R. S. DAHIYA

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Certain theorems on n-dimensional operational calculus

by

R. S. Dahiya

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In the present paper we obtain a few theorems in n-dimensional operational calculus starting from given operational relations in one variable. These theorems are next applied to the evaluation of a few operational results. In the theorems, we shall take $D(\sigma_{10}, \ldots \sigma_{n0})$ to represent the set of points for which $R(P_i) > R(P_{i0}) = \sigma_{i0} > 0$ and unless otherwise stated, D will represent the set of points for which $R(P_i) > 0$, $i = 1, 2, \ldots, n$.

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THEOREM 1. Let

(i) $$\phi(p) \subset f(x)$$

(ii) $$p\sigma(p) \subset f(x^2)$$

(iii) $$H(p) \subset x^{-n}\phi(1/x^2)$$

where $f(x)$, $f(x^2)$ and $x^{-n}\phi(1/x^2)$ are each integrable L in $(0, \infty)$ or the definition integral in each of (i), (ii), (iii) is absolutely convergent for $R(P) > 0$.

If $x_1, \ldots, x_n$, be a set of real numbers > 0, then

$$\frac{\sigma(\sqrt{1/x_1 + \ldots + 1/x_n})}{(x_1 \ldots x_n)^{\frac{3}{2}}} \supset_n \frac{\pi^{(n-1)/2} 2^n (p_1 \ldots p_n)}{\sqrt{p_1} + \ldots + \sqrt{p_n}} \cdot H(\sqrt{p_1} + \ldots + \sqrt{p_n}) \tag{2.1}$$

provided that

(a) the integral in (2.1) exists as an absolutely convergent integral for $(p_1, \ldots, p_n) \in D(\sigma_{10}, \ldots, \sigma_{n0})$; or

(b) the original in (2.1) is integrable L in $(0, \infty)$ with respect to the variables $x_1, \ldots, x_n$; (the operational variables $p_1, \ldots, p_n$ corresponding to $x_1, \ldots x_n$).

From (i), we have

$$(2.2) \qquad f(x^2) \supset \frac{p}{\sqrt{\pi}} \int_0^\infty e^{-\frac{1}{4}p^2u^2} \phi\left(\frac{1}{u^2}\right) du$$

where the integrals on the right is absolutely convergent. So that, by (ii), we can write (2.2) as

$$(2.3) \qquad \sigma(p) = \frac{1}{\sqrt{\pi}} \int_0^\infty e^{-\frac{1}{4}p^2u^2} \phi\left(\frac{1}{u^2}\right) du$$

writing (2.3) in the form

$$(2.4) \qquad \frac{\pi^{(1-n)/2}\sigma(\sqrt{1/x_1 + \ldots + 1/x_n})}{(x_1 \ldots x_n)^{\frac{3}{2}}} = \pi^{-n/2} \int_0^\infty (x_1 \ldots x_n)^{-\frac{3}{2}} \exp\left(-\tfrac{1}{4} \sum \frac{u^2}{x_i}\right) \phi\left(\frac{1}{u^2}\right) du.$$

We multiply both sides by $\exp(-\sum p_i x_i)$, integrate with respect to x_i between the limits $(0, \infty)$ and then change the order of integration in the resulting integral on the right, permissible by Fubini's theorem, on account of the absolute convergence.

This gives

$$(2.5) \qquad \pi^{(1-n)/2} \int_0^\infty \cdots \int_0^\infty \exp(-\sum p_i x_i) \frac{\sigma(\sqrt{1/x_1 + \ldots + 1/x_n})}{(x_1 \ldots x_n)^{\frac{3}{2}}} dx_1 \ldots dx_n$$
$$= \pi^{-n/2} \int_0^\infty \phi\left(\frac{1}{u^2}\right) du \int_0^\infty x_1^{-\frac{3}{2}} \exp\left(-\tfrac{1}{4}\frac{u^2}{x_1} - p_1 x_1\right) dx_1 \ldots$$
$$\int_0^\infty x_n^{-\frac{3}{2}} \exp\left(-p_n x_n - \tfrac{1}{4}\frac{u^2}{x_n}\right) dx_n.$$

Evaluating the inner integrals on the right by

$$(\mathrm{A}) \qquad \int_0^\infty x^{-\frac{3}{2}} e^{-px - \frac{1}{4}(a^2)/x} dx = \frac{2\sqrt{\pi}}{a} e^{-a\sqrt{p}}.$$

We obtain, on account of the operational relation (iii), the required theorem.

