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by

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Introduction

The field of real numbers may be introduced in one of two ways. In the so-called "constructive" or "genetic" method [6, p. 26], one defines the real numbers directly from the rational numbers as infinite decimals, Dedekind cuts, Cauchy sequences, nested interval sequences or some other similar objects. In the "axiomatic" or "postulational" method, on the other hand, one simply takes the real numbers to be any system of objects which satisfies the axioms for a "complete ordered field". (If we postulate "Cauchy-completeness" rather than "order-completeness", we must also require the field to be archimedean [4, Ch. II, Sec. 9—10].)

These two methods do not contradict each other, but are in fact complementary. The Dedekind construction furnishes an existence proof for the axiomatic approach. Similarly, the axiomatic characterization provides a certain justification for the seemingly arbitrary choice of any particular construction; for we can show that any two complete ordered fields are isomorphic [4, Ch. II, Sec. 9—10].

In each of the above-mentioned genetic approaches to the theory of real numbers, essential use is made of the concept of an arbitrary sequence of rational numbers. Turing in [20] first attempted to constructivize the theory by restricting the functions that appear in the definition through decimal expansions to be "computable", or equivalently [6, Ch. XIII] "general recursive". Here the general recursive functions are assumed to comprise all in-

[1] This paper is Part I of the author's Ph. D. Thesis at the University of Wisconsin written under the direction of Prof. S. C. Kleene. Part of the material appeared in [11], written under the direction of Dr. D. Kreider (now at Dartmouth College), and was presented at Prof. H. Rogers' seminar in logic at MIT in the summer of 1960. I wish to express my sincerest appreciation to all three above-named persons for their help and encouragement.

tuitively effectively computable functions, in accordance with "Church's thesis" [6, Ch.'s XII, XIII]. Thus Turing calls a real number α in the unit interval [0, 1] *computable*, if there is a general recursive function $f(x)$ such that $0 \leqq f(x) \leqq 9$ for each natural number x and

$$\alpha = \sum_{x \geqq 1} f(x) \cdot 10^{-x}.$$

Turing's definition has been extended, in the obvious way to real numbers outside the interval [0, 1], and his basic idea of restricting the functions (or sets) in the classical definition to be recursive has been applied to the other genetic approaches, i.e. Dedekind cuts, Cauchy sequences, etc. [18]. R. M. Robinson first observed in [19] that the constructive versions of these approaches all lead, as classically, to the same set of real numbers. This set of *recursive real numbers* was shown by H. G. Rice [18] to be a denumerable subfield of the classical real number which becomes algebraically closed on the adjunction of the imaginary unit $\sqrt{-1}$. Moreover, the field operations are "computable" on this subfield, under any of several definitions of computability of functions of (recursive) real arguments (see Lemma 4 below).

In this paper we develop a constructive version of the axiomatic approach to the theory of real numbers. We shall give suitable constructive definitions of the relevant classical concepts, and prove for the recursive real numbers uniqueness theorems similar to the classical characterizations of the real numbers. Our method, we believe, provides a general tool for constructivizing various classical axiomatic theories (cf. our forthcoming [12]). Since it is this method rather than the specific results that interests us, we shall at many points digress from the main argument to illustrate other applications.

A constructive theory of fields has been developed in Fröhlich and Shepherdson [3] and Rabin [17], along the lines of Van der Waerden's work on "explicitly given fields" [21, Sec. 42]. The approach in these papers is algebraic and has proved powerful in studying the connection between algebra and computability theory. Our definitions may be viewed as generalizations of those of Fröhlich-Shepherdson and Rabin, necessitated by the fact that the field of recursive real numbers is not a "computable ordered field" (see § 4, below, for a precise statement). In an effort to make this paper as self-contained as possible, we shall present the definitions of Rabin at the appropriate places and compare them to ours.

For the theory of recursive real numbers we shall refer whenever possible to Rice's [18], which gives a concise exposition of most of the results we need. For recursive function theory, we assume familiarity with Kleene's [6]; our notation and terminology will be those of [6], unless otherwise specified.

1. Notation systems

It is easy to show that there exist primitive recursive functions sign(x), num(x) and den(x), with sign$(x) = 0$ or 1, such that the mapping $x \to r_x$ given by

$$(1.1) \qquad r_x = \mathrm{r}(x) = (-1)^{\mathrm{sign}(x)}\, \mathrm{num}(x)/\mathrm{den}(x)$$

is a one-to-one correspondence of the set N of natural numbers $0, 1, \ldots$ onto the set Q of fractions in lowest terms [8, p. 396, 5, § 3]. Utilizing this correspondence, we can associate with each real number α at least one number-theoretic function $\mathrm{f}(x)$ such that, for all x and y,

$$(1.2) \qquad |\mathrm{r}(\mathrm{f}(x)) - \mathrm{r}(\mathrm{f}(x+y))| < 2^{-x}$$

and

$$(1.3) \qquad \alpha = \lim_{x\to\infty} \mathrm{r}(\mathrm{f}(x)).$$

Conversely, any number-theoretic function $\mathrm{f}(x)$ satisfying (1.2) determines exactly one real number α defined by (1.3).

Here we obtain the real numbers as limits of certain Cauchy sequences of rational numbers.

The natural way to constructivize this approach to real numbers is by restricting the function $\mathrm{f}(x)$ to be general recursive. Thus a real number α is *recursive* if there is a general recursive function $\mathrm{f}(x)$ satisfying (1.2) and (1.3). See [18] for references and alternative definitions.

By the normal form theorem [6, §§ 58, 63], for each partial recursive function $\mathrm{f}(x_1, \ldots, x_n)$ (where we take $n \geqq 1$), there is a natural number f (called a *Gödel number* of $\mathrm{f}(x_1, \ldots, x_n)$) such that

$$(1.4) \quad \mathrm{f}(x_1, \ldots, x_n) \simeq U(\mu y T_n(f, x_1, \ldots, x_n, y)) \simeq \{f\}(x_1, \ldots, x_n).$$

Here $U(y)$ is a particular primitive recursive function and for each n, $T_n(f, x_1, \ldots, x_n, y)$ is a particular primitive recursive predicate. The complete equality $\simeq$ [6, pp. 327—328] is replacable by $=$ if $\mathrm{f}(x_1, \ldots, x_n)$ is general recursive.

In (1.4) we anticipated a convention which we now state. We shall use single letters or combinations of letters in Roman type as names for number-theoretic functions, e.g. $\mathrm{f}(x)$, $\mathrm{g}(x, y)$, $\mathrm{sign}(x)$. If a function thus named is proved or assumed to be (partial) recursive, its name in italic type will denote some Gödel number of it in the sense of (1.4), e.g. f, g, *sign*.

If the general recursive function $\mathrm{f}(x)$ with Gödel number f determines the recursive real number α in the sense of (1.2) and (1.3), we call f an R-index of α and write $\alpha = \alpha_f$.

The set R of natural numbers which are R-indices of recursive real numbers is characterized by

$$(1.5) \quad f \in R \equiv (x)(Ez)T_1(f, x, z) \;\&\; (x)(y)(z)(t)[T_1(f, x, z) \;\&\; T_1(f, x+y, t) \rightarrow |\mathrm{r}(U(z)) - \mathrm{r}(U(t))| < 2^{-x}].$$

Each element f of R determines one real number α_f. This correspondence, however, is not one-to-one, since different Gödel numbers may represent the same function and different functions may determine the same real number. We thus have a natural equivalence relation on R,

$$f \sim_{\mathrm{R}} g \equiv \alpha_f = \alpha_g,$$

which we can express without reference to real numbers by

$$(1.6) \quad f \sim_{\mathrm{R}} g \equiv f \in R \;\&\; g \in R \;\&\; (x)(y)(t)[T_1(f, x, y) \;\&\; T_1(g, x, t) \rightarrow |\mathrm{r}(U(y)) - \mathrm{r}(U(t))| \leqq 2^{-x+1}].$$

If we want to develop the theory of recursive real numbers independently of the classical theory, we replace the real number α_f (for $f \in R$) by the equivalence class of f in R under $\sim_{\mathrm{R}}$; we denote this equivalence class by $\bar{f}^{\mathrm{R}}$ or $[f]^{\mathrm{R}}$. From this point of view, the essential object of study or recursive analysis is the set R (or some other set playing a similar role) with the equivalence relation $\sim_{\mathrm{R}}$ on it.

The ordered pair $(R, \sim_{\mathrm{R}})$ forms in a certain sense a system of notations for the recursive real numbers. We abstract from this example a general concept of a "notation system".

DEFINITION 1. A *notation system* is an ordered pair $\mathbf{T} = (T, \sim_{\mathrm{T}})$ where T is a set of natural numbers and $\sim_{\mathrm{T}}$ is an equivalence relation on T.

For $x \in T$, $\bar{x}^{\mathrm{T}}$ or $[x]^{\mathrm{T}}$ shall be the equivalence class under $\sim_{\mathrm{T}}$ of x in T; x will be called a T-*index* of $\bar{x}^{\mathrm{T}}$. We shall always take $x \sim_{\mathrm{T}} y$ to be false unless both x and y are members of T.

We mentioned in the introduction that the field operations are "computable" on the set of recursive real numbers. One way in which this can be made precise in the case of addition is as follows (see Lemma 4): there is a partial (in fact, primitive) recursive function $f_+(x, y)$ such that, for all $x, y \in R$, $f_+(x, y) \in R$ and

$$[\mathrm{f}_+(x, y)]^{\mathrm{R}} = \bar{x}^{\mathrm{R}} + \bar{y}^{\mathrm{R}}. \tag{1.7}$$

If a definition of addition of real numbers is presupposed, (1.7) asserts that it is a computable operation on the recursive reals; i.e. we have an algorithm f_+ which, when supplied with R-indices x, y of recursive real numbers, furnishes an R-index $\mathrm{f}_+(x, y)$ of their sum. Alternatively, (1.7) with a suitable f_+ can be considered as the definition of addition of recursive real numbers.

In order to abstract from this example a general concept of a "recursive operator" from one notation system into another, we introduce some convenient notations.

We shall use Greek letters α, β, γ as variables over equivalence classes of notation systems. If α is an equivalence class of $\mathrm{T} = (T, \sim_{\mathrm{T}})$, we write $\alpha \in \mathrm{T}$ and we call α an *element* of T. We are thus viewing T dually as an ordered pair and as the collection of its equivalence classes.

If $\mathbf{F}(\alpha_1, \ldots, \alpha_n)$ is a partial function defined for some n-tuples of elements of some set $(n \geqq 1)$, we abbreviate "$\mathbf{F}(\alpha_1, \ldots, \alpha_n)$ is defined" by "$\mathbf{F}(\alpha_1, \ldots, \alpha_n)\downarrow$".

