rangle$ is the predicate defined by:

$$x \in \langle R\rangle \Leftrightarrow \overline{\omega} R x.$$

If $Q_s R = R$, and R a P-Q-, respectively a P-Q-T-relation, then $\langle R\rangle$ is called a P-Q-, respectively a P-Q-T-predicate of countability or countability-predicate.

Remark 3.5: To a certain extent the countability-predicates have already been introduced by Brouwer in [1]; Brouwer's countability-predicates have been studied extensively by Heyting in [2]. We restrict ourselves to P-Q-countability-predicates.

Theorem 3.3: *Scheme VI contains all P-Q-countability-predicates and their intersections.*



Scheme VI

PROOF: The P-Q-countability-predicates of scheme VI are obtained by taking the relations R of scheme II which satisfy $Q_sR = R$. The following intersections have to be considered:

a) $\langle Q_sP_s\rangle \cap \langle Q\rangle$, b) $\langle Q_sP\rangle \cap \langle Q\rangle$, c) $\langle Q_sP\rangle \cap \langle QP_s\rangle$,
d) $\langle PQ_s\rangle \cap \langle QP_s\rangle$, e) $\langle PQ_s\rangle \cap \langle QP\rangle$, f) $\langle PQ_s\rangle \cap \langle Q\rangle$.

We now extend a theorem of [2] to:

$$\overline{\omega}QP_sa,\ a \text{ discrete} \Rightarrow \overline{\omega}Q_sP_sa.$$

In fact, suppose $\overline{\omega}Qa''$, $a'' = \{\varphi(1), \varphi(2), \ldots\}$, $a''P_sa$, then we construct a':

$$a' = \{i : i \in a \;\&\; (j)(j < i \rightarrow \varphi(i) \neq \varphi(j))\}$$

and we get $\overline{\omega}P_sQ_sa$. This reduces b), f), to a) and c), d) to $\langle Q_sP_s\rangle$.

The difference between many countability-predicates of scheme VI can be demonstrated easily by means of counterexamples, as is done in [2].

In [2] the following countability-predicates and intersections of these predicates are shown to be different: $\langle PQT_s\rangle$ (Dutch: uittelbaar, German: auszählbar), $\langle QT_s\rangle$ (Dutch: doortelbaar, German: durchzählbar), $\langle PQ\rangle$ (Dutch: overtelbaar, German: überzählbar), $\langle PQ\rangle \cap \langle QT_s\rangle$, $\langle Q\rangle$ (Dutch: opsombaar, German: aufzählbar), $\langle PQ_s\rangle$ (Dutch: aftelbaar, German: abzählbar), $\langle PQ_s\rangle \cap \langle QT_s\rangle$, $\langle P_sQ_s\rangle$ (Dutch: telbaar, German: zählbar), $\langle Q_sP_s\rangle \cap \langle Q\rangle$.

The countability-predicates can be used to refine the P-Q-T-notions of finiteness; we demonstrate this in a few examples.

REMARK 3.6: If we define: $x \in [\emptyset] \Leftrightarrow x = \emptyset$, we have:

$[Q_s] \subset [\emptyset] \cup (\langle P_sQ_s\rangle \cap \langle Q\rangle)$;
$[Q] \subset [\emptyset] \cup \langle Q\rangle$; $[QP] \subset \langle QP\rangle$;
$[Q_sP] \subset \langle Q_sP\rangle$; $[PQ] \subset \langle PQ\rangle$;
$[PQ_s] \subset \langle PQ_s\rangle$; $([QP] \cap [PQ_s]) \subset (\langle PQ_s\rangle \cap \langle QP\rangle)$.

Example 3.6: $[QP]$ is not contained in $[QP] \cap \langle Q\rangle$.
Compare

$$t_1 = \{x : x = r_1 \vee x = r_2 \vee (x = 3 \;\&\; (Ey)\Pi^1_1(y))\}.$$
$$t_2 = \{x : x = r_1 \vee x = r_2 \vee (x = 3 \;\&\; \neg(Ey)\Pi^1_1(y))\}.$$

$t_1 \in [QP] \cap \langle Q\rangle$, since we can define a mapping φ:

$$\varphi(1) = r_1, \quad \varphi(2) = r_2, \quad n > 2: \begin{cases} \varphi(n) = r_2 \text{ if } \neg(Ey)(\Pi_1^1(y) \,\&\, y < n). \\ \varphi(n) = 3 \text{ if } (Ey)(\Pi_1^1(y) \,\&\, y < n). \end{cases}$$

We are not able to construct such a mapping from $\overline{\omega}$ onto t_2.

