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by

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The notion of a uniformly distributed sequence has been generalized by L. A. Rubel [4] from compact groups to locally compact groups. A locally compact group that contains a uniformly distributed sequence is called monogenic. Our main result is the characterization of discrete monogenic groups as direct sums of divisible groups and distinguished subgroups of certain cartesian products of finite cyclic groups and p-adic integers.

The following definitions are included for the reader's convenience. All groups are abelian.

DEFINITION. A topological group G is *monothetic* if it contains a dense cyclic subgroup $[x_0]$. A generator of such a subgroup is called a *monothetic generator* of G.

DEFINITION. Let G be a compact group and $M(G)$ the space of bounded Borel measures on G. Let $\{x_i\}$ be a sequence in G, and let δ_i be the point mass at x_i (i.e., mass 1 at x_i). The sequence $\{x_i\}$ is *uniformly distributed* in G if the sequence $\mu_n = 1/n$ $(\delta_1 + \ldots + \delta_n)$ converges weak * in $M(G)$ to normalized Haar measure ν.

DEFINITION. (Rubel) Let G be a locally compact group, and let $\{x_i\}$ be a sequence in G. The sequence $\{x_i\}$ is *uniformly distributed* in G if $\{\varphi(x_i)\}$ is uniformly distributed in $\varphi(G)$ whenever $\varphi(G)$ is compact and φ is a continuous open homomorphism.

DEFINITION. A locally compact group G is *monogenic* if it contains an element x_0 such that $\{x_0, 2x_0, 3x_0, \ldots\}$ is uniformly distributed; such an element x_0 is called a *monogenic generator* of G.

The proof of the following lemma is immediate.

LEMMA 0. *Let x_0 be a monogenic generator of a locally compact group G, H a closed subgroup, and φ: $G \to G/H$ the natural map. Then G/H is monogenic with monogenic generator $\varphi(x_0)$.*

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Eckmann [1] has shown that if G is compact, then G is monogenic if and only if G is monothetic; moreover, x_0 is a monogenic generator if and only if x_0 is a monothetic generator. It is easy to see by duality that a locally compact group is monogenic if and only if there is an element $x_0 \in G$ (the monogenic generator) that separates every discrete subgroup of G^*, the character group of G.

It is easy to construct examples of a direct product of two monogenic groups that is not monogenic, and of an open subgroup of a monogenic group that is not monogenic. Here is an example of a continuous homomorphic image of a monogenic group that is not monogenic. Let R_d be the discrete reals, and let R be the reals in the usual topology; it is easy to see that R_d is monogenic and that R is not monogenic, but the identity map is a continuous homomorphism from R_d onto R.

From now on, all groups are discrete unless stated otherwise.

LEMMA 1. *A discrete group G is monogenic if and only if there is a cyclic subgroup A of G such that, whenever K is a finite group and $f\colon G \to K$ is an epimorphism, then $K = f(A)$.*

PROOF: In the definition of monogenic, we consider all $\varphi\colon G \to \varphi(G)$ such that $\varphi(G)$ is compact and φ is open. If G is discrete, then $\varphi(G)$ is discrete, so that we need only consider finite images of G. By Eckmann [1], all such images are cyclic with generator $\varphi(x_0)$, where $[x_0] = A$.

It follows that every finite image of a monogenic group is cyclic.

PROPOSITION 2. A discrete group G is monogenic if and only if G contains a cyclic subgroup A with G/A divisible.

PROOF: Let G be monogenic with monogenic generator x_0, and let $A = [x_0]$. Then G/A has no finite images and so is divisible.

Conversely, let K be finite and $f\colon G \to K$ be onto. Then f induces an epimorphism $G/A \to K/f(A)$. Since G/A is divisible, so is its image $K/f(A)$. But $K/f(A)$ is finite, hence 0. Therefore $K = f(A)$ and G is monogenic.

By Lemma 0, any homomorphic image of a monogenic group is monogenic. In particular, any direct summand of a monogenic group is monogenic.