DEFINITION 2. Let $\mathrm{T} = (T, \sim_{\mathrm{T}})$, $\mathrm{S} = (S, \sim_{\mathrm{S}})$ be notation systems and let $\mathbf{F}(\alpha_1, \ldots, \alpha_n)$ be a function or operator defined on n-tuples of elements of T and taking elements of S as values. Then $\mathbf{F}(\alpha_1, \ldots, \alpha_n)$ is *recursive* (an n-ary recursive operator from T into S), if there is a partial recursive function $\mathrm{f}(x_1, \ldots, x_n)$ such that, whenever $a_1, \ldots, a_n$ are T-indices of $\alpha_1, \ldots, \alpha_n$ respectively, then $\mathrm{f}(a_1, \ldots, a_n)\downarrow$, $\mathrm{f}(a_1, \ldots, a_n) \in S$ and

$$\mathbf{F}(\alpha_1, \ldots, \alpha_n) = [\mathrm{f}(a_1, \ldots, a_n)]^{\mathrm{S}}. \tag{1.8}$$

We say then that $\mathrm{f}(x_1, \ldots, x_n)$ *determines* $\mathbf{F}(\alpha_1, \ldots, \alpha_n)$ (or *defines* $\mathbf{F}(\alpha_1, \ldots, \alpha_n)$ *recursively*), and we call any Gödel number f of $\mathrm{f}(x_1, \ldots, x_n)$ an *index* of $\mathbf{F}(\alpha_1, \ldots, \alpha_n)$ (as a recursive operator on n variables from T into S).

To illustrate these definitions, we consider some specific examples of notation systems that have been used in the literature.

The simplest example of a notation system is the set N of natural numbers with identity as the equivalence relation,

$\mathbf{N} = (N, \lambda xy\, x{=}y)$. Here the recursive operators are the general recursive functions.

A more interesting example is the usual notation system for the general recursive functions. We define

$$(1.9) \qquad x \in F \equiv (u)(Et)T_1(x, u, t),$$

$$(1.10) \qquad \begin{aligned} x \sim_{\mathrm{F}} y \equiv x \in F \;\&\; y \in F \;\& \\ (u)(v)(t)[T_1(x, u, v) \;\&\; T_1(y, u, t) \rightarrow U(v) = U(t)]. \end{aligned}$$

The notation system $\mathbf{F} = (F, \sim_{\mathrm{F}})$ gives us one way of referring to general recursive functions; an F-index of a general recursive function α is a Gödel number of α in the sense of (1.4).

An example of a computable (binary) operation on general recursive functions is pointwise addition, i.e. the operation which assigns to two functions $\mathrm{f}(x)$ and $\mathrm{g}(x)$ the sum function $\mathrm{f}(x)+\mathrm{g}(x)$. We see from the formula

$$\alpha+\beta = [\Lambda x\{a\}(x)+\{b\}(x)]^{\mathrm{F}} \quad (a \in \alpha \in \mathrm{F},\ b \in \beta \in \mathrm{F})$$

that pointwise addition is a recursive operator. (Here the $+$ on the right is ordinary number addition, and the Λ is that of [6, Th. XXIII].)

The recursive operators from F into N have been called (numerical-valued) *effective operations* by Myhill-Shepherdson [15] and Kreisel-Lacombe-Schoenfield [9]. It is shown im [9, Th. 1 and § 3, Remark 1] that they are exactly all restrictions to F of the *partial recursive functionals* (in the sense of Kleene [6, § 63]) whose domain contains F.[2]

Another example of a notation system is $\mathbf{O} = (O, \lambda xy\, |x|{=}|y|)$, where O is the set of ordinal notations of Kleene [7] and, for $x \in O$, $|x|$ is the ordinal that x represents. Several recursive operators from O into O have been studied in the theory of constructive ordinals; for example, we see from the formula

$$\alpha+\beta = [a+_o b]^{\mathrm{O}} \qquad (a \in \alpha \in \mathrm{O},\ b \in \beta \in \mathrm{O})$$

that addition of constructive ordinals is a (binary) recursive operator [7, (XVI)].

2. Recursive groups

The concepts of notation system and recursive operator that we introduced in the preceding section allow a natural con-

[2] Actually a stronger theorem is proved in [9]. In [12] we shall give a further generalization of this Kreisel-Lacombe-Schoenfield result in what appears to be its natural setting.

structivization of several classical axiomatic theories. In this section we treat briefly, as an example, the case of groups.

DEFINITION 3. A *recursive group* is a notation system $G = (G, \sim_G)$ together with two recursive operators $\alpha \cdot \beta$ and α^{-1} (binary and unary respectively) from G into G which satisfy the classical group axioms.

In order to show that every countable group is (classically) isomorphic to a recursive group we need a simple lemma, which will also be useful later.

Let A be a set of natural numbers and let $\mathscr{F}$ be a set of (possibly partial) number-theoretic functions. We define inductively the set $A^{\mathscr{F}}$ by the following three clauses.

(2.1.a) $a \in A \rightarrow a \in A^{\mathscr{F}}$.

(2.1.b) $a_1, \ldots, a_n \in A^{\mathscr{F}}$ & $\mathrm{f}(x_1, \ldots, x_n) \in \mathscr{F}$ & $\mathrm{f}(a_1, \ldots, a_n)\downarrow$
$\rightarrow \mathrm{f}(a_1, \ldots, a_n) \in A^{\mathscr{F}}$.

(2.1.c) $a \in A^{\mathscr{F}}$ only as required by (2.1.a) and (2.1.b).

Clearly $A^{\mathscr{F}}$ is the smallest set containing A and closed under the operations of $\mathscr{F}$; we call $A^{\mathscr{F}}$ the *functional closure* of A by $\mathscr{F}$.

For each natural number e, let W_e be the domain of the partial recursive function $\{e\}(x)$; thus

$$(2.2) \qquad x \in W_e \equiv (Ey)T_1(e, x, y).$$

Now $W_0, W_1, \ldots$ is an enumeration (with repetitions) of all recursively enumerable sets (including the empty set; cf. [6, Th.'s XIV, XVIII, XIX]).

LEMMA 1. *For each m, let $\mathscr{F}_m$ be the set of partial recursive functions $\{\{z\}(x_1, \ldots, x_n) : 2^z 3^n \in W_m\}$. There is a primitive recursive function* $\mathrm{fc}(e, m)$ *such that, for each e, and m,*

$$W_{\mathrm{fc}(e,m)} = W_e^{\mathscr{F}_m}.$$

In particular, the functional closure of a recursively enumerable set by a finite set of partial recursive functions is recursively enumerable.

PROOF. We first define a function $\mathrm{g}(e, m, t)$ so that, for fixed e and m, the domain of $\mathrm{g}(e, m, t)$ is the union of W_e and the set of numbers obtainable from W_e by using once an operation of $\mathscr{F}_m$. Thus $\mathrm{g}(e, m, t)$ is to be undefined, unless

$$t \in W_e \vee (Ez)(En)(Ex)[2^z 3^n \in W_m \;\&\; (i)_{1 \leq i \leq n}(x)_i \in W_e \;\&\; \{z\}((x)_1, \ldots, (x)_n) \simeq t].$$

To see that $\mathrm{g}(e, m, t)$ can be taken to be partial recursive, we use [6, p. 287 (22)] to rewrite $\{z\}((x)_1, \ldots, (x)_n) \simeq t$ as

$$(Ey)[T(z, 2^n \cdot \Pi_{1 \leqq i \leqq n} p_i^{(x)_i+1}, y) \ \& \ U(y) = t];$$

then, upon advancing the existential quantifiers (including those in W_e) and contracting them (by [6, p. 285 (17)]), the condition of definition of $\mathrm{g}(e, m, t)$ assume the form $(Ey)R(e, m, t, y)$ with a (primitive) recursive R. Now we take

$$\mathrm{g}(e, m, t) \simeq \mu y R(e, m, t, y).$$

Next we define $\mathrm{f}(k, e, m)$ by the recursion

$$\mathrm{f}(0, e, m) = e,$$
$$\mathrm{f}(k+1, e, m) = S_1^2(g, \mathrm{f}(k, e, m), m),$$

where g is a Gödel number of g. For each $k \geqq 1$, $\mathrm{f}(k, e, m)$ is a Gödel number of a function with domain the set of numbers obtainable from members of W_e by $\leqq k$ uses of functions of $\mathscr{F}_m$. Now let $\mathrm{h}(e, m, t) \simeq \mu y[(y)_0 > 0 \ \& \ T_1(\mathrm{f}((y)_0, e, m), t, (y)_1)]$ and $\mathrm{fc}(e, m) = \Lambda t\, \mathrm{h}(e, m, t)$.

THEOREM 1. *Every countable group is isomorphic to a recursive group whose set of indices consists of all the natural numbers.*

PROOF. Let $\mathbf{s}_0, \mathbf{s}_1, \ldots$ be an enumeration of the given group, and denote the group operations by $\mathbf{s}_i \cdot \mathbf{s}_j$ and $\mathbf{s}_i^{-1}$. We associate with each element $\mathbf{s}_i$ of the group the natural number 7^i. Let A be the (recursive) set of numbers of the form 7^i, and let $\mathscr{F}$ be the finite set of functions $\mathrm{f}_1(i, j) = 3^i 5^j$, $\mathrm{f}_2(i) = 2^i$. By Lemma 1, the set $A^{\mathscr{F}}$ is recursively enumerable; it is clearly infinite. Let $\mathrm{f}(x)$ be a general recursive function which enumerates $A^{\mathscr{F}}$ without repetitions.

We define a function $\phi(x)$ from the set G of all natural numbers into the given group by induction on the form of definition of $A^{\mathscr{F}}$:

$$\phi(x) = \mathbf{s}_i \text{ if } \mathrm{f}(x) = 7^i,$$
$$\phi(x) = \phi(\mathrm{f}^{-1}(i)) \cdot \phi(\mathrm{f}^{-1}(j)) \text{ if } \mathrm{f}(x) = 3^i 5^j,$$
$$\phi(x) = \phi(\mathrm{f}^{-1}(i))^{-1} \text{ if } \mathrm{f}(x) = 2^i.$$

On G we define an equivalence relation $\sim_{\mathrm{G}}$ by

$$x \sim_{\mathrm{G}} y \equiv \phi(x) = \phi(y).$$

It is now easy to verify that the functions $\mathrm{f}^{-1}(\mathrm{f}_1(\mathrm{f}(x), \mathrm{f}(y)))$ and $\mathrm{f}^{-1}(\mathrm{f}_2(\mathrm{f}(x)))$ determine recursive operators on $\mathrm{G} = (G, \sim_{\mathrm{G}})$

which satisfy the group axioms, and that $\phi(x)$ induces in a natural way an isomorphism of this recursive group with the given group.