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THEOREM 2. *Let*

(i) $$\phi(p) \subset f(x)$$

(ii) $$\psi(p) \subset x^{-\frac{1}{2}}\phi\left(\frac{1}{x}\right)$$

(iii) $$\lambda(p) \subset x^{1-n}f(x^4)$$

where $f(x)$ $x^{-\frac{1}{2}}\phi(1/x)$ *&* $x^{1-n}f(x^4)$ *are each integrable L in* $(0, \infty)$ *or the definition integral in each of* (i), (ii) *and* (iii) *is absolutely convergent for* $R(P) > 0$. *If* $x_1, \ldots, x_n$ *be a set of positive real numbers, then*

$$\frac{\psi[\frac{1}{64}(1/x_1+\ldots+1/x_n)^2]}{(x_1\ldots x_n)^{\frac{3}{2}}(1/x_1+\ldots+1/x_n)^2} \supset_n \frac{2^{n-4}\pi^{(n+1)/2}(p_1\ldots p_n)}{\sqrt{p_1}+\ldots+\sqrt{p_n}} \cdot \lambda(\sqrt{p_1}+\ldots+\sqrt{p_n} \tag{3.1}$$

provided that

(*a*) *the definition integral in* (3.1) *is absolutely convergent for* $(p_1, \ldots, p_n) \in D(\sigma_{10}, \ldots, \sigma_{n0})$; *or*

(*b*) *the original in* (3.1) *is integrable L in* $(0, \infty)$ *with respect to the variables* $x_1, \ldots, x_n$; (*the operational variables* $p_1, \ldots, p_n$ *corresponding to* $x_1, \ldots, x_n$).

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THEOREM 3. *Let*

(i) $$\phi(p) \subset f(x)$$

(ii) $$\phi(\sqrt{p}) \subset F\left(\frac{1}{x}\right)$$

(iii) $$H(p) \subset \frac{1}{x^n} f(x).$$

Then

$$\frac{F(1/x_1+\ldots+1/x_n)}{(x_1\ldots x_n)^{\frac{3}{2}}(1/x_1+\ldots+1/x_n)^{\frac{1}{2}}} \supset_n \frac{2^n\pi^{(n-1)/2}(p_1\ldots p_n)}{\sqrt{p_1}+\ldots+\sqrt{p_n}} \cdot H(\sqrt{p_1}+\ldots+\sqrt{p_n}). \tag{4.1}$$

The proofs of theorems 2 and 3 are on the same lines.

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As applications of theorems 1, 2 and 3, we shall find some two-dimensional correspondences.

(a) Let

$$f(x) = J_{2n,n}(3\sqrt[3]{x}); \text{ so that } \phi(p) = J_{2n}\left(\frac{2}{\sqrt{p}}\right)$$

$$f(x^2) = J_{2n,n}(3\sqrt[3]{x^2}); \text{ so that } \sigma(p) = \frac{1}{p}\, e^{-(2/p^2)} I_n\left(\frac{2}{p^2}\right)$$

and $n = 2$, then

$$x^{-n}\phi\left(\frac{1}{x^2}\right) = x^{-2}J_{2n}(2x);$$

so that

$$H(p) = \frac{\Gamma(2n-1)}{\Gamma(2n+1)p^{2n-2}}\, {}_2F_1\left[\begin{matrix} n-\frac{1}{2}, n; \\ 2n+1; \end{matrix}\; -\frac{4}{p^2}\right].$$

Hence we obtain, from (2.1)

$$\frac{e^{-(2xy)/(x+y)}}{xy(x+y)^{\frac{1}{2}}} I_n\left(\frac{2xy}{x+y}\right) \supset_2 \frac{2\sqrt{\pi}}{n(2n-1)}\, \frac{pq}{(\sqrt{p}+\sqrt{q})^{2n-1}}$$
$$\cdot\, {}_2F_1\left[\begin{matrix} n-\frac{1}{2}, n; \\ 2n+1; \end{matrix}\; -\frac{4}{(\sqrt{p}+\sqrt{q})^2}\right].$$