Recall that a group G is *reduced* if it contains no non-zero divisible subgroups. An arbitrary group $G = D \oplus R$, where D is divisible and R is reduced. The following proposition lets us restrict our attention to reduced groups.

PROPOSITION 3. If G is monogenic and D is divisible, then $G \oplus D$ is monogenic. If G is monogenic and $G = E \oplus R$, where E is divisible and R is reduced, then R is monogenic.

PROOF: Suppose A is a cyclic subgroup of G with G/A divisible. Then $(G \oplus D)/A \approx (G/A) \oplus D$ which is divisible, so that $G \oplus D$ is monogenic. The second part follows from Lemma 0, for R is an image of G.

DEFINITION. A subgroup S of G is *pure* in G in case

$$S \cap nG = nS$$

for every integer n.

Two examples of pure subgroups are the torsion subgroup and any direct summand. A partial converse is [3, p. 18]: if S is a pure subgroup of bounded order (i.e., $nS = 0$ for some $n \neq 0$), then S is a direct summand.

PROPOSITION 4. Let G be a reduced monogenic group with torsion subgroup tG. Then tG can be embedded in Q/Z, the rationals modulo one.

PROOF: We prove the following statement from which the proposition follows easily: for every prime p, the p-primary component T_p of tG is cyclic. If $T_p \neq 0$, then G contains a cyclic p-primary direct summand C [2, p. 80]. Suppose $C \neq T_p$. If $G = C \oplus H$, then H is monogenic, and it contains elements of order p. Thus, $H = C' \oplus H'$, where C' is a p-primary cyclic, and so $G = C \oplus C' \oplus H'$. But now $C \oplus C'$ is a finite image of G that is not cyclic, contradicting the fact that G is monogenic.

COROLLARY 5. If G is reduced and monogenic, then

$$tG = \sum_{p \in P} \sigma(p^{k(p)}),$$

where P is a set of primes, $k(p)$ is a positive integer, and $\sigma(n)$ denotes the cyclic group of order n.

COROLLARY 6. If G is reduced monogenic with a monogenic generator x_0 of finite order, then $G = [x_0]$.

PROOF: Since $tG \subset Q/Z$, every finite subgroup of G, e.g., $[x_0] = A$, is contained in a finite direct summand H of tG. But H is also a direct summand of G, by the result referred to above [3, p. 18]. Therefore $G = H \oplus M$, so that $G/A \cong (H/A) \oplus M$ is divisible. Hence $M = 0$ (since G is reduced), $H = A$ (since H is finite), and $G = A$.

PROPOSITION 7. If G has a monogenic generator of finite order, then $G = A \oplus D$, where D is divisible and A is finite cyclic.

We now furnish G with a topology making it a topological group.

DEFINITION. The *Prüfer topology* [2]) on G is the topology in which a base of neighborhoods of 0 is the set of subgroups of the form $n!G$, where $n > 0$.

The following proposition is well known, but we include a brief proof for the reader's convenience.

PROPOSITION 8. Let G have the Prüfer topology. A subgroup B of G is dense if and only if G/B is divisible.

PROOF: Let $x \in G$ and $n > 0$. Since B is dense, B meets every open set. In particular, $B \cap (-x+n!G) \neq \emptyset$. There is thus an element $b \in B$ with $b = -x+n!g$ for some $g \in G$. Therefore $x+B = x+b+B = n!g+B$ and G/B is divisible.

For the converse, it suffices to prove $B \cap (x+n!G) \neq \emptyset$ for all $x \in G$ and $n > 0$. Since G/B is divisible, there is an element $h \in G$ with $x = n!h+b$, for some $b \in B$. Therefore $b = x-n!h \in B \cap (x+n!G)$.

COROLLARY 9. A discrete group G is monogenic if and only if, when it is given the Prüfer topology, it contains a dense cyclic subgroup.

PROOF: This follows immediately from Propositions 2 and 8.