Theorem 1 shows that the concept of a recursive group is as wide as that of a classical countable group to within a classical isomorphism. We cannot expect to enrich the algebraic structure of groups by imposing the restriction of recursiveness. Two comments, however, are relevant here.

First, a recursive group is endowed with a specific recursive structure. We call two recursive groups G_1 and G_2 *recursively isomorphic*, if there is a recursive operator $F : G_1 \to G_2$ which is a classical isomorphism from G_1 onto G_2 and whose inverse $F^{-1} : G_2 \to G_1$ is also a recursive operator. In a constructive theory it is natural to identify two recursive groups only if they are recursively isomorphic, and to study those properties of a given group that are related to and can be expressed in terms of the given recursive structure on the group, e.g. "recursive subgroups", "recursive automorphisms" etc. It is easy to construct examples of recursive groups that are classically, but not recursively, isomorphic.

A second and more fundamental restriction to the applicability of Theorem 1 is that until now we have imposed no constructivity restrictions on the set T and the equivalence relation $\sim_T$ of a notation system. It is clear that from the constructive point of view one should require the predicates $x \in T$ and $x \sim_T y$ to be "constructively definable" in some sense. We plan to discuss this problem in a paper which is now in preparation.

Rabin in [17] imposes this latter restriction in a very strong form by requiring the set of indices G to be recursive and the equivalence relation $\sim_G$ to be simply identity, $x \sim_G y \equiv x = y$. (Because of this he does not need to postulate the computability of the inverse operation, which follows from the computability of multiplication.)

Another difference in Rabin's approach is that he considers abstract classical groups rather than specific recursive representations of them. A classical group Γ is *computable* (in the sense of Rabin), if there is a recursive set G and a one-to-one function $i \to \mathbf{s}_i$ from G onto Γ such that the number-theoretic function $\mathrm{f}(i, j)$ defined by the formula

$$\mathbf{s}_{\mathrm{f}(i,j)} = \mathbf{s}_i \cdot \mathbf{s}_j$$

is general recursive. The inverse of the function $i \to \mathbf{s}_i$ is called an *admissible indexing* of Γ. The recursive group (in our sense) that $\mathrm{G} = (G, \lambda xy\, x = y)$ forms with $\mathrm{f}(i, j)$ and the corresponding inverse operation is called a *recursive realization* of Γ.

Let us call a notation system *discrete* if its equivalence relation is simply identity. Any computable group is then isomorphic to a discrete recursive group, namely any recursive realization of it.

LEMMA 2. *A finitely generated discrete recursive group is computable.*

We shall omit the proof since the lemma is very similar to Rabin's [17, Th. 3]. The idea is that in this case the set of indices G must be the functional closure of a set of generators by the finite set of group operations and is thus, by Lemma 1, recursively enumerable. We obtain a recursive realization of the group by enumerating G with a general recursive function, as in the proof of Theorem 1.

Rabin constructs in [17, § 1.5] a finitely generated group which is not computable. By Lemma 2 this group cannot be isomorphic to any discrete recursive group.

Theorem 1 shows that the generality we allow for the predicate $x \in G$ is unnecessary for the classical aspects of the theory of recursive groups. Every countable group is isomorphic to a recursive group G, where G is recursively enumerable. This is not the case with the equivalence relation $\sim_{\mathrm{G}}$; if we insist that it be simply identity, we shall exclude from consideration some finitely generated groups.

3. Recursive fields

Let us try to define recursive fields after the pattern of Definition 1. We have to be careful in handling the multiplicative inverse function, which in the case of fields is undefined at **0**. There are various ways in which partial recursive operators may be defined on notation systems. We prefer a definition which, by analogy with the case of partial recursive functions, places some restriction on the domain.

DEFINITION 4. Let $\mathrm{T} = (T, \sim_{\mathrm{T}})$, $\mathrm{S} = (S, \sim_{\mathrm{S}})$ be notation systems and let $\mathbf{F}(\alpha_1, \ldots, \alpha_n)$ be a partial function or operator defined on some n-tuples of elements of T and taking elements of S as values. Then $\mathbf{F}(\alpha_1, \ldots, \alpha_n)$ is *partial recursive* (an n-ary partial recursive operator from T into S), if there is a partial recursive function $\mathrm{f}(x_1, \ldots, x_n)$ such that, whenever $a_1, \ldots, a_n$

are T-indices of $\alpha_1, \ldots, \alpha_n$ respectively, then

$$(3.1.a) \qquad \mathrm{f}(a_1, \ldots, a_n)\downarrow \equiv \mathbf{F}(\alpha_1, \ldots, \alpha_n)\downarrow$$

and

$$(3.1.b) \qquad \begin{array}{l} \mathrm{f}(a_1, \ldots, a_n)\downarrow \rightarrow [\mathrm{f}(a_1, \ldots, a_n) \in S \ \& \\ \mathbf{F}(\alpha_1, \ldots, \alpha_n) = [\mathrm{f}(a_1, \ldots, a_n)]^{\mathrm{S}}]. \end{array}$$

We say then that $\mathrm{f}(x_1, \ldots, x_n)$ *determines* $\mathbf{F}(\alpha_1, \ldots, \alpha_n)$, and we call any Gödel number f of $\mathrm{f}(x_1, \ldots, x_n)$ an *index* of $\mathbf{F}(\alpha_1, \ldots, \alpha_n)$ (as a partial recursive operator on n variables from T into S).

Intuitively, a partial operator $\mathbf{F}(\alpha)$ from T into S is partial recursive if there is an algorithm which, when supplied with any T-index a of α, terminates if and only if $\mathbf{F}(\alpha)\downarrow$ and in that case produces an S-index of $\mathbf{F}(\alpha)$.

Definition 5. A *recursive field* is a notation system $\mathrm{K} = (K, \sim_{\mathrm{K}})$, together with recursive operators $\alpha+\beta$, $-\alpha$, $\alpha\times\beta$ and a partial recursive operator α^{-1} whose domain is all of K except one element (to be called $\mathbf{0}$), which satisfy the classical field axioms.

We shall follow the algebraic practice of indicating the field operations by the symbols $+$, $-$, $\times$, $^{-1}$ in all fields. When there is a possibility of confusion we shall use superscripts $+^{\mathrm{K}}$, $-^{\mathrm{K}}$ etc. Similarly for the additive and multiplicative identities $\mathbf{0}$ and $\mathbf{1}$, or $\mathbf{0}^{\mathrm{K}}$ and $\mathbf{1}^{\mathrm{K}}$.

The requirement that α^{-1} be a partial recursive operator prohibits division by $\mathbf{0}$ in a very strong sense; the multiplicative inverse algorithm will not terminate if supplied with an index of $\mathbf{0}$. Because of this we cannot show that every countable field is isomorphic to a recursive field after the trivial fashion of Theorem 1. We shall only give an outline of this proof, which is a routine formalization of Van der Waerden's arguments in [21, § 42].

Theorem 2. *Every countable field is isomorphic to a discrete recursive field.*

Proof. Let Γ be the given field. We choose a (possibly finite) sequence $\mathbf{s}_1, \mathbf{s}_2, \ldots$ of elements of Γ such that the subfields of Γ defined by

$\Gamma_0 =$ the prime field of Γ,

$\Gamma_{n+1} = \Gamma_n(\mathbf{s}_{n+1}) =$ the smallest subfield of Γ containing both Γ_n and $\mathbf{s}_{n+1}$, form a strictly increasing ($\Gamma_n \subsetneqq \Gamma_{n+1}$), possibly finite, sequence of fields whose union is Γ.

The plan of the proof is to define a strictly increasing sequence of sets $K_0 \subsetneqq K_1 \subsetneqq, \ldots$ such that with suitable operations each K_n will be a discrete recursive field isomorphic to Γ_n. Then $K = \bigcup_n K_n$ will be a discrete recursive field isomorphic to Γ. (Though each K_n will be recursive, K will not in general be recursive.)

To start the inductive definition, we find a recursive set K_0, and a one-to-one function ϕ_0 from K_0 onto Γ_0, such that

(3.2.a) $0 \in K_0$ & $1 \in K_0$,

(3.2.b) if $x \in K_0$ and $x \neq 0, 1$, then x is of the form $2 \cdot 3^y$ ($y \neq 0$),

(3.2.c) K_0 with suitable operations is a discrete recursive field isomorphic to Γ_0 by ϕ_0,

(3.2.d) $\phi_0(0) = 0$, $\phi_0(1) = 1$.

If Γ has prime characteristic p, Γ_0 is finite and the construction of K_0 is trivial. If Γ has characteristic 0, Γ_0 is isomorphic to the rational numbers. In this case we define K_0 using a suitable indexing of the rational numbers and the function $\mathrm{r}(x)$ of (1.1).

Assume now that we have defined $K_0, \ldots, K_n$ such that

(3.3.a) $K_i \subsetneqq K_{i+1}$ for $0 \leqq i \leqq n-1$,

(3.3.b) each K_i, with suitable operations, is a recursive field isomorphic to Γ_i by some $\phi_i : K_i \to \Gamma_i$,

(3.3.c) the isomorphism $\phi_i : K_i \to \Gamma_i$ is the restriction of $\phi_{i+1} : K_{i+1} \to \Gamma_{i+1}$ to K_i, for $0 \leqq i \leqq n-1$.

The additional properties that the sets K_i have will become apparent from the construction of K_{n+1}.

Except for the trivial possibility that the union of $\Gamma_0, \ldots, \Gamma_n$ is already the whole of Γ, there are two cases.

Case 1. Γ_{n+1} is an algebraic extension of Γ_n. In this case, there is a polynomial

$$\mathbf{P}(\xi) = \mathbf{t}_0 + \mathbf{t}_1\xi + \ldots + \mathbf{t}_k\xi^k \qquad (k > 1,\ \mathbf{t}_k \neq 0) \tag{3.4}$$

with coefficients in Γ_n, irreducible in Γ_n, which has $\mathbf{s}_{n+1}$ as a root. Γ_{n+1} is isomorphic to the field of polynomials

$$\alpha_0 + \alpha_1\xi + \ldots + \alpha_{k-1}\xi^{k-1}$$

with coefficients in Γ_n, with the ordinary polynomial operations performed modulo $\mathbf{P}(\xi)$.

We can represent a polynomial by the sequences of its coefficients. Since the coefficients here are in the field Γ_n, we may use the natural numbers in K_n that correspond to these coefficients by ϕ_n. To represent a finite sequence of numbers $x_0, \ldots, x_n$ by a single number we use Kleene's

$$(3.5) \qquad \langle x_0, \ldots, x_n \rangle = p_0^{x_0} \ldots p_n^{x_n},$$

where $p_0, p_1, \ldots$ is the sequence of prime numbers with $p_0 = 2$.