(b) Let

$$f(x) = x^{\nu/2}J_\nu(2a\sqrt{x});$$

so that

$$\phi(p) = a^\nu p^{-\nu} e^{-(a^2/p)},$$

$$f(x^2) = x^\nu J_\nu(2ax);$$

so that

$$p\sigma(p) = \frac{a^\nu 2^{2\nu}\Gamma(\nu+\frac{1}{2})p}{\sqrt{\pi}(p^2+4a^2)^{\nu+\frac{1}{2}}},\quad R(\nu) > -\tfrac{1}{2},$$

$$x^{-n}\phi\left(\frac{1}{x^2}\right) = a^\nu x^{2\nu-2}e^{-a^2x^2};$$

so that

$$H(p) = \frac{\Gamma(2\nu-2)p}{(8a^2)^{\nu-\frac{1}{2}}}\, e^{(p^2/8a^2)} D_{2\nu-1}\left(\frac{p}{\sqrt{2}a}\right),\quad R(a) < 0.$$

Hence from (2.1), we get

$$\frac{(xy)^{\nu-1}}{(x+y+4a^2xy)^{\nu+\frac{1}{2}}} \supset_2 \frac{\sqrt{\pi}\,\Gamma(2\nu-1)}{a^{3\nu-1}2^{5\nu-\frac{7}{2}}}$$
$$pq\, e^{((\sqrt{p}+\sqrt{q})^2/8a^2)} D_{2\nu-1}\left(\frac{\sqrt{p}+\sqrt{q}}{\sqrt{2}a}\right),\quad \begin{matrix} R(\nu) > -\frac{1}{2} \\ R(a) < 0 \end{matrix}.$$

(c) Let

$$f(x) = \sqrt{x} L_1(\sqrt{x});$$

so that
$$\phi(p) = \frac{1}{2p} e^{\frac{1}{4}p} \operatorname{Erf} e\left(\frac{1}{2\sqrt{p}}\right)$$

$$f(x^2) = x L_1(x);$$

so that
$$p\sigma(p) = \frac{p}{(p^2-1)^{\frac{3}{2}}} - \frac{2\sqrt{2}\,p^{-\frac{1}{2}}}{\sqrt{\pi}(1-p^2)^{-\frac{3}{4}}} P_{-\frac{1}{2}}^{-\frac{3}{2}}\left(\frac{1}{p}\right)$$

$$x^{-n}\phi\left(\frac{1}{x^2}\right) = \tfrac{1}{2} e^{x^2/4} \operatorname{Erf} c\left(\frac{x}{2}\right);$$

so that
$$H(p) = \frac{p}{2\sqrt{\pi}} e^{-p^2} Ei(p^2).$$

Hence from (2.1), we get

$$2pq\, e^{-(\sqrt{p}+\sqrt{q})^2} Ei\,(\sqrt{p}+\sqrt{q}) \subset_2 \frac{1}{(x+y-xy)^{\frac{3}{2}}} - \frac{2\sqrt{2}(xy-x-y)^{\frac{3}{4}}}{\sqrt{\pi}(xy)^{\frac{3}{4}}(x+y)^{\frac{3}{4}}} P_{-\frac{1}{2}}^{-\frac{3}{2}}\left(\sqrt{\frac{xy}{x+y}}\right).$$

(d) Let

$$f(x) = \frac{x^{\nu}}{\Gamma(\nu+1)};$$

so that
$$\sigma(p) = \frac{1}{p^{\nu}}; \quad R(\nu) > -1.$$

$$f(x^2) = \frac{x^{2\nu}}{\Gamma(\nu+1)};$$

so that
$$\sigma(p) = \frac{\Gamma(2\nu+1)}{\Gamma(\nu+1)} p^{-1-2\nu}, \ R(\nu) > -\tfrac{1}{2}$$

and $n = 3$, then

$$x^{-n}\phi\left(\frac{1}{x^2}\right) = x^{2\nu-3}; \text{ so that } H(p) = \Gamma(2\nu-2)p^{3-2\nu}.$$