It follows that discrete monogenic groups can be topologized in such a way that they become (not necessarily locally compact) monothetic groups. Now there exist reduced groups that are not T_1 in the Prüfer topology. However, if

$$G^\omega = \bigcap_{n=1}^{\infty} n!G$$

is zero, then a norm can be defined on G analogous to the p-adic norm. If $x = 0$, set $||x|| = 0$; if $x \neq 0$, set $||x|| = e^{-n}$, where $x \in n!G$ but $x \notin (n+1)!G$. G becomes a metric space in the Prüfer topology, where distance is $||x-y||$.

LEMMA 10. *Let G be a group having no p-torsion.* Then G^ω is p-divisible, i.e., $G^\omega = pG^\omega$.

PROOF: If $x \in G^\omega$, then there are elements $y_1, y_2, \ldots$ in G with $x = py_1 = p^2y_2 = \ldots$. Since G has no elements of order p, the y_i

[2]) This is often called the n-adic topology; W. Krull suggested that it be named after Prüfer.

are unique. In particular, y_1 is divisible by every power of p. We must show that $y_1 \in G^\omega$. Suppose $(m, p) = 1$. There are then integers a and b with $am+bp = 1$, so that $y_1 = amy_1+bpy_1 = amy_1+bx \in mG$, since $x \in G^\omega$. Finally, if $n > 0$, we shall show that $y_1 \in nG$. Now $n = p^k m$, where $(m, p) = 1$. There are integers α and β with $\alpha m+\beta p^k = 1$, and so $y_1 = \alpha m y_1+\beta p^k y_1 = \alpha m(p^k g)+\beta p^k(mh) \in nG$. Therefore $y_1 \in G^\omega$.

COROLLARY 11. Let G be a group with p-torsion of bounded order for some p, i.e., there is an integer k so that $p^k x = 0$ for all x whose order is a power of p. Then G^ω is p-divisible.

PROOF: The group p^kG is a group having no p-torsion, so that $(p^kG)^\omega$ is p-divisible, by Lemma 10. But $(p^kG)^\omega = G^\omega$: if $x = n!y_n$ for all n, then $x = n!(p^k z_n)$ for all n, where $z_n = ((n+p^k)!/n!p^k)y_{n+p^k}$. The other inclusion is obvious.

PROPOSITION 12. If G is a reduced monogenic group, then $G^\omega = 0$. Hence G is metric in the Prüfer topology.

PROOF: If G has no p-torsion, then G^ω is p-divisible, by Proposition 10. If G does have p-torsion, it is a finite cyclic group, by Corollary 5; hence G^ω is p-divisible for this p by Corollary 11. Therefore G^ω is divisible, being p-divisible for all p. Since G is reduced, $G^\omega = 0$.

If G is metric (in the Prüfer topology), we denote its completion $G\#$.

Note that if S is a pure subgroup of a group G, then the Prüfer topology on S is the same as the relative topology on S induced from G. The following proposition gives a partial converse.

LEMMA 13. *If G is metric (in the Prüfer topology), then G is a pure subgroup of its completion $G\#$.*

PROOF: Suppose $g = ng\#$, where $g\# \in G\#$. There is a sequence $\{g_i\}$ in G with $g_i \to g\#$ in $G\#$, and so $ng_i \to ng\# = g$ in $G\#$. Since G is a subspace of $G\#$, $ng_i \to g$ in G. For large i, therefore, $ng_i - g \in n!G$, so that $g \in nG$ and G is pure.

LEMMA 14. *If G is metric (in the Prüfer topology), then*

$$G\# = (tG)\# \oplus F,$$

where F is torsion-free.

PROOF: Since tG is pure in G, it is a subspace, and so $(tG)\# \subset G\#$. Since tG is pure in G, $(tG)\#$ is pure in $G\#$. Every complete group is *algebraically compact*[3] [2, p. 84], which says that $(tG)^{\#}$ is a direct summand of $G^{\#}$. The complementary summand F is torsion-free, for $t(G^{\#}) \subset (tG)^{\#}$.

