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On the Fourier coefficients of a simple discontinuous function

by

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1

Let $f(x)$ be a function integrable in the sense of Lebesgue over the interval $(-\pi, \pi)$ and defined outside this by periodicity. Let its Fourier series be

$$\frac{1}{2}a_0 + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx) \equiv \sum A_n(x).$$

Then

$$\sum_{n=1}^{\infty} (b_n \cos nx - a_n \sin nx) \equiv \sum B_n(x)$$

is the conjugate series of the Fourier series of $f(x)$.

For a fixed x , we write

$$\psi(t) = \psi_x(t) = f(x+t) - f(x-t) - l.$$

2

Let $\sum_{n=1}^{\infty} a_n$ be an infinite series with the partial sums $\{s_n\}$. Let σ_n be the first arithmetical means of s_n , that is, $\sigma_n = (\sum_{\nu=1}^n s_\nu)/n$. Let $A : (a_{n,m})$ ($m = 1, 2, \dots; n \leq m$) be a regular triangular matrix. If

$$\lim_{m \rightarrow \infty} \sum_{n=1}^m a_{n,m} \sigma_n = \tau,$$

then we say that the series $\sum a_n$ or the sequence $\{s_n\}$ is summable A. (C, 1) to the sum τ .

3

In this note, we prove a theorem for the A method of summation of the Fourier coefficients of $f(t)$ connected with its jump at the point $t = x$.

THEOREM. Let the jump of $f(t)$ at $t = x$ be l , i.e.,

$$f(x+0) - f(x-0) = l.$$

If A satisfies

$$(i) \quad \sum_{n=1}^m n |a_{n,m} - a_{n+1,m}| = o(1) \quad [2],$$

and

$$(ii) \quad \sum_{n=1}^m |a_{n,m}| = o(1)$$

as $m \rightarrow \infty$, then the sequence

$$\{nB_n(x)\} \equiv \{n(b_n \cos nx - a_n \sin nx)\}$$

is summable $A. (C, 1)$ to l/π .

4

If we denote by σ_n the $(C, 1)$ -transformation of $\{nB_n(x)\}$, we have, after Mohanty-Nanda [1],

$$\sigma_n - \frac{l}{\pi} = \frac{1}{\pi} \int_0^1 \psi(t) \left\{ \frac{\sin nt}{nt^2} - \frac{\cos nt}{t} \right\} dt + o(1),$$

by Riemann-Lebesgue's theorem.

On account of the regularity of the A method, we need establish that

$$I = \frac{1}{\pi} \sum_{n=1}^m a_{n,m} \int_0^1 \psi(t) g_n(t) dt = o(1)$$

as $m \rightarrow \infty$, where

$$g_n(t) = \frac{\sin nt}{nt^2} - \frac{\cos nt}{t},$$

which is $O(n^2t)$ by expanding $\sin nt$ and $\cos nt$ into the power series of n and t . It is known that [3, p. 5]

$$\sum_{\nu=1}^n \frac{\sin \nu t}{\nu} = O(1).$$

Thus, if we write

$$\begin{aligned} I &= \frac{1}{\pi} \sum_{n=1}^m a_{n,m} \left\{ \int_0^{n^{-1}} + \int_{n^{-1}}^1 \right\} \psi(t) g_n(t) dt \\ &= \frac{1}{\pi} \sum_{n=1}^m a_{n,m} \{P + Q\}, \end{aligned}$$

say, then,

$$\begin{aligned}
 |P| &\leq \left(\sum_{n=1}^m |a_{n,m}| \right) \int_0^{n^{-1}} |\psi(t)| |g_n(t)| dt \\
 &= o(1) \cdot O\left(n^2 \int_0^{n^{-1}} t dt \right) \\
 &= o\left(n^2 \int_0^{n^{-1}} t dt \right) \\
 &= o(1).
 \end{aligned}$$

We denote

$$\begin{aligned}
 G_\nu(t) &= g_1(t) + g_2(t) + \dots + g_\nu(t) \\
 &= \frac{1}{t^2} \sum_{\mu=1}^\nu \frac{\sin \mu t}{\mu} - \frac{1}{t} \sum_{\mu=1}^\nu \cos \mu t \\
 &= O\left(\frac{1}{t^2}\right) - \frac{1}{t} D_\nu(t) \\
 &= O\left(\frac{1}{t^2}\right),
 \end{aligned}$$

where $D_\nu(t)$ is the Dirichlet kernel of the Fourier series of $f(t)$. It is known that [3, p. 49] $D_\nu(t) = O(t^{-1})$.

By considering that

$$\sum_{n=1}^m |a_{n,m}| = \sum_{n=1}^{m-1} n(|a_{n,m}| - |a_{n+1,m}|) + m|a_{m,m}|$$

and

$$\left| \sum_{n=1}^{m-1} n(|a_{n,m}| - |a_{n+1,m}|) \right| \leq \sum_{n=1}^m |a_{n,m} - a_{n+1,m}|,$$

we see that the conditions (i) and (ii) imply that $ma_{m,m} = o(1)$ as $m \rightarrow \infty$. So that, we write

$$\begin{aligned}
 \left| \frac{1}{\pi} \sum_{n=1}^m a_{n,m} Q \right| &= \left| \frac{1}{\pi} \sum_{n=1}^m a_{n,m} \int_{n^{-1}}^1 \psi(t) \{G_n(t) - G_{n-1}(t)\} dt \right| \\
 &\leq \left| \frac{1}{\pi} \sum_{n=1}^{m-1} (a_{n,m} - a_{n+1,m}) \int_{n^{-1}}^1 \psi(t) G_n(t) dt \right| \\
 &\quad + \left| \frac{1}{\pi} \sum_{n=2}^m a_{n,m} \int_{n^{-1}}^{(n-1)^{-1}} \psi(t) G_{n-1}(t) dt \right| + o(1) \\
 &= J_1 + J_2 + o(1),
 \end{aligned}$$

say, by Abel's transformation. Now,

$$\begin{aligned} J_1 &= O \left\{ \sum_{n=1}^{m-1} |a_{n,m} - a_{n+1,m}| \int_{n^{-1}}^1 \frac{dt}{t^2} \right\} \\ &= O \left\{ \sum_{n=1}^m n |a_{n,m} - a_{n+1,m}| \right\} \\ &= o(1) \end{aligned}$$

by the condition (i). And,

$$\begin{aligned} J_2 &= O \left\{ \sum_{n=2}^m |a_{n,m}| \int_{n^{-1}}^{(n-1)^{-1}} \frac{dt}{t^2} \right\} \\ &= O \left\{ \sum_{n=1}^m |a_{n,m}| \right\} \\ &= o(1) \end{aligned}$$

as $m \rightarrow \infty$ by the condition (ii) of the theorem.

The theorem is thus completely proved.

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