Thus we obtain, from (2.1)

$$\frac{(xyz)^{\nu-1}}{(xy+yz+zx)^{\nu+\frac{1}{2}}} \supset_3 \frac{8\pi\Gamma(2\nu-2)\Gamma(\nu+1)}{\Gamma(2\nu+1)} \times \frac{pqr}{(\sqrt{p}+\sqrt{q}+\sqrt{r})^{2\nu-2}}, \ R(\nu) > -\tfrac{1}{2}.$$

(e) Let

$$f(x) = (2x)^{\nu-1}e^{-2\sqrt{x}};$$

so that $$\phi(p) = \Gamma(2\nu)p^{1-\nu}e^{1/2p}D_{-2\nu}\left(\sqrt{\frac{2}{p}}\right)$$

$$x^{-\frac{1}{2}}\phi(x) = \Gamma(2\nu)x^{\nu-\frac{3}{2}}e^{x/2}D_{-2\nu}(\sqrt{2x});$$

so that $$\psi(p) = \frac{\sqrt{2\pi}\,\Gamma(2\nu-1)p}{2^{\nu-\frac{1}{2}}(p-1)^{\nu-\frac{1}{2}}}\,P_{-1}^{1-2\nu}\left(\frac{1}{\sqrt{p}}\right),\ R(\nu) > \tfrac{1}{2}$$

and $n = 2$, then

$$x^{1-n}f(x^4) = 2^{\nu-1}x^{4\nu-5}e^{-2x^2};$$

so that $$\lambda(p) = \frac{\Gamma(4\nu-4)}{2^{3(\nu-1)}}\,p\,e^{(p^2/16)}D_{4-4\nu}\left(\frac{p}{2}\right).$$

Hence from theorem 2, we obtain

$$\frac{(xy)^{2\nu-\frac{5}{2}}}{[(x^2+y^2)-64x^2y^2]^{\nu-\frac{1}{2}}}\,P_{-1}^{1-2\nu}\left(\frac{xy}{8(x+y)}\right)$$
$$\supset_2 \frac{\pi\Gamma(4\nu-4)pq}{2^{8\nu-q}\Gamma(2\nu-1)}\,e^{((\sqrt{p}+\sqrt{q})^2/16)}D_{4-4\nu}\left(\frac{\sqrt{p}+\sqrt{q}}{2}\right),\ R(\nu) > +\tfrac{1}{2}.$$

(f) Let

$$f(x) = x^{\nu-1}e^{-(x^2/8a)};$$

so that $$\phi(p) = 2^{\nu}a^{\nu/2}\Gamma(\nu)p\,e^{ap^2}D_{-\nu}(2\sqrt{a}p)$$

$$\phi(\sqrt{p}) = \Gamma(\nu)2^{\nu}a^{\nu/2}\sqrt{p}\,e^{ap}D_{-\nu}(2\sqrt{ap});$$

so that $$F\left(\frac{1}{x}\right) = \frac{\Gamma(\nu)(2a)^{\nu/2}}{\Gamma((\nu+1)/2)}\,x^{(\nu-1)/2}(x+2a)^{-\nu/2}$$

and $n = 2$, then

$$x^{-n}f(x) = x^{\nu-3}e^{-(x^2/8a)};$$

so that $$H(p) = 2^{\nu-2}a^{(\nu/2)-1}\Gamma(\nu-2)p\,e^{ap^2}D_{2-\nu}(2\sqrt{a}p).$$

Hence from (4.1), we obtain

$$\frac{(xy)^{(\nu-3)/2}}{(x+y)^{\nu/2}(xy/(x+y)+2a)^{(\nu+1)/2}} \supset_2 \frac{\sqrt{\pi}\,2^{\nu/2}\Gamma((\nu+1)/2)}{a(\nu-1)(\nu-2)}\,pq$$
$$\exp\,(a(\sqrt{p}+\sqrt{q})^2]D_{2-\nu}[2\sqrt{a}(\sqrt{p}+\sqrt{q})\ R(\nu) > -2.$$

