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Note on approximation theorems

by

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This note is concerned with three approximation theorems for a class of definite integrals and sine series with monotone decreasing coefficients. S. Aljančić, R. Bojanić and M. Tomić [1] have considered previously the asymptotic value of the definite integral

$$\Phi(\lambda) = \int_a^b f(t)L(\lambda t)dt, \quad 0 \leqq a < b < \infty, \quad \lambda \to \infty,$$

where $f(t) \in L(a, b)$ and $L(t)$ is a slowly increasing function in the sense of Karamata. In this note, we shall consider the asymptotic behaviour of definite integrals of a more general type:

$$\Phi(\lambda) = \int_0^\infty f(t)\phi(\lambda t)dt,$$

where $\phi(t)$ is such that $\phi(t)/t^{p_1}$ is non-decreasing and $\phi(t)/t^{p_2}$ is non-increasing as t is increasing in $(0, \infty)$ for $0 \leqq p_1 \leqq p_2 < \infty$. In another paper [2], Aljančić, Bojanić and Tomić have also considered the asymptotic behaviour of $g(x) = \sum_1^\infty \lambda_n \sin nx$, $\lambda_n \searrow 0$, as $x \to +0$, under the condition that $\lambda_n \sim An^{-\alpha}L(n)$, as $n \to \infty$. We shall now consider the asymptotic behaviour of the same series when λ_n is asymptotically defined by a more general class of functions.

NOTATION. By $\phi(x) \sim [p_1, p_2]$, $0 \leqq p_1 \leqq p_2 < \infty$ or $-\infty < p_1 \leqq p_2 \leqq 0$, we denote the non-negative even function $\phi(x)$, such that $\phi(x)x^{-p_1}$ is non-decreasing and $\phi(x)x^{-p_2}$ is non-increasing, as x is increasing in $(0, \infty)$. By $\phi(x) \sim \langle p_1, p_2\rangle$, we mean $\phi(x) \sim [p_1+\varepsilon, p_2-\varepsilon]$ for some $\varepsilon > 0$. We define $\phi(x) \sim [p_1, p_2\rangle$, $\phi(x) \sim \langle p_1, p_2]$ in a similar way. By $a(x) \simeq b(x)$ and $a(x) \asymp b(x)$, $x \to c$, we mean $a(x)/b(x) \to 1$ and $K_1 < a(x)/b(x) < K_2$, respectively, as $x \to c$.

THEOREM 1. If $\phi(x) \sim [p_1, p_2]$, $0 \leqq p_1 \leqq p_2 < \infty$, and if $f(t) \geqq 0$, $t^{p_1}f(t) \in L(0, 1)$, $t^{p_2}f(t) \in L(1, \infty)$, then $\Phi(\lambda) \asymp \phi(\lambda)$ as $\lambda \to \infty$. More precisely, we have

(1) $\quad K_1(f, p_1, p_2)\phi(\lambda) \leqq \Phi(\lambda) = \int_0^\infty f(t)\phi(\lambda t)dt \leqq K_2(f, p_1, p_2)\phi(\lambda),$

uniformly for $0 < \lambda < \infty$, where

(2) $K_1 = \int_0^1 t^{p_2} f(t)dt + \int_1^\infty t^{p_1} f(t)dt, \quad K_2 = \int_0^1 t^{p_1} f(t)dt + \int_1^\infty t^{p_2} f(t)dt.$

In fact, if $t > 1$, then $\phi(t\lambda)(t\lambda)^{-p_2} \leqq \phi(\lambda)\lambda^{-p_2}$, and $\phi(\lambda)\lambda^{-p_1} \leqq \phi(t\lambda)(t\lambda)^{-p_1}$, i.e. $t^{p_1}\phi(\lambda) \leqq \phi(t\lambda) \leqq t^{p_2}\phi(\lambda)$. Similarly, if $0 < t < 1$, then $t^{p_2}\phi(\lambda) \leqq \phi(t\lambda) \leqq t^{p_1}\phi(\lambda)$. It follows that

$$\phi(\lambda)\left\{\int_0^1 t^{p_2} f(t)dt + \int_1^\infty t^{p_1} f(t)dt\right\} \leqq \int_0^\infty f(t)\phi(\lambda t)dt \leqq \phi(\lambda)\left\{\int_0^1 t^{p_1} f(t)dt + \int_1^\infty t^{p_2} f(t)dt\right\}. \tag{3}$$

THEOREM 2. Let

$$g(x) = \sum_1^\infty \lambda_n \sin nx, \tag{4}$$

where λ_n decreases steadily to zero. If $\phi(x) \sim \langle -1, 0 \rangle$ and if $\lambda_n \simeq \phi(n)$, as $n \to \infty$, then $g(x) \asymp 1/x\phi(1/x)$, as $x \to +0$.

PROOF. Since $\phi(x) \sim \langle -1, 0 \rangle$, $\varepsilon\phi(x)/x \leqq -\phi'(x) \leqq (1-\varepsilon)\phi(x)/x$ for some $\varepsilon > 0$. It follows that

$$g(x) = \sum_{n=1}^\infty \lambda_n \sin nx = \sum_{1 \leqq n \leqq [1/x]} + \sum_{n > 1/x} = T_1 + T_2,$$

where

$$\left\{\begin{aligned} |T_1| &\leqq K\int_1^{1/x} \phi(t)dt \leqq K\left(\frac{1}{x}\right)\phi\left(\frac{1}{x}\right) - K\int_1^{1/x} t\phi'(t)dt \\ &\leqq K\left(\frac{1}{x}\right)\phi\left(\frac{1}{x}\right) + (1-\varepsilon)\int_1^{1/x}\phi(t)dt \leqq K\left(\frac{1}{x}\right)\phi\left(\frac{1}{x}\right), \end{aligned}\right. \tag{5}$$

as $x \to +0$.