Example 3.7: $[QT_s]$ is not contained in $[QT_s] \cap \langle Q \rangle$. This is demonstrated by a modification of example 3.4. A species e is given by the following definition:

$$e = \{x : x \in c \,\&\, \neg(Ey)\Pi_1^3(y)\}.$$

c is the species defined in example 3.4. Since it is unknown whether e contains an element or not, $e \in \langle Q \rangle$ cannot be proved.

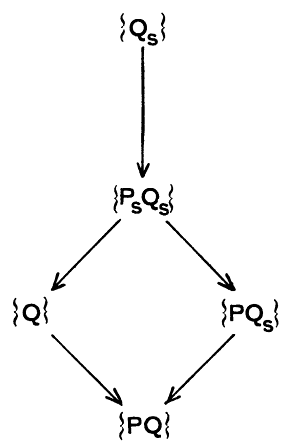
D. Notions of infinity.

Definition 3.6: If R is a comparison-relation, a predicate $\{R\}$ is defined by:

$$x \in \{R\} \Leftrightarrow xR\overline{\omega}.$$

If $RQ_s = R$, and R a P-Q- or a P-Q-T-relation, R is called a P-Q or a P-Q-T-notion of infinity.

Theorem 3.4: *Scheme VII contains all P-Q-notions of infinity.*



Scheme VII

Proof: If R is a P-Q-notion of infinity, then we may suppose $RQ_s = R$. This is the case for $R = Q_s$, Q, P_sQ_s, P_sQ, PQ_s, PQ. Now $\{P_sQ\}$ is always equivalent to $\{Q\}$ as is seen by the following argument:

Suppose $aP_sQ\overline{\omega}$, so $a_1 \subset a$, $\varphi a_1 = \overline{\omega}$; then we define φ' by:

$$\varphi'(x) = \varphi(x) \text{ for } x \in a_1,$$
$$\varphi'(x) = 1 \quad \text{ for } x \notin a_1.$$

In the same manner we prove: $\{P_sQ_s\} \subset \{Q\}$.

Remark 3.7: $\{P_sQ_s\}$, respectively $\{PQ_s\}$ correspond to Brouwers notions "reduzierbar unendlich", respectively "unendlich" (see also [4], 3.2.5).

We demonstrate that $\{Q\}$ is a new notion by two examples.

Example 3.8: $\{PQ_s\}$ is not contained in $\{Q\}$.

The segment [0, 1] cannot be mapped onto $\overline{\omega}$; in fact, [0, 1] may be represented by a fan (finitary spread, see [3], 3.1); a mapping onto $\overline{\omega}$ would have assigned a natural number to every element of the fan, and so [0, 1] would have been split up into denumerably many detachable subspecies, each of which contains at least one element. This is impossible because of the fan-theorem; compare [3], 3.4.3, th. 2.

Example 3.9: $\{Q\}$ is not contained in $\{P_sQ_s\}$.

The union of continua, $v = \bigcup_{n=1}^{\infty} [n, n+\frac{1}{2}]$ can be mapped onto $\overline{\omega}$, by taking $\varphi(x) = n$ for $x \in [n, n+\frac{1}{2}]$; v cannot belong to $\{P_sQ_s\}$, since this would require a proper subspecies of a continuum with at least one element to be detachable in this continuum.

Theorem 3.5: $a \in \{PQ\} \Rightarrow a \notin [Q]$.

Proof: Suppose $a \in \{PQ\} \cap [Q]$. Then $a = \{b_1, \ldots, b_n\}$, $b_1, \ldots, b_n$ not necessarily different. There are also $x_1, \ldots, x_{n+1}$, such that $\varphi(x_i) = i$, $x_i \in a$; so all x_i are different; hence we obtain a contradiction, and $a \notin [Q]$.

