COMPOSITIO MATHEMATICA

A. J. STAM

On shifting iterated convolutions I

Compositio Mathematica, tome 17 (1965-1966), p. 268-280
<http://www.numdam.org/item?id=CM_1965-1966__17__268_0>

On shifting iterated convolutions I

by

A. J. Stam

1. Introduction

Throughout this paper P, Q, R, with or without indices, denote probability measures on the Borel sets of the real line, PQ denotes the convolution of P and Q and P^n the n^{th} iterated convolution of P. So $U_a P^n$, where U_a is the probability measure degenerate at a, is the n^{th} convolution of P, shifted to the right over a distance a.

The problem considered in this paper is to describe the set L_0 of those values a for which

$$(1.1) \qquad \lim_{n\to\infty} ||P^n - U_a P^n|| = 0.$$

Here $||M||$, for any finite signed measure M, is the total variation of M. It is well known that, for any two finite signed measures M and N,

$$(1.2) \qquad ||M+N|| \leqq ||M|| + ||N||,$$

$$(1.3) \qquad ||MN|| \leqq ||M||\,||N||,$$

MN denoting convolution as before.

In section 5 we consider the following property, weaker than (1.1):

$$(1.4) \qquad \lim_{n\to\infty} ||P^n Q - U_a P^n Q|| = 0$$

for every absolutely continuous Q. This holds for every a if P is not a lattice distribution.

Our main results on (1.1) are the following. The limit in (1.1) always exists and is either 0 or 2. The set L_0 is the real line if and only if P^n for some n has an absolutely continuous component. If P is purely discrete, L_0 is the additive group generated by the set of differences of those y for which $P(\{y\}) > 0$.

For the case that every P^n is purely singular, the author only found examples of a countable L_0 and an uncountable L_0.

The restriction to probability measures is essential. If $||P|| < 1$, the problem is trivial since then $\lim_{n\to\infty} ||P^n|| = 0$. If P is a measure with $P(-\infty, +\infty) > 1$, we may expect $L_0 = \{0\}$, since for probability measures the convergence in (1.1) and (1.4), if present, is of order $n^{-\frac{1}{2}}$ (see lemma 6 below).

2. Preliminary results

Lemma 1. *The set of all a for which*

$$\lim_{n\to\infty} ||P^n R - U_a P^n R|| = 0, \tag{2.1}$$

is an additive group.

Proof. The additivity is immediate by (1.2). Moreover, if (2.1) holds for a, the same is true for $-a$.

Lemma 2. *The sequence $||P^n R - U_a P^n R||$, $n = 1, 2, \ldots$, is non-increasing.*

Proof. The assertion follows from (1.3) since $||P|| = 1$.

Lemma 3. *Let Q be any probability measure on the real line. Then $||Q - U_a Q|| < 2$ if and only if there exist probability measures Q_0 and Q_1 and real numbers α, β with $\alpha > 0$, $\beta \geqq 0$, $a+\beta = 1$, such that*

$$Q = \alpha(\tfrac{1}{2}U_0 + \tfrac{1}{2}U_a)Q_0 + \beta Q_1. \tag{2.2}$$

Proof. That (2.2) is sufficient follows from the inequality

$$\begin{aligned} ||Q - U_a Q|| &= ||\tfrac{1}{2}\alpha U_0 Q_0 + \beta Q_1 - \tfrac{1}{2}\alpha U_{2a} Q_0 - \beta U_a Q_1|| \\ &\leqq \tfrac{1}{2}\alpha + \beta + \tfrac{1}{2}\alpha + \beta = 1 + \beta < 2. \end{aligned}$$

To prove necessity, let A, B be a Hahn decomposition of $(-\infty, +\infty)$ with respect to $Q - U_a Q$. (Halmos [1], § 29). Then for every Borel set E we have, putting $R \overset{\text{df}}{=} U_a Q$:

$$Q(E) = M_1(E) + M_0(E), \quad R(E) = M_2(E) + M_0(E),$$

with

$$\begin{aligned} M_1(E) &\overset{\text{df}}{=} Q(AE) - R(AE), \\ M_2(E) &\overset{\text{df}}{=} R(BE) - Q(BE), \\ M_0(E) &\overset{\text{df}}{=} Q(BE) + R(AE). \end{aligned}$$

By definition of a Hahn decomposition, M_1 and M_2 are (non-negative) measures. The measure M_0 does not vanish, since this would imply $Q(B) = R(A) = 0$ in contradiction with the assumption $||Q-R|| < 2$.

From $Q = M_1+M_0$ and $Q = U_{-a}R = U_{-a}M_2+U_{-a}M_0$ it follows that

$$Q = (\tfrac{1}{2}U_0+\tfrac{1}{2}U_a)U_{-a}M_0+\tfrac{1}{2}(M_1+U_{-a}M_2).$$

Since M_0, M_1, M_2 are measures and M_0 does not vanish, (2.2) holds with $Q_0 = U_{-a}M_0/||M_0||$ and Q_1 either vanishing or equal to $(M_1+U_{-a}M_2)/||M_1+M_2||$.