From Abel's transformation, it is easy to see that

$$|T_2| = \left|\sum_{n > 1/x} \lambda_n \sin nx\right| \leqq K\phi\left(\frac{1}{x}\right)x^{-1}. \tag{6}$$

It remains to show that $|g(x)| > K(1/x)\phi(1/x)$, as $x \to +0$. In fact,

$$\left\{\begin{aligned} g(x) &= \sum_1^\infty \Delta\lambda_n \frac{\sin^2(n+\frac{1}{2})\frac{x}{2}}{\sin\frac{x}{2}} - \frac{\lambda_1}{2}\operatorname{tg}\frac{x}{4} \\ &= \sum_1^\infty \Delta\lambda_n \frac{\sin^2(n+\frac{1}{2})\frac{x}{2}}{\sin\frac{x}{2}} + o\left\{\frac{1}{x}\phi\left(\frac{1}{x}\right)\right\} = I + II, \end{aligned}\right. \tag{7}$$

where

$$
(8)\quad \begin{cases} I > \dfrac{K}{x} \displaystyle\sum_{\pi/2x \leqq n \leqq 3\pi/2x} \Delta\phi(n) \geqq \dfrac{K}{x} \left\{\phi\left(\dfrac{\pi}{2x}\right) - \phi\left(\dfrac{3\pi}{2x}\right)\right\} \\ = \dfrac{K}{x} \left\{\phi\left(\dfrac{\pi}{2x}\right) - \phi\left(\dfrac{\lambda\pi}{2x}\right)\right\} \qquad (\lambda = \tfrac{3}{2}) \\ \geqq \dfrac{K}{x} (1-\lambda^{-\varepsilon})\phi\left(\dfrac{\pi}{2x}\right) \geqq K\left(\dfrac{1}{x}\right)\phi\left(\dfrac{1}{x}\right), \end{cases}
$$

and $II = o\{1/x\phi(1/x)\}$, as $x \to +0$. Hence $|g(x)| > K(1/x)\phi(1/x)$, as $x \to +0$.

Theorem 3. If $\phi(x) \sim \langle -2, -1]$, $\lambda_n \searrow 0$, $\lambda_n \cong \phi(n)$, as $n \to \infty$, then $g(x) \asymp (1/x)\phi(1/x)$, as $x \to +0$.

Proof. From (7), together with $\phi(x) \sim \langle -2, -1]$, when $x \simeq 1/n$ we also have

$$
(9)\quad \begin{cases} |g(x)| = \left| \displaystyle\sum_1^\infty \Delta\lambda_n \dfrac{\sin^2(n+1)\dfrac{x}{2}}{\sin\dfrac{x}{2}} \right| + o\left\{\dfrac{1}{x}\phi\left(\dfrac{1}{x}\right)\right\} \\ \leqq \dfrac{1}{x} \displaystyle\sum_1^n \Delta\lambda_n n^2 x^2 + \dfrac{\lambda_n}{x} \\ \leqq x \displaystyle\sum_1^n k^2 \Delta\lambda_k + \dfrac{K}{x}\phi\left(\dfrac{1}{x}\right) \\ \leqq x \displaystyle\sum_1^n (2k-1)\lambda_k + \dfrac{K}{x}\phi\left(\dfrac{1}{x}\right) \\ \leqq Kx \displaystyle\sum_1^n (2k-1)\phi(k) + \dfrac{K}{x}\phi\left(\dfrac{1}{x}\right). \end{cases}
$$

Write $\tau(x) = x\phi(x)$, then $\tau(x) \sim \langle -1, 0]$, since $\tau(x)x^{1-\varepsilon}$ is non-decreasing. It follows that

$$
(10)\qquad \int_1^n \tau(x)dx = n\tau(n) - \tau(1) - \int_1^n x\tau'(x)dx.
$$

But since $x^{1-\varepsilon}\tau(x)$ is non-decreasing, $(1-\varepsilon)x^{-\varepsilon}\tau(x) + x^{1-\varepsilon}\tau'(x) \geqq 0$, and $-x\tau'(x) \leqq (1-\varepsilon)\tau(x)$. Substituting this in (10) and it follows then from (9) that

$$
(11)\quad \int_1^n \tau(x)dx \leqq Kn\tau(n), \quad |g(x)| \leqq Kx \cdot n^2\phi(n) \leqq K\left(\frac{1}{x}\right)\phi\left(\frac{1}{x}\right),
$$

as $x \simeq 1/n \to 0$, $n \to \infty$. By slight modification in (7) and (8), it may be readily shown that

$$|g(x)| > K\left(\frac{1}{x}\right)\phi\left(\frac{1}{x}\right). \tag{12}$$

Hence $g(x) \asymp (1/x)\phi(1/x)$, as $x \to +0$.

Added in proof. The author has just observed that the argument in the proof of Theorem 3 actually gives a more general result, viz. with $\phi(x) \sim \langle -2, -1]$ replaced by $\phi(x) \sim \langle -2, 0\rangle$. This includes both Theorem 2 and Theorem 3 as particular cases.

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