We finish this paragraph by pointing out some interesting conclusions from corollary 2.8.1. $[PQ]$, $\langle PQ \rangle$, $[PQT_s]$, $\langle PQT_s \rangle$ are invariant with respect to P, Q; the last two predicates also with respect to T. It is easy to see that the union of two species from respectively $[PQ]$, $\langle PQ \rangle$, $[PQT_s]$, $\langle PQT_s \rangle$ again belongs to $[PQ]$ etc.

$\{PQ\}$ is invariant with respect to union with an arbitrary species.

4. Other notions of finiteness

Definition 4.1: (Brouwer, see [3] 3.4.4) A species is called bounded in number by n (in short: bounded by n), if b cannot contain more than n different elements; in other words, if b does not contain a finite subspecies with $n+1$ elements. If b is bounded by n for a certain n, b is said to be bounded in number.

DEFINITION 4.2: (de Iongh, [4]) A species b is said to be determined in number (by n), if b is bounded by n, and contains finite subspecies with n elements.

NOTATION 4.1: If x is a species, we use the notation:

$$\neg\neg(x \in [Q_s]) \Leftrightarrow x \in [Q_s]''.$$

$$x \text{ is bounded in number} \Leftrightarrow x \in [N].$$

Now we are able to prove:

THEOREM 4.1: a) $[N]$ is invariant with respect to PQT.
b) x is determined in number $\Leftrightarrow x \in [Q_s T_s]$.

PROOF: In [4] de Iongh proved the invariance of $[N]$ with respect to P and Q so the invariance with respect to T remains to be proved.

Suppose aTb, with a bounded by n. For arbitrary species c we introduce the predicate $P_n(c)$, saying that c contains at least n different elements:

$$P_n(c) \Leftrightarrow (Ex_0)(Ex_1)\ldots(Ex_{n-1})\Big(\bigwedge_{\substack{i,j=0\\ i\neq j}}^{n-1} x_i \neq x_j \;\&\; \bigwedge_{i=0}^{n-1} x_i \in c\Big).$$

So we have $\neg P_{n+1}(a)$.

Suppose $P_{n+1}(b)$. This implies the existence of $n+1$ different elements $b_0, \ldots, b_n \in b$; hence

$$\bigwedge_{i=0}^{n} \neg\neg b_i \in b \;\&\; \bigwedge_{\substack{i,j=0\\ i\neq j}}^{n} b_i \neq b_j;$$

we get:

$$\neg\neg\Big(\bigwedge_{\substack{i,j=0\\ i\neq j}}^{n} b_i \neq b_j \;\&\; \bigwedge_{i=0}^{n} b_i \in a\Big),$$

hence

$$\neg\neg(Ex_0)\ldots(Ex_n)\Big(\bigwedge_{\substack{i,j=0\\ i\neq j}}^{n} x_i \neq x_j \;\&\; \bigwedge_{i=0}^{n} x_i \in a\Big),$$

and this is equivalent to $\neg\neg P_{n+1}(a)$. A contradiction arises, so $\neg P_{n+1}(b)$ and $b \in [N]$.

To prove b) of our theorem, we suppose a to be determined in number by n, aPb, $b = \{b_1, \ldots, b_n\}$, $i \neq j \rightarrow b_i \neq b_j$. We have:

$$(x \in a \;\&\; \neg(x = b_1 \vee x = b_2 \vee \ldots x = b_n))$$
$$\rightarrow x \neq b_1 \;\&\ldots\&\; x \neq b_n \;\&\; x \in a.$$

A contradiction is the result, hence

$$x \in a \rightarrow \neg\neg(x = b_1 \vee \ldots \vee x = b_n).$$

Therefore, $bT_s a$, so $a \in [Q_s T_s]$.

Remark 4.1: In [4] de Iongh also proved $[Q_s]''$ to be invariant with respect to PQ; for species with a stable equality he proved $[N] \subset [Q_s]''$.

Theorem 4.2: If a species b with stable equality is bounded by 1, then $b \in [PQ]$.