Let $t_0, \ldots, t_k \in K_n$ be such that $(i)_{0 \leq i \leq k}\, \phi(t_i) = \mathbf{t}_i$. Let

$$(3.6) \qquad \begin{aligned} &x \in K_{n+1} \equiv x \in K_n \vee [x \text{ is of the form} \\ &\quad \langle n+2, \langle t_0, \ldots, t_k \rangle, \langle x_0, \ldots, x_{k-1} \rangle \rangle, \text{ where} \\ &\quad (i)_{0 \leq i \leq k-1}\, x_i \in K_n \,\&\, (Ei)_{0 < i \leq k-1}\, x_i \neq 0]. \end{aligned}$$

We extend $\phi_n(x)$ to the members of $K_{n+1} - K_n$ by

$$\phi_{n+1}(x) = \phi_n(x_0) + \phi_n(x_1)\xi + \ldots + \phi_n(x_{k-1})\xi^{k-1},$$

where $x, x_0, \ldots, x_{k-1}$ satisfy the second disjunct of the right side of (3.6).

CASE 2. Γ_{n+1} is a transcendental extension of Γ_n. In this case the field Γ_{n+1} is isomorphic to the field of rational functions over Γ_n. Each rational function can be represented uniquely as the quotient $\mathbf{P}(\xi)/\mathbf{Q}(\xi)$ of two polynomials, if we agree that the leading coefficient of $\mathbf{Q}(\xi)$ is 1 and that $\mathbf{P}(\xi)$ and $\mathbf{Q}(\xi)$ are relatively prime. Using the same method of indexing as before, we define

$$(3.7) \qquad \begin{aligned} &x \in K_{n+1} \equiv x \in K_n \vee [x \text{ is of the form} \\ &\quad \langle n+2, 1, \langle x_0, \ldots, x_k \rangle, \langle y_0, \ldots, y_m \rangle \rangle, \text{ where} \\ &\quad (i)_{0 \leq i \leq k}\, x_i \in K_n \,\&\, x_k \neq 0 \,\&\, (j)_{0 \leq j \leq m}\, y_j \in K_n \,\&\, y_m = 1 \,\& \\ &\quad k+m > 0, \text{ and the polynomials } \phi_n(x_0) + \phi_n(x_1)\xi + \ldots + \\ &\quad \phi_n(x_k)\xi^k \text{ and } \phi_n(y_0) + \phi_n(y_1)\xi + \ldots + \phi_n(y_{m-1})\xi^{m-1} + \xi^m \text{ are} \\ &\quad \text{relatively prime over } \Gamma_n]. \end{aligned}$$

The condition $k+m > 0$ prevents us from re-indexing an element of Γ_n in K_{n+1} as a quotient of its index in K_n by 1. The isomorphism $\phi_{n+1}(x)$ is defined in the obvious way.

It remains to show that each K_n is a recursive set and a recursive field under the operations of Γ_n transferred by the isomorphism. We can do this inductively following the scheme suggested by Van der Waerden.

In Case 1, the members of K_{n+1} represent polynomials with coefficients in K_n. We perform the field operations by first operating with the polynomials in the usual way and then reducing the result modulo the defining irreducible polynomial of the extension. It is clear from [21, Sec. 32] that this can be done effectively if we know the defining polynomial. On the other hand, because of our indexing, we can find effectively from any $x \in K_{n+1}$ the defining polynomial of the extension by looking at $(x)_1$.

In Case 2, the elements of K_{n+1} represent rational functions over K_n which we can add, subtract, multiply or divide effectively in the usual fashion, if the field operations are effective on K_n. The fact that we can always reduce the result to the quotient of two relatively prime polynomials follows from [21, Sec. 18]. It is shown there (essentially) that the Euclidean algorithm for polynomials over a discrete recursive field is recursive.

The same remarks apply to the inductive proof that K_n is a recursive set. We shall not present the details of these computations, which add nothing to Van der Waerden's intuitive argument.

To show that $K = \bigcup_n K_n$ is a recursive field, we only need notice that, for all x,

$$x \in K \equiv x \in K_{(x)_0 - 1}.$$

Thus, given $x, y \in K$, we can operate on them with the operations of $K_{\max[(x)_0, (y)_0]-1}$.

It is clear that in most situations the set K will not be recursive, or even recursively enumerable. For each n, let u_n be the index of the element ξ of Γ_n in K_n (here ξ represents either a polynomial or a rational function over Γ_{n-1}, depending on whether Γ_n is an algebraic or transcendental extension of Γ_{n-1}). The function $\psi(n) = (u_n)_1$ gives us a certain measure of the degree of unsolvability of K. In particular, if $\psi(n)$ is recursive, then K is recursively enumerable. $\psi(n)$ clearly depends on the initial choice of the sequence $\mathbf{s}_1, \mathbf{s}_2, \ldots$ that we made in the proof. It might be interesting to study in what way the different $\psi(n)$'s that are possible for a given field determine the "non-constructivity" of the field.

Remark. Let Γ_0 be the prime field of Γ, Δ the algebraic closure of the extension of Γ_0 by countably many indeterminates. Rabin's [17, Th. 7] implies that Δ is isomorphic to a discrete recursive field. Since Γ is imbeddable in Δ, Γ is itself isomorphic to a discrete recursive field. (This argument was suggested to us by Rabin). Our much longer elementary proof defines, in effect, a canonical imbedding of Γ into Δ.

Rabin in [17] calls a field *computable*, if it is isomorphic to a discrete recursive field $\mathbf{K} = (K, \lambda xy\, x = y)$ with K recursive. Using the next lemma we shall construct a countable field which is not isomorphic to any recursive field $\mathbf{K} = (K, \sim_{\mathbf{K}})$ with K recursively enumerable.

LEMMA 3. *For each* $n \geqq 0$, *let* $a_n = \sqrt{p_n}$, *where* $p_0, p_1, \ldots$ *is the sequence of primes. Let* Γ_{-1} *be the field of rational numbers. For each* $n \geqq 0$, *let* $\Gamma_n = \Gamma_{n-1}(a_n) =$ *the smallest subfield of the real numbers containing* $a_0, \ldots, a_n$. *For each n, let* $\Delta_n = \Gamma_{-1}(a_0, \ldots, a_{n-1}, a_{n+1}, a_{n+2}, \ldots) =$ *the smallest subfield of the real numbers containing the square root of every prime except (perhaps)* p_n.

(a) *If* $m < k$, *then* $a_k \notin \Gamma_m$.

(b) *For every n*, $a_n \notin \Delta_n$.

PROOF. We shall prove (a) by contradiction. Let k be the smallest number with the property $a_k \in \Gamma_{k-1}$, and for that k let m be the smallest number such that $a_k \in \Gamma_m$.

It is easy to verify that, for distinct prime numbers $q_0, \ldots, q_n$, the equation

$$\sqrt{q_0} = \xi\sqrt{q_1 \ldots q_n} \tag{3.8}$$

has no rational solution in ξ.

Since for every n, a_n is irrational, $m \geqq 0$. Since the equation $\Gamma_m = \Gamma_{m-1}$ contradicts the choice of m, Γ_m is an extension of degree two over Γ_{m-1}, and hence $a_k \in \Gamma_m$ implies

$$a_k = \alpha+\beta a_m \qquad (0 \leqq m < k,\ \beta \neq 0,\ \alpha, \beta \in \Gamma_{m-1}). \tag{3.9}$$

If $\alpha \neq 0$, we can square (3.9) and solve for a_m,

$$a_m = \frac{1}{2\alpha\beta}\{a_k^2-\alpha^2-\beta^2 a_m^2\} \in \Gamma_{m-1}. \tag{3.10}$$

But (3.10) contradicts the choice of k. So $\alpha = 0$, and

$$a_k = \beta a_m \qquad (\beta \in \Gamma_{m-1}). \tag{3.11}$$

Let m_1 be the smallest number such that $\beta \in \Gamma_{m_1}$. By (3.8) $0 \leqq m_1 < m$, and by an argument similar to the above

$$a_k = (\alpha_1+\beta_1 a_{m_1})a_m \qquad (0 \leqq m_1 < m < k,\ \beta_1 \neq 0,\ \alpha_1, \beta_1 \in \Gamma_{m_1-1}). \tag{3.12}$$

If $\alpha_1 \neq 0$, we can solve for a_{m_1} as in (3.10) and contradict the choice of k; thus $\alpha_1 = 0$ and

$$a_k = \beta_1 a_{m_1} a_m \qquad (0 \leqq m_1 < m < k,\ \beta_1 \in \Gamma_{m_1-1}). \tag{3.13}$$

Applying (3.8) again and continuing in the same fashion, we reach a contradiction in at most m steps.

To prove (b), we notice that $a_n \in \Delta_n$ implies that for some $m \neq n$

$$a_n = \alpha + \beta a_m, \tag{3.14}$$

where $\beta \neq 0$ and $\alpha, \beta \in \Gamma_{-1}(a_0, \ldots, a_{n-1}, a_{n+1}, \ldots, a_{m-1})$. By (a), $n < m$; solving (3.14) for a_m, we contradict (a).

We can now define in the following way a field which is not isomorphic to any recursive field $\mathrm{K} = (K, \sim_{\mathrm{K}})$ with K recursively enumerable.

Let A be any set of natural numbers such that the predicate $x \notin A$ is not expressible in the form $(u)(Ev)R(u, v, x)$ with $R(u, v, x)$ recursive; for example we may take

$$x \in A \equiv (u)(Ev)T_2(x, x, u, v). \tag{3.15}$$

Let $x_0, x_1, \ldots$ be some enumeration of A, and define

$$\Gamma = \Gamma_{-1}(a_{x_0}, a_{x_1}, \ldots).$$

By Part (b) of the lemma,

$$x \in A \equiv a_x \in \Gamma \equiv \text{the polynomial } \xi^2 - p_x \text{ has a root in } \Gamma, \tag{3.16}$$

and on taking negations

$$x \notin A \equiv (\xi)[\xi \in \Gamma \rightarrow \xi^2 - p_x \neq 0]. \tag{3.17}$$

Now assume that Γ is isomorphic to some recursive field $\mathrm{K} = (K, \sim_{\mathrm{K}})$, with K recursively enumerable,

$$u \in K \equiv (Ev)R(v, u). \tag{3.18}$$

Let $\mathrm{g}_+(u, v)$, $\mathrm{g}_-(u)$, $\mathrm{g}_\times(u, v)$ and $\mathrm{g}_{-1}(u)$ be partial recursive functions which determine the field operations of K, let k_0 and k_1 be K-indices of the **0** and the **1** of K.