(g) Let

$$f(x) = x^{\nu+1} J_\nu(ax);$$

so that
$$\phi(p) = \frac{2^{\nu+1}}{\sqrt{\pi}} \frac{\Gamma(\nu+\frac{3}{2}) a^\nu p^2}{(p^2+a^2)^{\nu+\frac{3}{2}}}, \; R(\nu) > -\tfrac{1}{2}.$$

$$\phi(\sqrt{p}) = \frac{2^{\nu+1} \Gamma(\nu+\frac{3}{2})}{\sqrt{\pi}} \frac{a^\nu p}{(p+a^2)^{\nu+\frac{3}{2}}};$$

so that
$$F\left(\frac{1}{x}\right) = \frac{2^{\nu+1} a^\nu}{\sqrt{\pi}} x^{\nu+\frac{1}{2}} e^{-a^2 x}$$

and $n = 2$, then

$$x^{-n} f(x) = x^{\nu-1} J_\nu(ax);$$

so that
$$H(p) = \frac{\Gamma(2\nu) p}{(p^2+a^2)^{+(\nu/2)}} P_{\nu-1}^{-\nu}\left(\frac{p}{\sqrt{p^2+a^2}}\right).$$

Hence from (4.1), we get

$$\frac{(xy)^{\nu-\frac{1}{2}}}{(x+y)^{\nu+1}} e^{-a^2 xy/(x+y)} \supset_2 \frac{\pi \Gamma(2\nu)}{2^{\nu-1} a^\nu [(\sqrt{p}+\sqrt{q})^2+a^2]^{-(\nu/2)}}$$
$$P_{\nu-1}^{-\nu}\left[\frac{\sqrt{p}+\sqrt{q}}{\sqrt{(\sqrt{p}+\sqrt{q})^2+a^2}}\right], \; R(\nu) > -\tfrac{1}{2}.$$

(h) Let

$$f(x) = x^3 J_{\frac{3}{2}}^2(x);$$

so that
$$\phi(p) = \frac{5 \cdot 2^5}{\pi p^6} \, 2F_1\left[2, \tfrac{7}{2}; \tfrac{5}{2}; -\frac{4}{p^2}\right]$$

$$\phi(\sqrt{p}) = \frac{5 \cdot 2^5}{\pi p^3} \, 2F_1\left[2, \tfrac{7}{2}; \tfrac{5}{2}; -\frac{4}{p}\right];$$

so that
$$F\left(\frac{1}{x}\right) = \frac{5 \cdot 2^5 x^3}{\pi \Gamma(4)} \, 2F_2 \left[2, \tfrac{7}{2}; \tfrac{5}{2}, 4; -4x\right]$$

and $n = 2$, then

$$x^{-n} f(x) = x J_{\frac{3}{2}}^2(x);$$

so that
$$H(p) = \frac{2^4}{\pi \Gamma(4) p^5} \, 2F_1\left[([2, 3; 4, -\frac{4}{p^2}\right].$$

Hence from (4.1), we get

$$\frac{(xy)^2}{(x+y)^{\frac{7}{2}}}\, {}_2F_2\left[2, \tfrac{7}{2}; \tfrac{5}{2}, 4; -\frac{4(x+y)}{xy}\right] \supset_2 \frac{4\sqrt{\pi}\, pq}{5(\sqrt{p}+\sqrt{q})^5}\, {}_2F_1\left[2, 3; 4; -\frac{4}{(\sqrt{p}+\sqrt{q})^2}\right].$$

(i) In the theorem taking $n = 3$ and let

$$f(x) = \frac{x^\nu}{\Gamma(\nu+1)};$$

so that $$\phi(p) = \frac{1}{p^\nu},\ R(\nu) > -1$$

$$x^{-n}f(x) = \frac{x^{\nu-3}}{\Gamma(\nu+1)};$$

so that $$H(p) = \frac{\Gamma(\nu-2)}{\Gamma(\nu+1)p^{\nu-3}}$$

$$\phi(\sqrt{p}) = p^{-(\nu/2)};$$

so that $$F\left(\frac{1}{x}\right) = \frac{x^{\nu/2}}{\Gamma((\nu+2)/2)}.$$

Hence from (4.1), we get

$$\frac{(xyz)^{(\nu-2)/2}}{(xy+yz+zx)^{(\nu+1)/2}} \supset_3 \frac{8\pi\Gamma((\nu+2)/2)}{\nu(\nu-1)(\nu-2)} \times \frac{pqr}{(\sqrt{p}+\sqrt{q}+\sqrt{r})^{\nu-2}},\ R(\nu) > -2.$$

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Deptt. of Maths.
Birla college of Engineering
Pilani (Raj.) India.