DEFINITION. Let P be a set of distinct primes, and for each $p \in P$, let C_p denote either a p-primary cyclic group or the additive group of p-adic integers. We call the abstract group $\prod C_p$ a *universal monogenic group*. An element $x \in \prod C_p$ is called a *main diagonal* if, for all p, its pth coordinate is a generator of C_p when C_p is cyclic or a p-adic unit otherwise.

PROPOSITION 15. A reduced group G is a complete monogenic group if and only if it is isomorphic to a universal monogenic group. Moreover, a monogenic generator is a main diagonal.

PROOF: Suppose G is monogenic and $G = G^{\#}$. By Lemma 14, $G = (tG)^{\#} \oplus F$, where F is torsion-free (and monogenic). Since G is monogenic, $tG = \sum C_p$, where C_p is a cyclic p-primary group. Now $\prod C_p$ is complete, it contains $tG = \sum C_p$ as a pure subgroup (hence as a subspace), and it contains $\sum C_p$ as a dense subgroup (for $\prod C_p / \sum C_p$ is divisible). Therefore $(tG)^{\#} = \prod C_p$.

Let x be a monogenic generator of F, and let E be the pure subgroup of F generated by x, i.e.,

$$E = \{y \in F: my \in [x] \text{ for some } m \neq 0\}.$$

Now E is a subspace of F, and it is dense in F since it contains $[x]$. Therefore $E^{\#} = F^{\#} = F$ (for a direct summand of a complete group is complete). It is well known that $E^{\#} = \prod I^{\#}_p$ ($I^{\#}_p = p$-adic integers) where p ranges over all primes for which $pE \neq E$. Therefore G is a universal monogenic group.

A universal monogenic group G is complete in the Prüfer topology (it is compact); we show that G is monogenic with a main diagonal x as a monogenic generator. If $\varphi: G \to K$ is onto, where K is a finite group of order n, then $nG \subset \ker \varphi$, and there is an epimorphism $G/nG \to K$. But $nG = \prod nC_p$, and $G/nG = \prod (C_p/nC_p)$. If $(p, n) = 1$, however, $C_p = nC_p$, so that G/nG is a finite sum of C_p/nC_p. It follows easily that G/nG is cyclic with generator $\varphi(x)$. Therefore G is monogenic. (It is easy to see that if x is not a main diagonal, then x is not a monogenic generator.)

THEOREM. *A discrete group G is monogenic if and only if $G = D \oplus H$, where D is divisible and H is a pure subgroup of a universal monogenic group that contains a main diagonal.*

[3] A group is algebraically compact if it is a direct summand whenever it is pure.

PROOF: By Proposition 3, we may assume that G is reduced.

If G is monogenic, then G is a pure dense subgroup of $G^{\#}$, by Lemma 13. By Proposition 15, $G^{\#}$ is a universal monogenic group. If x is a monogenic generator of G, then $[x]$ is dense in G, and so $[x]$ is dense in $G^{\#}$. Therefore x is a main diagonal, by Proposition 15.

Conversely, suppose G is a pure subgroup of M containing x, where M is a universal monogenic group with monogenic generator x. Then $G/[x]$ is a pure subgroup of the divisible group $M/[x]$, and so it is divisible. Therefore G is monogenic.

References

B. ECKMANN,

[1] Über monothetische Gruppen, Comm. Math. Helv., 16, (1943), pp. 249—263.

L. FUCHS,

[2] Abelian Groups, Publishing House of the Hungarian Academy of Sciences, Budapest (1958).

I. KAPLANSKY,

[3] Infinite Abelian Groups, University of Michigan Press, Ann Arbor (1954).

L. A. RUBEL,

[4] Uniform distribution on locally compact groups, Comm. Math. Helv. (to appear).

W. RUDIN,

[5] Fourier analysis on groups, Interscience, (Now John Wiley) (1960) New York.

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