We can enumerate effectively the "K-natural numbers" of any recursive field K by means of the recursive function

$$\mathrm{n}^{\mathrm{K}}(0) = k_0, \tag{3.19.a}$$

$$\mathrm{n}^{\mathrm{K}}(x+1) = \mathrm{g}_+(\mathrm{n}^{\mathrm{K}}(x), k_1). \tag{3.19.b}$$

The equivalence (3.17) can now be stated as

$$(3.20)\quad x \notin A \equiv (u)(u \in K \rightarrow [\mathrm{g}_+(\mathrm{g}(u, u), \mathrm{g}_-(\mathrm{n}^{\mathrm{K}}(p_x)))]^{\mathrm{K}} \neq \mathbf{0}].$$

Using the fact that the multiplicative inverse function is a partial recursive operator undefined at **0**,

$$(3.21)\quad x \notin A \equiv (u)[u \in K \rightarrow \mathrm{g}_{-1}(\mathrm{g}_+(\mathrm{g}(u, u), \mathrm{g}_-(\mathrm{n}^{\mathrm{K}}(p_x))))\downarrow].$$

This last equivalence implies that $x \notin A$ is expressible in the form $(u)(Ev)R(u, v, x)$, with $R(u, v, x)$ recursive, which contradicts our assumption about A.

It is clear how this counterexample can be modified (by raising the degree of unsolvability of $x \in A$) to produce fields that cannot be isomorphic to any recursive field $\mathrm{K} = (K, \sim_{\mathrm{K}})$ with a specified degree of unsolvability for $u \in K$. These counterexamples make strong use of our requirement that α^{-1} be a partial recursive operator undefined at **0**. If we had placed no restriction on the behaviour of the algorithm $\mathrm{g}_{-1}(u)$ for input an index of **0**, we could prove by the method of Theorem 1 that any countable field is isomorphic to a recursive field (in this sense) $\mathrm{K} = (K, \sim_{\mathrm{K}})$ with K recursively enumerable.

Here again the comments following Theorem 1 are relevant. Moreover, the representation of a given field in the fashion outlined in the proof of Theorem 2 is usually very hard and void of implication. This is the case, in particular, for recursive ordered fields which are of special interest to us.

LEMMA 4. *The notation system* $\mathrm{R} = (R, \sim_{\mathrm{R}})$ *for the recursive real numbers is a recursive field under the usual operations.*

We shall omit an explicit construction of the partial recursive functions that determine the field operations. The idea of the proof is that the field operations are computable on the rational numbers, and that the proofs of the classical theorems

$$\lim_{n\to\infty}(\alpha_n * \beta_n) = \lim_{n\to\infty} \alpha_n * \lim_{n\to\infty} \beta_n$$

(where $*$ can be $+$, $-$, $\times$ or $\div$) are constructive (see [18, Th. 4] and [5, § 3 Satz 7, § 6 Satz 3]).

In the sequel $\mathrm{f}_+(x, y)$, $\mathrm{f}_-(x)$, $\mathrm{f}(x, y)$ and $\mathrm{f}_{-1}(x)$ will be specific partial recursive functions that determine the field operations as recursive operators on R. (In fact we may take the first three to be primitive recursive, and $\mathrm{f}_{-1}(x)$ to be the restriction of a primitive recursive function to a recursively enumerable set.)

4. Recursive ordered fields

One of the fundamental properties of the field of real numbers is that it is an ordered field; in fact it contains an isomorphic copy of every archimedean ordered field. In this section we shall prove the corresponding statement about "recursive ordered fields".

Let us first note in what sense the ordering $\alpha < \beta$ is constructive on the notation system R.

LEMMA 5. *There is a partial recursive function* less (x, y) *such that, for any* R*-indices a and b of recursive real numbers* α *and* β,

$$\text{less}\,(a, b)\downarrow \equiv \alpha < \beta. \tag{4.1}$$

PROOF. We first verify that $\mathrm{r}(x)-\mathrm{r}(y) > 2^{-z+1}$ is a primitive recursive predicate of x, y, z, and then set

$$\text{less}\,(x, y) \simeq \mu t[T_1(x, (t)_0, (t)_1) \mathbin{\&} T_1(y, (t)_0, (t)_2) \mathbin{\&} \mathrm{r}(U((t)_2))-\mathrm{r}(U((t)_1)) > 2^{-(t)_0+1}].$$

(This lemma is essentially equivalent to Rice's [18, Th. 1].)

Thus we have an algorithm which, when applied to any R-indices of two recursive real numbers α and β, terminates if and only if $\alpha < \beta$. This does not give us a decision procedure for the predicate $\alpha < \beta$; in fact no such decision procedure exists (see Lemma 9). Nevertheless predicated with this property on R resemble in many ways recursively enumerable predicates of natural numbers. We shall state the definitions for an arbitrary notation system.

DEFINITION 6. (a) A predicate $P(\alpha_1, \ldots, \alpha_n)$ defined on n-tuples of elements of a notation system T is *listable* (on T), if there is a partial recursive function $\mathrm{f}(x_1, \ldots, x_n)$ such that, for every n-tuple $a_1, \ldots, a_n$ of T-indices of $\alpha_1, \ldots \alpha_n$ respectively,

$$P(\alpha_1, \ldots, \alpha_n) \equiv \mathrm{f}(a_1, \ldots, a_n)\downarrow. \tag{4.2}$$

We say then that $\mathrm{f}(x_1, \ldots, x_n)$ *determines* $P(\alpha_1, \ldots, \alpha_n)$ and we call any Gödel number f of $\mathrm{f}(x_1, \ldots, x_n)$ an *index* of $P(\alpha_1, \ldots, \alpha_n)$ (as a listable predicate of n variables on T).

(b) $P(\alpha_1, \ldots, \alpha_n)$ is *recursive* (on T) if both it and its negation are listable on T.

By a routine construction, we can establish:

LEMMA 6. (a) *A predicate* $P(\alpha_1, \ldots, \alpha_n)$ *is listable on* T *if and only if the set of n-tuples of elements of* T *on which it is true is the* *domain of some partial recursive operator from* T *into* $\mathbf{N} = (N, \lambda xy\, x = y)$.

(b) *A predicate* $P(\alpha_1, \ldots, \alpha_n)$ *is recursive on* T *if and only if its characteristic operator*

$$\mathbf{C}_P(\alpha_1, \ldots, \alpha_n) = \begin{cases} 0 & \text{if } P(\alpha_1, \ldots, \alpha_n), \\ 1 & \text{if } \bar{P}(\alpha_1, \ldots, \alpha_n) \end{cases}$$

from T into N *is recursive.*

Definition 7. A *listable (linear, irreflexive) ordering* of a notation system T is a predicate $\alpha \prec \beta$ listable on T which satisfies the classical axioms for a linear, irreflexive ordering, i.e. for all $\alpha, \beta, \gamma \in \mathrm{T}$, $\alpha \neq \beta \equiv \alpha \prec \beta \vee \beta \prec \alpha$ and $\alpha \prec \beta \;\&\; \beta \prec \gamma \rightarrow \alpha \prec \gamma$.

Definition 8. *A recursive ordered field (ROF)* is a recursive field K together with a listable ordering $\alpha \prec \beta$ of K such that, for all $\alpha, \beta, \gamma \in \mathrm{K}$,

$$\alpha \prec \beta \rightarrow \alpha + \gamma \prec \beta + \gamma, \tag{4.3.a}$$

$$\alpha \prec \beta \;\&\; \gamma \succ \mathbf{0} \rightarrow \alpha \times \gamma \prec \beta \times \gamma. \tag{4.3.b}$$

A K-rational number of a ROF is any element of the prime field of K. We call K *archimedean* if it has no elements greater than all the K-rational numbers.

It is clear that R forms an archimedean ROF with the usual ordering.

Theorem 3. *Let* K *be an archimedean ROF. There is a recursive operator* $\mathbf{F} : \mathrm{K} \rightarrow \mathrm{R}$ *which is a one-to-one order-preserving isomorphism of* K *with a subfield of* R.

Proof. The usual proof of the corresponding classical result is constructive and goes through in our case. We shall outline some of the details.

Let $g_+(u, v)$, $g_-(u)$, $g_\times(u, v)$ and $g_{-1}(u)$ be partial recursive functions which determine the field operations on K, let k_0 and k_1 be K-indices of the $\mathbf{0}$ and the $\mathbf{1}$ of K. We capture the "K-natural numbers" using the recursive function $n^{\mathrm{K}}(x)$ defined by (3.19.a) and (3.19.b). If $\mathbf{n}_x^{\mathrm{K}}$ is the equivalence class of $n^{\mathrm{K}}(x)$ in K, we can show as a consequence of the other properties (4.3.a) and (4.3.b) that

$$x < y \equiv \mathbf{n}_x^{\mathrm{K}} \prec \mathbf{n}_x^{\mathrm{K}}. \tag{4.4}$$

To capture the K-rational numbers, we define the recursive function

$$(4.5)\quad \rho^{K}(x) = \begin{cases} g_{\times}(n^{K}(\mathrm{num}(x)),\ g_{-1}(n^{K}(\mathrm{den}(x)))) & \text{if } \mathrm{sign}(x) = 0, \\ g_{-}(g_{\times}(n^{K}(\mathrm{num}(x)),\ g_{-1}(n^{K}(\mathrm{den}(x))))) & \text{if } \mathrm{sign}(x) = 1. \end{cases}$$

If $\boldsymbol{\rho}_x^K$ is the equivalence class of $\rho^K(x)$ in K, we can show as a consequence of (4.4) and the properties of the function $r(x)$ that

$$(4.6)\qquad r_x < r_y \equiv \boldsymbol{\rho}_x^K \prec \boldsymbol{\rho}_y^K,$$

and that the function $\phi(r_x) = \boldsymbol{\rho}_x^K$ is an isomorphism of the field of rational numbers with the prime field of K.

An element of a linearly ordered set is, of course, completely determined by its position in the ordering relative to any subset "dense in the ordering". Since K is archimedean, the K-rational numbers form such a set. Since the ordering is listable, we can effectively determine the position of any element of K relative to the K-rational numbers and then find the corresponding element of R using the functions $\rho^K(x)$ and $r(x)$. This is the idea behind the formula below.

Let $\mathrm{less}^K(u, v)$ be a partial recursive function with Gödel number $less^K$ which determines the listable ordering of K. Let

$$(4.7)\quad \begin{aligned} f(u) = \Lambda t\, (\mu y[&T_2(less^K, \rho^K((y)_0), u, (y)_2)\ \& \\ &T_2(less^K, u, \rho^K((y)_1), (y)_3)\ \&\ r((y)_1) - r((y)_0) < 2^{-t}])_0. \end{aligned}$$

It is easy to verify that $f(u)$ determines a recursive operator

$$(4.8)\qquad \mathbf{F}(\alpha) = [f(a)]^R \qquad (a \in \alpha \in K)$$

from K into R, and that this operator is the required order-preserving isomorphic imbedding.

