COMPOSITIO MATHEMATICA

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*Compositio Mathematica*, tome 18, nº 1-2 (1967), p. 7-12
<http://www.numdam.org/item?id=CM_1967__18_1-2_7_0>

# On proximate type of entire functions

by

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## 1

In studying the growth of an entire function $f(z)$ of finite order $\rho$, use is made of a comparison function $\rho(r)$ called the proximate order [1, p. 54] of $f(z)$, which possesses the following properties:

i) $\rho(r)$ is real, continuous and piecewise differentiable for $r > l$,

ii) $\rho(r) \to \rho$ as $r \to \infty$,

iii) $\rho'(r)r \log r \to 0$ as $r \to \infty$, where $\rho'(r)$ is either the right or left hand derivative at points where they are different,

(iv) $\limsup_{r\to\infty} \dfrac{\log M(r)}{r^{\rho(r)}} = 1$

where

$$M(r) = \max_{|z|=r} |f(z)|.$$

It is evident that $\rho(r)$ has been linked with the order $\rho$ and $\log M(r)$ to give information about the growth of $f(z)$. Besides the order and the lower order there are two other constants, viz., the type $T$ and the lower type $t$ of $f(z)$ which give a more precise information about the growth than given by the order and lower order. These are determined as

$$\lim_{r\to\infty} \begin{matrix}\sup\\ \inf\end{matrix} \frac{\log M(r)}{r^\rho} = \frac{T}{t}, \qquad (0 < \rho < \infty).$$

Since the proximate order $\rho(r)$ is not linked with the type of $f(z)$ it becomes natural to search for another comparison function, $T(r)$, say, which should take into account the type of the function and be closely linked with its maximum modulus $M(r)$. In analogy with the proximate order we call this function $T(r)$ as a proximate type of the entire function $f(z)$.

In this paper we first define proximate type of an entire function and then prove its existence on lines similar to those of Shah [2] for the case of proximate order. The idea is further extended by defining a lower proximate type. Finally, we demonstrate that $r^{-\rho} \log M(r)$ is a proximate type for a certain class of entire functions.

## 2

DEFINITION. *A function $T(r)$ is said to be a proximate type of an entire function $f(z)$ of order $\rho$ $(0 < \rho < \infty)$ and finite type $T$ if it has the following properties*:

(2.1) *$T(r)$ is real, continuous and piecewise differentiable for $r > l$,*

(2.2) $T(r) \to T$ *as* $r \to \infty$,

(2.3) $rT'(r) \to 0$ *as* $r \to \infty$, *where $T'(r)$ is either the right or the left hand derivative at points where they are different,*

(2.4) $$\limsup_{r\to\infty} \frac{M(r)}{\exp\{r^\rho T(r)\}} = 1, \text{ where } M(r) = \max_{|z|=r} |f(z)|.$$

LEMMA. $\exp\{r^\rho T(r)\}$ *is an increasing function of $r$ for $r > r_0$.* By (2.1), (2.2) and (2.3) we have

$$\frac{d}{dr} \exp\{r^\rho T(r)\} = \{rT'(r)+\rho T(r)\}r^{\rho-1} \exp\{r^\rho T(r)\} > 0$$

for $r > r_0$, so the result follows.

THEOREM 1. *For every entire function $f(z)$ of order $\rho$ $(0 < \rho < \infty)$ and finite type $T$ there exists a proximate type $T(r)$.*

PROOF. Let $S(r) = r^{-\rho} \log M(r)$. Then two cases arise. Either (A) $S(r) > T$ for a sequence of values of $r$ tending to infinity, or (B) $S(r) \leqq T$ for all large $r$. In case (A), we define $Q(r) = \max_{x \geqq r} \{S(x)\}$. Since $S(x)$ is continuous, $\limsup_{x\to\infty} S(x) = T$ and $S(x) > T$ for a sequence of values of $x$ tending to infinity, $Q(r)$ exists and is a non-increasing function of $r$.

Let $r_1$ be a number such that $r_1 > e^e$ and $Q(r_1) = S(r_1)$. Such values will exist for a sequence of values of $r$ tending to infinity. Next, suppose $T(r_1) = Q(r_1)$ and let $t_1$ be the smallest integer not less than $1+r_1$ such that $Q(r_1) > Q(t_1)$ and set $T(r) = T(r_1) = Q(r_1)$ for $r_1 < r \leqq t_1$. Define $u_1$, as follows

$$
\begin{aligned}
&u_1 > t_1 \\
&T(r) = T(r_1) - \log\log r + \log\log t_1 \quad \text{for } t_1 \leqq r \leqq u_1, \\
&T(r) = Q(r) \quad \text{for } r = u_1,
\end{aligned}
$$