Proof: Take as a subspecies of $\{1\}$:

$$\{x : x = 1 \;\&\; (Ey)(y \in b)\}.$$

If $p \in b$, we take $\varphi(1) = p$. φ is unique since

$$x \in b \;\&\; x' \in b \rightarrow \neg\neg\, x = x';\Rightarrow x = x'.$$

Theorem 4.3: *Suppose a is a species bounded by n. If there is a partial ordering for the elements of a, which fulfils the condition*:

$$\neg(x < x') \;\&\; \neg(x' < x) \rightarrow x = x',$$

then $a \in [PQT]$.

Proof: Define $b \subset \bar{n}$ in the following manner:

$$b = \{i : i \in \bar{n} \;\&\; (Ex)(x \in a \;\&\; H_{i-1}(x) \;\&\; \neg H_i(x))$$

with:

$$H^0_0(x) \leftrightarrow (A \rightarrow A)$$
$$H^0_i(x) \leftrightarrow (Ex_1)\ldots(Ex_i)(x_1 < x_2 < \ldots < x_i < x \;\&\; \bigwedge_{j=1}^{i} \neg\neg x_j \in a)$$
$$H_i(x) \leftrightarrow \neg\neg\, H^0_i(x).$$

Now we are mapping b onto c:

$$i \in b \Rightarrow \varphi(i) = p \text{ if } p \in a \;\&\; H_{i-1}(p) \;\&\; \neg H_i(p).$$

We have to show this mapping to be unique.

For this reason we suppose:

$$H_{i-1}(p) \;\&\; \neg\, H_i(p) \;\&\; H_{i-1}(p') \;\&\; \neg\, H_i(p').$$

This is equivalent to:

$$\neg\neg\left(H^0_{i-1}(p)\ \&\ \neg H^0_i(p)\ \&\ H^0_{i-1}(p')\ \&\ \neg H^0_i(p')\right).$$

Thus we get:

$$p \in a\ \&\ p < p'\ \&\ H^0_{i-1}(p) \rightarrow H^0_i(p')$$
$$p' \in a\ \&\ p' < p\ \&\ H^0_{i-1}(p') \rightarrow H^0_i(p)$$

So $p, p' \in a\ \&\ H^0_{i-1}(p)\ \&\ \neg H^0_i(p)\ \&\ H^0_{i-1}(p')\ \&\ \neg H^0_i(p')$ implies $\neg(p < p')\ \&\ \neg(p' < p)$; therefore $p = p'$.

As a result we have:

$$p, p' \in a\ \&\ \neg\neg\left(H^0_{i-1}(p)\ \&\ \ldots\ \&\ \neg H^0_i(p')\right) \rightarrow \neg\neg p = p'.$$

From our supposition $\neg(x < x')\ \&\ \neg(x' < x) \rightarrow x = x'$ follows the stability of the equality, hence $p = p'$.

Next we have to show aTc. aPc is trivial, so we only have to prove: $x \in a \rightarrow \neg\neg\, x \in c$.

For sake of convenience we introduce:

$$G_i(x) \Leftrightarrow x \in a\ \&\ H_{i-1}(x)\ \&\ \neg H_i(x).$$

Suppose $x \in a$, $x \notin c$. We obtain

$$G_i(x) \rightarrow x \in c \qquad i = 1, 2, 3, \ldots$$
$$\neg G_1(x) \rightarrow \neg\neg H_1(x)$$
$$\neg G_2(x)\ \&\ H_1(x) \rightarrow \neg\neg H_2(x) \text{ etc.}$$

Finally,

$$\neg G_n(x)\ \&\ H_{n-1}(x) \rightarrow \neg\neg H_n(x)$$

Thus,

$$x \in a - c \Rightarrow H_n(x);\ \neg\neg H^0_n(x) \leftrightarrow H_n(x).$$

Since a is bounded in number, we have also:

$$T_{n+1} : (x_1)(x_2)\ldots(x_{n-1})\Big((\neg\neg(x_1 < x_2 < \ldots < x_{n+1})\ \&\ \neg\neg \bigwedge_{j=1}^{n} x_j \in a) \rightarrow \neg\, x_{n+1} \in a\Big).$$

T_{n+1} contradicts $H^0_n(x)$, hence also $\neg\neg H^0_n(x)$.

Therefore, $x \in a \rightarrow \neg\neg\, x \in c$, q.e.d..

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