Corollary 3.1. (a) *A countable field of real numbers is order-isomorphic to some ROF if and only if all its elements are recursive real numbers.*

(b) *There are countable ordered fields which are not isomorphic to any ROF.*

Proof. Let $\mathscr{R}_1$ be a denumerable subfield of the real number $\mathscr{R}$ order-isomorphic to a ROF K by $\mathbf{G} : \mathscr{R}_1 \to K$. Then K is necessarily archimedean. If $\mathbf{F} : K \to R$ is the recursive imbedding of K into the field of recursive real numbers supplied by the theorem, the composite mapping $\mathbf{F} \circ \mathbf{G} : \mathscr{R}_1 \to R$ is an order-preserving isomorphism of $\mathscr{R}_1$ into R. Since it is easy to verify that the only order-preserving isomorphism of one subfield of $\mathscr{R}$ into another is the identity, $\mathscr{R}_1$ must be contained in R.

To prove (b), we choose a countable field $\mathscr{R}_2$ of real numbers which contains a square root of every positive member of itself, and which is strictly larger than R. Any field isomorphism $\mathbf{H} : \mathscr{R}_2 \rightarrow \mathscr{R}$ is order-preserving, since for $\alpha > 0$,

$$\mathbf{H}(\alpha) = \mathbf{H}(\sqrt{\alpha} \cdot \sqrt{\alpha}) = \mathbf{H}(\sqrt{\alpha}) \cdot \mathbf{H}(\sqrt{\alpha}) > 0.$$

Thus $\mathscr{R}_2$ cannot be isomorphic to any ROF; for then it would be isomorphic to a subfield of R and thus order-isomorphic to a proper subfield of itself, which is impossible.

Definition 9. A set A of elements of a notation system $\mathbf{T} = (T, \sim_{\mathbf{T}})$ is *recursively enumerable* (on T), if there is a recursively enumerable set W_e such that

$$W_e \subset T \tag{4.9.a}$$

and, for all $\alpha \in \mathbf{T}$,

$$\alpha \in \mathbf{A} \equiv (Ex)[x \in W_e \ \& \ \alpha = \bar{x}^{\mathbf{T}}]. \tag{4.9.b}$$

We say then that W_e *determines* A and we call e an *index* of A (as a recursively enumerable subset of T). If T itself (as the set of its equivalence classes) is recursively enumerable, we call T a *recursively enumerable notation system.*

It is easy to verify (as in the proof of Theorem 1) that a group is computable if and only if it is isomorphic to a recursively enumerable discrete recursive group. Similarly, a field is computable if and only if it is isomorphic to a recursively enumerable discrete recursive field. By analogy we call a countable ordered field *computable* if it is isomorphic to a recursively enumerable discrete ROF.

In order to show that the recursive real numbers do not form a computable ordered field, we need the following constructive version of Cantor's theorem on the non-enumerability of the set of real numbers.

Lemma 7. *There is a primitive recursive function* tr(e) *such that, if* $W_e \subset R$, *then* tr$(e) \in R$ *and* $[\mathrm{tr}(e)]^{\mathbf{R}}$ *does not belong to the recursively enumerable set of elements of* R *determined by* W_e, i.e.

$$(x)[x \in W_e \rightarrow \bar{x}^{\mathbf{R}} \neq [\mathrm{tr}(e)]^{\mathbf{R}}].$$

In particular, R *is not a recursively enumerable notation system.*

We could give an elementary proof of this lemma by translating Cantor's diagonal argument into the formalism of recursive functions. (A convenient way of doing this is outlined in [2, p. 28].)

A complete proof of a more general theorem will be given in our [12].

Now assume that R is isomorphic to a ROF $K = (K, \sim_K)$, where K is recursively enumerable. The natural imbedding $F : K \to R$ must be onto R, since R has no non-trivial automorphisms. If $f(u)$ determines $F(\alpha)$, the set $A = f(K)$ is recursively enumerable and determines R as a recursively enumerable notation system, contradicting Lemma 7. In particular, R is not a computable ordered field. (Dudley's [2, pp. 28—29] is referring to this result, if we understand it correctly.)

5. Recursive completeness

Two recursive fields K_1 and K_2 are *recursively isomorphic*, if there is a recursive operator $F : K_1 \to K_2$ which is a classical isomorphism and whose inverse $F^{-1} : K_2 \to K_1$ is also a recursive operator; *F*, in this case, is a *recursive isomorphism*. If K_1 and K_2 are ROF's, we further require F to be order-preserving.

We saw that for every archimedean ROF there is a natural recursive imbedding $F : K \to R$ which is an order-preserving isomorphism of K with a subfield of R. Two questions arise naturally:

(A) Under what conditions is F onto R, so that K is classically isomorphic to R?

(B) Under what conditions is F a recursive isomorphism, so that K is recursively isomorphic to R?

The answers we shall give to (A) will give a characterization of the set of recursive real numbers; the answers to (B) will further characterize the particular notation system R.

The corresponding classical problem is solved by requiring the given ordered field to be *complete*, either *order-complete* or *Cauchy-complete*. We shall constructivize these concepts and prove similar results.

DEFINITION 10. A sequence $\alpha_0, \alpha_1, \ldots$ of elements of a notation system T is *recursive*, if there is a general recursive function $f(x)$ such that, for all x, $\alpha_x = [f(x)]^T$. We say then that $f(x)$ *determines* the sequence, and we call any Gödel number f of $f(x)$ an index of the sequence.

For a fixed archimedean ROF K, let $\mathbf{2}^{-t}$ be the K-rational number that corresponds to 2^{-t}, i.e.

$$\mathbf{2}^{-t} = [\rho^K(r^{-1}(2^{-t}))]^K.$$

DEFINITION 11. (a) A sequence $\alpha_0, \alpha_1, \ldots$ of elements of an archimedean ROF K is *recursively Cauchy*, if there is a general recursive function $g(t)$ such that, for all x, y, t,

$$x, y \geqq g(t) \to -2^{-t} \prec \alpha_x - \alpha_y \prec 2^{-t}.$$

We then call $g(t)$ a *recursive Cauchy criterion* for $\alpha_0, \alpha_1, \ldots$, and we call any Gödel number g of $g(t)$ an *r.c. index* of $\alpha_0, \alpha_1, \ldots$.

(b) If $\alpha_0, \alpha_1, \ldots$ is a sequence of elements of an archimedean ROF, $\lim_{x\to\infty} \alpha_x = \alpha$, if for every t, there is an m such that, for all x,

$$x \geqq m \to -2^{-t} \prec \alpha - \alpha_x \prec 2^{-t}.$$

(c) An archimedean ROF is *weakly recursively complete*, if every recursive recursively Cauchy sequence of elements of K has a limit.

(d) An archimedean ROF K is (*strongly*) *recursively complete*, if there is a partial recursive function $c^K(f, g)$ (a *completeness function* for K) such that, if f is an index of a recursive sequence of elements of K with r.c. index g, then $c^K(f, g)\downarrow$, $c^K(f, g) \in K$ and $\lim_{x\to\infty} [f(x)] = [c^K(f, g)]^K$.

LEMMA 8 (Rice's [18, Th. 5]). R *is recursively complete.*

PROOF. $c^R(f, g) = c(f, g) = \Lambda t\{\{f\}(\{g\}(t+2))\}(t+1)$.

THEOREM 4. *Let* K *be an archimedean ROF. If* K *is weakly recursively complete, then the natural imbedding* $F : K \to R$ *is onto* R, *so that* K *is classically isomorphic to* R. *If* K *is recursively complete, then* $F : K \to R$ *is a recursive isomorphism.*

PROOF. A recursive real number $\bar{x}^R$ is the limit of the recursive recursively Cauchy sequences of rational numbers $r(\{x\}(0))$, $r(\{x\}(1)), \ldots$. The corresponding sequence $\rho^K_{\{x\}(0)}, \rho^K_{\{x\}(1)}, \ldots$ of elements of K is recursive and recursively Cauchy. If K is weakly recursively complete, this sequence must converge to some $\alpha \in K$; we may easily verify that $F(\alpha) = \bar{x}^K$.

For the second statement we need to find a K-index of this α recursively. We can do this by setting

$$g(x) \simeq c^K(\Lambda t\, \rho^K(\{x\}(t)), \Lambda t\, t).$$

It is now easy to verify that $g(x)$ determines a recursive operator $G : R \to K$ which is the inverse of $F : K \to R$.

Classically an ordered field is *order-complete*, if every non-void bounded subset of it has a least upper bound. We have some difficulty in constructivizing this concept because of the following fact, which Markov [10, Th. 8.1] credits to some E. M. Levinson. [3]

LEMMA 9. *No non-trivial predicate $P(\alpha)$ defined on the recursive real numbers is recursive on* R. [4]

The result of Lemma 9 generalizes trivially by induction to predicates of n variables. In particular, R cannot be ordered by a recursive predicate.

A set A of elements of a notation system is *listable* (*recursive*), if the corresponding predicate $\alpha \in A$ is listable (recursive).

LEMMA 10. *Let* K *be an archimedean ROF. If every non-void recursive subset of* K *has a least upper bound, then the natural imbedding* $F : K \to R$ *is onto* R *and* K *is isomorphic to* R. *In particular, this is true if* K *has no recursive subsets.*

PROOF. By contradiction. Assume that the recursive real number α_0 is not in the range of $F(\alpha)$. The subset $\{\alpha \in K : F(\alpha) < \alpha_0\}$ of K is then recursive and has no least upper bound.

This result is weaker than the corresponding Theorem 4 in the approach through Cauchy sequences, since we cannot in general prove that $F(\alpha)$ has a recursive inverse. Requiring the passage to the least upper bound of a non-void, bounded recursive set to be effective will not remedy the situation, since in general K will not have any recursive subsets. We could get around this difficulty by restricting attention to sets of K-rational numbers, or by giving a weaker definition of recursiveness for subsets of K. We prefer an alternative approach, which we have found useful in similar situations.

The position of an element in a linearly ordered set completely characterizes the element; no other member of the set has exactly the same order relation to every member of the set. Classically, this is a trivial observation. The constructive version of this property of linear orderings is not trivial.

DEFINITION 12. Let T be a notation system listably ordered by $\alpha \prec \beta$. We say that $\alpha \prec \beta$ *describes* T, if there is a partial recursive function (a *description function*) de(f) such that, if for all $x \in T$ and some $\alpha \in T$,

[3] The proof outlined by Markov depends on some rather deep properties of *recursive real functions*, i.e. recursive operators from R into R. Lemma 9 follows almost trivially from [12, Th. 2].