but

$$T(r) > Q(r) \quad \text{for } t_1 \leqq r < u_1.$$

Let $r_2$ be the smallest value of $r$ for which $r_2 \geqq u_1$ and $Q(r_2) = S(r_2)$. If $r_2 > u_1$ then let $T(r) = Q(r)$ for $u_1 \leqq r \leqq r_2$. Since $Q(r)$ is constant for $u_1 \leqq r \leqq r_2$, therefore $T(r)$ is constant for $u_1 \leqq r \leqq r_2$. We repeat the argument and obtain that $T(r)$ is differentiable in adjacent intervals. Further, $T'(r) = 0$, or $(-1/r \log r)$ and $T(r) \geqq Q(r) \geqq S(r)$ for all $r \geqq r_1$. Further, $T(r) = S(r)$ for an infinity of values of $r = r_1, r_2, \ldots$; $T(r)$ is non-increasing and $\lim_{r\to\infty} Q(r) = T$. Hence,

$$\limsup_{r\to\infty} T(r) = \lim_{r\to\infty} T(r) = T,$$

and since $M(r) = \exp\{r^\rho S(r)\} = \exp\{r^\rho T(r)\}$ for an infinity of $r$, $M(r) < \exp\{r^\rho T(r)\}$ for the remaining $r$, therefore

$$\limsup_{r\to\infty} \frac{M(r)}{\exp\{r^\rho T(r)\}} = 1.$$

Case (B). Let $S(r) \leqq T$ for all large $r$. Here there are two possibilities

(B.1) $$S(r) = T$$

for at least a sequence of values of $r$ tending to infinity;

(B.2) $$S(r) < T$$

for all large values of $r$.

In case (B.1) we take $T(r) = T$ for all values of $r$.

In case (B.2) let $P(r) = \max_{X \leqq x \leqq r} \{S(x)\}$ where $X > e^e$ is such that $S(x) < T$ for $x \geqq X$. $P(r)$ is non-decreasing. Take a suitably large value of $r_1 > X$ and let

$$T(r_1) = T,\ T(r) = T + \log\log r - \log\log r_1, \quad \text{for } s_1 \leqq r \leqq r_1,$$

where $s_1 < r_1$ is such that $P(s_1) = T(s_1)$. If $P(s_1) \neq S(s_1)$, then we take $T(r) = P(r)$ upto the nearest point $t_1 < s_1$, at which $P(t_1) = S(t_1)$. $T(r)$ is then constant for $t_1 \leqq r \leqq s_1$. If $P(s_1) = S(s_1)$, then let $t_1 = s_1$.

Choose $r_2 > r_1$ suitably large and let $T(r_2) = T$,

$$T(r) = T + \log\log r - \log\log r_2 \quad \text{for } s_2 \leqq r \leqq r_2$$

where $s_2$ $(< r_2)$ is such that $P(s_2) = T(s_2)$. If $P(s_2) \neq S(s_2)$ then let $T(r) = P(r)$ for $t_2 \leqq r \leqq s_2$ where $t_2(< s_2)$ is the point nearest to $s_2$ at which $P(t_2) = S(t_2)$.

If $P(s_2) = S(s_2)$, then let $t_2 = s_2$. For $r < t_2$ let

$$T(r) = T(t_2)+\log\log t_2-\log\log r \quad \text{for } u_1 \leqq r \leqq t_2$$

where $u_1(< t_2)$ is the point of intersection of $y = T$ with

$$y = T(t_2)+\log\log t_2-\log\log r.$$

Let $T(r) = T$ for $r_1 \leqq r \leqq u_1$. It is always possible to choose $r_2$ so large that $r_1 < u_1$. We repeat the procedure and note that

$$T(r) \geqq P(r) \geqq S(r)$$

and $T(r) = S(r)$ for $r = t_1, t_2, t_3, \ldots$. Hence

$$\lim_{r\to\infty} T(r) = T,$$

and

$$\limsup_{r\to\infty} \frac{M(r)}{\exp\{r^\rho T(r)\}} = 1.$$

Remark: It is possible to have a (smaller) class of functions $T(r)$ satisfying the conditions (2.1) to (2.4) and the relation

$$\lim_{r\to\infty} rT'(r)l_1rl_2r\ldots l_kr = 0.$$

The only change required in proving the existence of such functions is to take curves of the form

$$y = A\pm l_{k+2}r \quad (A \text{ is a contsant, } l_1r = \log r, \text{ etc.})$$

instead of $y = A\pm l_2r$ in our construction for $T(r)$.