[4] Turing [20, § 10 (v)] proves that every recursive subset of R which defines a Dedekind cut has a least upper bound, apparently unaware of the fact that no such subsets of R exist.

$$[\{f\}(x) \simeq 0 \equiv \bar{x}^{\mathrm{T}} \prec \alpha] \;\&\; [\{f\}(x) \simeq 1 \equiv \alpha \prec \bar{x}^{\mathrm{T}}],$$

then $\mathrm{de}(f)\downarrow$, $\mathrm{de}(f) \in T$ and $[\mathrm{de}(f)]^{\mathrm{T}} = \alpha$.

Intuitively, T is described by its ordering if, from the knowledge of the position of an element in the ordering, we can effectively find an index of the element.

Lemma 11. R *is described by its natural ordering.*

Proof. Let $\rho(x) = \rho^{\mathrm{R}}(x)$, and set

$$\begin{aligned}\mathrm{de}(f) = \Lambda t \big(\mu y[&T_1(f, \rho((y)_0), (y)_1) \;\&\; U((y)_1) = 0 \;\&\\ &T_1(f, \rho((y)_2), (y)_3) \;\&\; U((y)_3) = 1 \;\&\\ &\mathrm{r}((y)_2) - \mathrm{r}((y)_0) < 2^{-t}]\big)_0.\end{aligned}$$

Lemma 12. *Let* K *be an archimedean ROF described by its ordering* $\alpha \prec \beta$ *and such that the natural imbedding* $\mathbf{F} : \mathrm{K} \to \mathrm{R}$ *is onto* R. *Then* F *has a recursive inverse, and thus* K *is recursively isomorphic to* R.

Proof. Let $\mathrm{de}^{\mathrm{K}}(f)$ be a description function for K, $\mathrm{f}(x)$ a function with Gödel number f which determines the operator $\mathbf{F} : \mathrm{K} \to \mathrm{R}$, and less (x, y) a partial recursive function with Gödel number *less* which determines the listable ordering on R (Lemma 5). We can easily verify that, for every $\alpha \in \mathrm{R}$ and $a \in \alpha$,

$$\begin{aligned}\mathbf{F}^{-1}(\alpha) = [\mathrm{de}^{\mathrm{K}}\big(\Lambda x\, (\mu y[[&T_2(\mathit{less}, \mathrm{f}(x), a, (y)_1) \;\&\; (y)_0 = 0] \;\&\\ &[T_2(\mathit{less}, a, \mathrm{f}(x), (y)_1) \;\&\; (y)_0 = 1]]\big)_0]^{\mathrm{K}}.\end{aligned}$$

We combine Lemmas 10 and 12:

Theorem 5. *Up to recursive isomorphism,* R *is the only archimedean ROF which has no recursive subsets and is described by its ordering.*

From our point of view, this theorem gives a better characterization of R than Theorem 4, since it avoids all topological assumptions. This possibility of substituting constructivity assumptions for topological axioms is not an isolated peculiarity of this situation, but rather a general rule. In our [14] we plan to discuss the relation between the topological and the recursive structure on a notation system. We show there that on every notation system there is a *natural* (*recursive*) *topology*, which in most cases of interest coincides with the topology we would naturally put on the notation system from structural considerations. A notation system with no recursive subsets is "connected" in this topology.

6. Notation systems for the recursive real numbers

We initially chose the notation system **R** to represent the recursive real numbers, because its definition was a natural constructivization of the classical introduction of real numbers as Cauchy sequences of rational numbers. We next define notation systems $\mathbf{R}_d$ and $\mathbf{R}_{dc}$ which similarly constructivize the approaches to real numbers through decimal expansions and Dedekind cuts respectively (for F, cf. (1.9)).

$$(6.1)\quad x \in R_d \equiv [(x)_0 = 0 \vee (x)_0 = 1] \ \& \ (x)_2 \in F \ \& \ (u)[\{(x)_2\}(u) \leqq 9].$$

$$(6.2)\quad x \in R_{dc} \equiv x \in F \ \& \ (u)(v)[\mathrm{r}(u) < \mathrm{r}(v) \ \& \ \{x\}(v) = 0 \rightarrow \{x\}(u) = 0] \ \& \ (u)[\{x\}(u) = 0 \rightarrow (Ev)[\mathrm{r}(u) < \mathrm{r}(v) \ \& \ \{x\}(v) = 0]] \ \& \ (Eu)[\{x\}(u) = 0] \ \& \ (Eu)[\{x\}(u) \neq 0].$$

LEMMA 13. (a) *There is a primitive recursive function* $\mathrm{f}_d(x)$ *such that: If* $x \in R_d$, *then* $\mathrm{f}_d(x) \in R$, *and*

$$(6.3)\qquad [\mathrm{f}_d(x)]^{\mathrm{R}} = (-1)^{(x)_0} \cdot (x)_1 + \sum_{u \geqq 1} \{(x)_2\}(u) \cdot 10^{-u}.$$

Moreover, for every recursive real number α, *there is an* $x \in R_d$ *such that* $\alpha = [\mathrm{f}_d(x)]^{\mathrm{R}}$.

(b) *There is a primitive recursive function* $\mathrm{f}_{dc}(x)$ *such that: If* $x \in R_{dc}$, *then* $\mathrm{f}_{dc}(x) \in R$, *and for all* u

$$(6.4)\qquad \mathrm{r}(u) < [\mathrm{f}_{dc}(x)]^{\mathrm{R}} \equiv \{x\}(u) = 0.$$

Moreover, for every recursive real number α, *there is an* $x \in R_{dc}$ *such that* $\alpha = [\mathrm{f}_{dc}(x)]^{\mathrm{R}}$.

PROOF. For an explicit construction of $\mathrm{f}_d(x)$ and $\mathrm{f}_{dc}(x)$, see [2, Ch. 3]. The second statements of both (a) and (b) are consequences of Robinson's equivalence theorem [19, 18].

We define equivalence relations on the sets R_d and R_{dc} by

$$(6.5)\qquad x \sim_{\mathrm{R}_d} y \equiv x \in R_d \ \& \ y \in R_d \ \& \ \mathrm{f}_d(x) \sim_{\mathrm{R}} \mathrm{f}_d(y),$$

$$(6.6)\qquad x \sim_{\mathrm{R}_{dc}} y \equiv x \in R_{dc} \ \& \ y \in R_{dc} \ \& \ \mathrm{f}_{dc}(x) \sim_{\mathrm{R}} \mathrm{f}_{dc}(y).$$

Now $\mathrm{f}_d(x)$ determines a recursive operator $\mathbf{F}_d(\alpha)$ which maps $\mathbf{R}_d = (R_d, \sim_{\mathrm{R}_d})$ one-to-one onto **R**; likewise, $\mathrm{f}_{dc}(x)$ determines a recursive operator $\mathbf{F}_{dc}(\alpha)$ which maps $\mathbf{R}_{dc} = (R_{dc}, \sim_{\mathrm{R}_{dc}})$ one-to-one onto **R**. We show that neither of these recursive operators has a recursive inverse.

LEMMA 14. (a) *There is no partial recursive function* $\mathrm{f}(x)$ *such that, for every* $x \in R$, $\mathrm{f}(x)\downarrow$, $\mathrm{f}(x) \in R_{\mathrm{d}}$ *and*

$$\bar{x}^{\mathrm{R}} = [\mathrm{f}_{\mathrm{d}}(\mathrm{f}(x))]^{\mathrm{R}}.$$

(b) *Similarly, with* R_{dc}, f_{dc} *in place of* R_{d}, f_{d}.

PROOF. To prove (a) by contradiction, assume that $\mathrm{f}(x)$ is a partial recursive function, with Gödel number f, such that if $x \in R$, then $\mathrm{f}(x)\downarrow$, $\mathrm{f}(x) \in R_{\mathrm{d}}$ and $\bar{x}^{\mathrm{R}} = [\mathrm{f}_{\mathrm{d}}(\mathrm{f}(x))]^{\mathrm{R}}$. Let

$$\mathrm{g}(m, t) = \begin{cases} \mathrm{r}^{-1}(\frac{1}{2}) & \text{if } \overline{(Eu)}_{u \leq t}[T_1(f, m, (u)_0) \;\&\; T_1((U((u)_0))_2, 1, (u)_1)], \\ \mathrm{r}^{-1}(\frac{1}{2}+2^{-s}) & \text{if } (Eu)_{u \leq t}[T_1(f, m, (u)_0) \;\&\; T_1((U((u)_0))_2, 1, (u)_1) \;\&\; U((u)_1) \leq 4], \\ \mathrm{r}^{-1}(\frac{1}{2}-2^{-s}) & \text{otherwise,} \end{cases}$$

where $s \simeq \mathrm{s}(m) \simeq \mu u[T_1(f, m, (u)_0) \;\&\; T_1((U((u)_0)_2, 1, (u)_1)]$.

By Kleene's second recursion theorem [6, § 66], there is a number m such that, for all t, $\mathrm{g}(m, t) \simeq \{m\}(t)$. Then by the definition of $\mathrm{g}(m, t)$, $m \in R$, and thus by assumption $\mathrm{f}(m)\downarrow$ and $(\mathrm{f}(m))_2 \in F$, which together imply that $s \simeq \mathrm{s}(m)\downarrow$. Thus for $t \geq s$, one of the last two clauses in the definition of $\mathrm{g}(m, t)$ must apply, and hence $\bar{m}^{\mathrm{R}} \neq \frac{1}{2}$. CASE 1: $\bar{m}^{\mathrm{R}} < \frac{1}{2}$. Then $\{(\mathrm{f}(m))_2\}(1) \leq 4$. Thus, since $\{(\mathrm{f}(m))_2\}(1) = U((s)_1)$, the second clause of the definition of $\mathrm{g}(m, t)$ applies for $t \geq s$, so $\bar{m}^{\mathrm{R}} > \frac{1}{2}$, contradicting the case hypothesis. CASE 2: $\bar{m}^{\mathrm{R}} > \frac{1}{2}$. Similarly.

The proof of (b) is similar. (Another proof of this lemma is given in [2, Ch. 3].

Lemma 14 asserts that the notation systems R_{d} and R_{dc} are "stronger" than R. From an R_{d}-index of a recursive real number we can effectively find an R-index of the real number, but not inversely. We shall prove that neither of these notation systems is a recursive field under the ordinary operations on real numbers.

A set A of recursive real numbers is *additively generating*, if there is a recursive function $\mathrm{h}(x)$ (an *additively generating function* for A) such that, if $x \in R$, then $\mathrm{h}(x) = \langle u, v\rangle$ where $u, v \in R$, $\bar{u}^{\mathrm{R}}$, $\bar{v}^{\mathrm{R}} \in \mathrm{A}$ and $\bar{x}^{\mathrm{R}} = \bar{u}^{\mathrm{R}} + \bar{v}^{\mathrm{R}}$ (for $\langle u, v\rangle$, cf. (3.5)).