## 3

Let $f(z)$ be an integral function of order $\rho(0 < \rho < \infty)$, finite type $T$ and lower type $t$. We consider the class of functions $t(r)$ satisfying the following conditions:

(3.1) $t(r)$ is a non-negative continuous function of $r$ for $r > r_0$,

(3.2) $t(r)$ is differentiable for $r > r_0$ except at isolated points at which $t'(r-0)$ and $t'(r+0)$ exist,

(3.3) $$\lim_{r\to\infty} rt'(r) = 0,$$

(3.4) $$\lim_{r\to\infty} t(r) = t,$$

$$\liminf_{r\to\infty} \frac{M(r)}{\exp\{(r^\rho M(r)\}} = 1. \tag{3.5}$$

These functions are defined in the same way, except for (3.4) and (3.5), as the proximate type defined in § 2. We call $t(r)$ a *lower proximate type* for the function $f(z)$. The existence of such functions can be proved in the same way as proved for $T(r)$ and so we omit the proof.

## 4

In this section we construct a proximate type for a class of entire functions. It is known [3, p. 27] that

$$\log M(r) = \log M(r_0) + \int_{r_0}^{r} x^{-1} W(x) dx \tag{4.1}$$

where $W(r)$ is a positive, indefinitely increasing function. Hence differentiating we get $M'(r)/M(r) = W(r)/r$ where $M'(r)$ is the derivative of $M(r)$ which exists for almost all values of $r$.

LEMMA. *If*

$$\lim_{r\to\infty} \begin{matrix}\sup\\ \inf\end{matrix} \frac{W(r)}{r^\rho} = \begin{matrix}\alpha\\ \beta\end{matrix} \qquad (0 < \rho < \infty) \tag{4.2}$$

*then*

$$\beta \leqq \rho \liminf_{r\to\infty} \frac{\log M(r)}{r^\rho} \leqq \rho \limsup_{r\to\infty} \frac{\log M(r)}{r^\rho} \leqq \alpha. \tag{4.3}$$

PROOF. For any $\varepsilon > 0$ and $r > r_0' = r_0'(\varepsilon)$, we have from (4.2)

$$\beta - \varepsilon < \frac{W(r)}{r^\rho} < \alpha + \varepsilon.$$

So, for $r > \max(r_0, r_0')$, we have

$$(\beta-\varepsilon) r^{\rho-1} < \frac{M'(r)}{M(r)} < (\alpha+\varepsilon) r^{\rho-1}.$$

Integrating the above inequalities between suitable limits and then dividing by $r^\rho$ and proceeding to limits we get the result in (4.3).

We are now in a position to prove the following:

THEOREM 2. *Let $f(z)$ be an integral function of order $\rho$ $(0 < \rho < \infty)$ and type $T$ $(0 < T < \infty)$ and let $M(r) = \max_{|z|=r} |f(z)|$ and $W(r)$ be given by (4.1). If $\lim_{r\to\infty} (W(r))/r^\rho$ exists then $(\log M(r))/r^\rho$ is a proximate type of $f(z)$.*

PROOF. Let

$$T(r) = \frac{\log M(r)}{r^\rho}. \tag{4.4}$$

Since $\log M(r)$ is a real, continuous, increasing function of $r$, which is differentiable in adjacent intervals, it follows that $T(r)$ satisfies (2.1). Since $\lim_{r\to\infty} r^{-\rho} W(r)$ exists, (4.3) shows that $\lim_{r\to\infty} r^{-\rho} \log M(r)$ also exists and so $T(r) \to T$ as $r \to \infty$. Further $T(r)$ is piecewise differentiable and it has right and left hand derivatives where they are different, so

$$T'(r)r^\rho + \rho r^{\rho-1} T(r) = \frac{M'(r)}{M(r)}$$

or,

$$\lim_{r\to\infty} rT'(r) = \lim_{r\to\infty} \left[\frac{M'(r)}{M(r)r^{\rho-1}} - \rho T(r)\right]$$

$$= \lim_{r\to\infty} \left[\frac{W(r)}{r^\rho} - \rho T(r)\right] = 0.$$

Thus, $T(r)$ satisfies the condition (2.3) also. Finally

$$\limsup_{r\to\infty} \frac{M(r)}{\exp\,[r^\rho T(r)]} = 1$$

follows from (4.4). Hence the theorem is established.

## REFERENCES

CARTWRIGHT, M. L.

[1] Integral Functions, Cambridge Tract, C.U.P., 1956.

SHAH, S. M.

[2] On proximate orders of integral functions. Bull. Amer. Math. Soc. 52, (1946), 326—328.

VALIRON, G.

[3] Lectures on the general theory of integral functions. Chel. Publ. Co. New York, (1949).

(Oblatum 24-10-65)