Intuitively, A is additively generating, if every recursive real number can be effectively written as the sum of two members of A.

LEMMA 15. *The complement of a recursively enumerable subset of* R *is additively generating.*

PROOF. This is a direct consequence of Lemmas 1 and 7. Let $f_+(x, y)$ and $f_-(x)$, with Gödel numbers f_+ and f_-, be as in Lemma 4, and choose m so that $W_m = \{\langle f_+, 2\rangle, \langle f_-, 1\rangle\}$. Let A be a recursively enumerable subset of R, determined by W_e. Let

$$h_1(x, t) \simeq \mu y[t = x \vee T_1(e, t, y)],$$

and set $h(x) = \langle u, v\rangle$, where

$$u = \mathrm{tr}(\mathrm{fc}(\Lambda t\, h_1(x, t), m)),$$
$$v = f_+(x, f_-(u)).$$

If $x \in R$, the set $W_{\mathrm{fc}(\Lambda t\, h_1(x,t), m)}$ is a subset of R, since it is the functional closure of $W_e \cup \{x\}$ by the functions f_+ and f_-. Thus it determines a certain recursively enumerable set B of elements of R. By Lemma 7, $u \in R$ and $\bar{u}^R \notin B$; in particular, $\bar{u}^R \notin A$, and there is no recursive real number $\xi \in A$ such that $\bar{u}^R = \bar{x}^R - \xi$. Thus $\bar{v}^R = \bar{x}^R - \bar{u}^R \notin A$ and the proof is complete.

A set A of elements of a notation system $T = (T, \sim_T)$ defines in a natural way a notation system $(A, \sim_A)$, where

(6.7.a) $$x \in A \equiv x \in T \;\&\; \bar{x}^T \in A,$$

(6.7.b) $$x \sim_A y \equiv x \in A \;\&\; y \in A \;\&\; x \sim_T y.$$

In accordance with the remarks preceding Definition 2, we use the same symbol A for this *natural sub-notation system of* T *defined by* A.

A (not necessarily recursive) operator from T into S is *recursive on the subset* A *of* T, if it coincides on A with a recursive operator from the natural sub-notation system of T defined by A into S.

LEMMA 16. *Let* K *be a recursive group, and let* G(α) *be an operator from* R *onto* K *which is an isomorphism of the abelian group of recursive real numbers with* K. *If* G(α) *is recursive on some additively generating subset of* R, *then it is a recursive operator.*

PROOF. Let $A \subset R$ have generating function $h(x)$, let $g_A(x)$ determine $G(\alpha)$ as a recursive operator from A into K, and let $g_+(u, v)$ determine addition on K. $G(\alpha)$ is determined from R into K by

$$g(x) \simeq g_+(g_A((h(x))_0), g_A((h(x))_1)).$$

THEOREM 6. (a) *Addition of real numbers is not a recursive operator on either* R_d *or* R_{dc}; *in particular neither of these notation systems is a recursive field under the ordinary operations on real numbers.*

(b) *Let* A *be a subset of* R, *and define the notation system* $\mathbf{R}_A = (R_A, \sim_{R_A})$ *by*

(6.8.a) $R_A = \{\langle x, y\rangle : x \in R \,\&\, [[\bar{x}^R \in A \,\&\, y = 0] \vee [\bar{x}^R \notin A \,\&\, y = 1]]\}$,

(6.8.b) $x \sim_{R_A} y \equiv x \in R_A \,\&\, y \in R_A \,\&\, (x)_0 \sim_R (y)_0$.

The function $f_{R_A}(x) = (x)_0$ *determines a recursive operator* $\mathbf{F}_{R_A}(\alpha)$ *which is one-to-one on* $\mathbf{R}_A$ *onto* R. *If either* A *or its complement is additively generating, then addition of real numbers is not a recursive operator on* $\mathbf{R}_A$.

PROOF. To prove (a), we first show by an elementary construction that the inverses of $\mathbf{F}_d(\alpha)$ and $\mathbf{F}_{dc}(\alpha)$ are both recursive on the set of irrational recursive real numbers. This set is additively generating by Lemma 15. Thus, if $\mathbf{R}_d$ and $\mathbf{R}_{dc}$ were recursive groups, Lemma 16 would imply that $\mathbf{F}_d(\alpha)$ and $\mathbf{F}_{dc}(\alpha)$ have recursive inverses, contradicting Lemma 14. (b) follows directly from Lemmas 14 and 16. (Another proof of (a) is in [2, Ch. 4 Th. 1].)

Theorem 6 shows that the most obvious attempts to construct a recursive ordered field K which is classically but not recursively isomorphic to R fail. In particular, we cannot force an additively generating subset A of R to be recursive and still retain the computability of addition.

We showed that the complement of a recursively enumerable set is additively generating. Actually Part (b) of the theorem covers many interesting cases besides the recursively enumerable sets. Instead of "additively generating" sets we can define *sets generating with respect to* (any function) $\mathbf{F}(\alpha_1, \ldots \alpha_n)$ and prove the analog of Lemma 16 in the same way. If $\mathbf{F}(\alpha_1, \ldots \alpha_n)$ is expressible in terms of the field operations and A is generating with respect to $\mathbf{F}(\alpha_1, \ldots, \alpha_n)$, then $\mathbf{R}_A$ (as defined by (6.8.a) and (6.8.b)) cannot be a recursive field. It can be shown ([13, Th. 1]) that any listable subset A of R is generating with respect to the function $\mathbf{F}(\alpha, \beta) = (\alpha - \alpha_0)/(\beta - \alpha_0)$ where $\alpha_0 \in A$. Thus the notation system $\mathbf{R}_A$ cannot be a recursive field if A is listable.

THEOREM 7. *There is an archimedean ROF* K *which is classically but not recursively isomorphic to* R.

PROOF. Let $u_0, u_1, \ldots$ be an enumeration without repetitions of some immune set I,[5] and let $v_0, v_1, \ldots$ be an enumeration of

[5] A set of natural numbers is *immune* if it is infinite and has no infinite recursively enumerable subsets. See [16] for the construction of some immune sets (as complements of *simple* sets), or [1] where the concept is isolated and studied.

some subset R_0 of R which contains at least one R-index of every recursive real number. Let E be the functional closure of the set $\{\langle u_0, v_0\rangle, \langle u_1, v_1\rangle, \ldots\}$ by the set of functions $\{5^x 7^y, 11^x, 13^x 17^y, 19^x\}$. Using the recursion theorem, we can define a partial recursive function $\mathrm{f}(x)$ such that, for all x and y,

$$\begin{aligned}
&\mathrm{f}(2^x 3^y) = y,\\
&\mathrm{f}(5^x 7^y) \simeq \mathrm{f}_+(\mathrm{f}(x), \mathrm{f}(y)),\\
&\mathrm{f}(11^x) \simeq \mathrm{f}_-(\mathrm{f}(x)),\\
&\mathrm{f}(13^x 17^y) \simeq \mathrm{f}_\times(\mathrm{f}(x), \mathrm{f}(y)),\\
&\mathrm{f}(19^x) \simeq \mathrm{f}_{-1}(\mathrm{f}(x)).
\end{aligned}$$

We define K by

$$x \in K \equiv x \in E \mathbin{\&} \mathrm{f}(x)\downarrow.$$

We can easily show, by induction on the form of definition of K, that if $x \in K$ then $\mathrm{f}(x) \in R$. This induces an equivalence relation on K,

$$x \sim_{\mathrm{K}} y \equiv x \in K \mathbin{\&} y \in K \mathbin{\&} \mathrm{f}(x) \sim_{\mathrm{R}} \mathrm{f}(x).$$

The function $\mathrm{f}(x)$ determines a one-to-one recursive operator $\mathbf{F}(\alpha)$ from $\mathrm{K} = (K, \sim_{\mathrm{K}})$ onto $\mathbf{R}$. The functions $\mathrm{g}_+(x, y) = 5^x 7^y$, $\mathrm{g}_-(x) = 11^x$ and $\mathrm{g}_\times(x, y) = 13^x 17^y$ determine recursive operators from K into K, and the function $\mathrm{g}_{-1}(x) \simeq 19^x + 0 \cdot \mathrm{f}_{-1}(\mathrm{f}(x))$ determines a partial recursive operator from K into K, with which K is a recursive field, isomorphic to R by $\mathbf{F}(\alpha)$. The ordering $\alpha \prec \beta \equiv \mathbf{F}(\alpha) < \mathbf{F}(\beta)$ makes K into a ROF, order-isomorphic to R by $\mathbf{F}(\alpha)$.

We want to show that K is not recursively isomorphic to R. Since R has no non-trivial automorphisms, it is enough for this purpose to verify that $\mathbf{F}(\alpha)$ does not have a recursive inverse.

For each $x \in E$ we define a finite subset $S(x)$ of the immune set I by the inductive clauses,

$$\begin{aligned}
&S(\langle u_n, v_n\rangle) = \{u_n\},\\
&S(5^x 7^y) = S(13^x 17^y) = S(x) \cup S(y),\\
&S(11^x) = S(19^x) = S(x).
\end{aligned}$$

We can easily verify that, for every recursively enumerable subset W_e of E, the set $S(W_e) = \bigcup_{x \in W_e} S(x)$ is recursively enumerable. Since it is a subset of the immune set I, it must be finite.

Let A be a recursively enumerable subset of K, determined by $W_e \subset K \subset E$. If $u_{n_1}, \ldots, u_{n_k}$ are the finitely many members of

$S(W_e)$, A is contained in the subfield of K generated by $\{[\langle u_{n_1}, v_{n_1}\rangle]^K, \ldots, [\langle u_{n_k}, v_{n_k}\rangle]^K\}$. Thus, $\mathbf{F}(A)$ is contained in the field of recursive real numbers generated by $\{\bar{v}^R_{n_1}, \ldots, \bar{v}^R_{n_k}\}$.

If $\mathbf{F}(\alpha)$ has a recursive inverse, then for each recursively enumerable subset B of R, the set $A = \mathbf{F}^{-1}(B)$ is recursively enumerable in K. For if $g(x)$ determines $\mathbf{F}^{-1}(\alpha)$ and W_m determines B, then the recursively enumerable set $g(W_m)$ determines A on K. By the preceding remarks it will suffice to exhibit a recursively enumerable subset of R which is not contained in any subfield of R generated by finitely many elements. Using a routine construction and Lemma 6, we can show that the set $\{\sqrt{p_0}, \sqrt{p_1}, \ldots\}$ has this property.

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