LEMMA 4. *If* $P = P_1P_2$ *and* $\lim_{n\to\infty} ||P_1^nR-U_aP_1^nR|| = 0$, *then*

$$\lim_{n\to\infty} ||P^nR-U_aP^nR|| = 0.$$

PROOF. Since $||P_2|| = 1$, the lemma follows immediately by (1.3) and the relation

$$P^nR-U_aP^nR = P_2^n(P_1^nR-U_aP_1^nR).$$

LEMMA 5. *For some* m *let*

$$P^m = \alpha P_1+\beta P_2, \tag{2.3}$$

with P_1 *and* P_2 *probability measures and* α, β *constants with* $\alpha > 0$, $\beta \geqq 0$, $\alpha+\beta = 1$. *If* P_1 *satisfies* (2.1), *the same is true for* P. *In fact, we have*

$$\lim_{n\to\infty} ||P^nR-U_aP^nR|| \leqq \lim_{n\to\infty} ||P_1^nR-U_aP_1^nR||. \tag{2.4}$$

PROOF. By lemma 2, with $Q \overset{\text{df}}{=} P^m$

$$\lim_{n\to\infty} ||P^nR-U_aP^nR|| = \lim_{n\to\infty} ||Q^nR-U_aQ^nR||.$$

Since the case $\beta = 0$ is trivial, we assume $\alpha < 1$.

By (1.2) and (1.3)

$$\begin{aligned}||Q^nR-U_aQ^nR|| &= ||\sum_{k=0}^{n} \tbinom{n}{k}\alpha^k\beta^{n-k}P_1^kP_2^{n-k}(R-U_aR)||\\ &\leqq \sum_{k=0}^{n} \tbinom{n}{k}\alpha^k\beta^{n-k}||P_1^kR-P_1^kU_aR||.\end{aligned}$$

Now $\lim_{k\to\infty} ||P_1^kR-P_1^kU_aR||$ exists, so by the Toeplitz theorem (Loève [2], § 16.3, p. 238) the relation (2.4) follows.

Lemma 5 will be fundamental in our proofs. If (2.3) holds, we will say that P^m *contains* P_1.

LEMMA 6. *Let* $P = \frac{1}{2}U_b + \frac{1}{2}U_{a+b}$. *Then, for* $n \to \infty$,

$$\|P^n - U_a P^n\| \sim cn^{-\frac{1}{2}}. \tag{2.5}$$

PROOF. Since P^n is a binomial distribution concentrated in the points $nb+ka$, $k = 0, 1, \ldots, n$,

$$\begin{aligned}\|P^n - U_a P^n\| &= \binom{n}{0} 2^{-n} + \sum_{k=1}^{n} \left| \binom{n}{k} - \binom{n}{k-1} \right| 2^{-n} + \binom{n}{n} 2^{-n} \\ &= \frac{4}{n+1} \sum_{k=0}^{n+1} \binom{n+1}{k} 2^{-n-1} \left| k - \frac{n+1}{2} \right| \\ &= \frac{4}{n+1} \int |x| dB_{n+1}(x) = 2(n+1)^{-\frac{1}{2}} \int |y| dB_{n+1}(\tfrac{1}{2}y\sqrt{n+1}),\end{aligned}$$

where B_m is the distribution function of the binomial distribution $b(\frac{1}{2}, m)$ centered at zero. Since $B_n(\frac{1}{2}y\sqrt{n})$ converges completely to the distribution function of $N(0, 1)$ and has second moment bounded with respect to n, we have (see Loève [2], § 11.4)

$$\lim_{n\to\infty} \int |y| dB_{n+1}(\tfrac{1}{2}y\sqrt{n+1}) = (2\pi)^{-\frac{1}{2}} \int |y| \exp(-\tfrac{1}{2}y^2) dy,$$

which concludes the proof.

3. The set L_0

In this section we consider the set L_0 of those a for which (1.1) holds.

THEOREM 1. *The value of* $\lim_{n\to\infty} \|P^n - U_a P^n\|$ *is either* 0 *or* 2.

PROOF. Obviously the limit is in $[0, 2]$. If it is not 2, then for some n

$$\|P^n - U_a P^n\| < 2,$$

and P^n by lemma 3 contains a probability measure of the form $(\frac{1}{2}U_0 + \frac{1}{2}U_a)Q_0$. So by applying lemma 6, 4 and 5 respectively, we see that $\lim_{n\to\infty} \|P^n - U_a P^n\| = 0$.

THEOREM 2. *The set* L_0 *is the real line if and only if* P^n *for some* n *has an absolutely continuous component.*

PROOF. Sufficiency: If P is absolutely continuous with density $p(x)$, then

$$\lim_{a\to 0} \|P - U_a P\| = \lim_{a\to 0} \int |p(x) - p(x-a)| dx = 0,$$

so that $||P-U_aP|| < 2$ if $a \in (-\varepsilon, \varepsilon)$ for some $\varepsilon > 0$. Therefore $L_0 \supset (-\varepsilon, \varepsilon)$ by lemma 2 and theorem 1. It follows from lemma 1 that $L_0 = (-\infty, +\infty)$.

If P^n has an absolutely continuous component, the assertion $L_0 = (-\infty, +\infty)$ follows from lemma 5 and what has been shown above. Necessity: Let Q be any absolutely continuous probability measure with density $q(y)$. Then A_n, B_n being a Hahn decomposition for P^n-QP^n, we have

$$
\begin{aligned}
||P^n-QP^n|| &= P^n(A_n)-QP^n(A_n)+QP^n(B_n)-P^n(B_n) \\
&= \int q(y)\{P^n(A_n)-U_yP^n(A_n)+U_yP^n(B_n)-P^n(B_n)\}dy \qquad (3.1)\\
&\leqq 2\int q(y)||P^n-U_yP^n||dy.
\end{aligned}
$$

Here $||P^n-U_yP^n||$ is a Borel function of y. This is seen by the following relation, $F(x)$ being the distribution function of P^n:

$$||P^n-U_yP^n|| = \sup \sum_{i=1}^{N-1} |F(b_{i+1})-F(b_{i+1}-y)-F(b_i)+F(b_i-y)|,$$

where the supremum is taken over $N = 2, 3, \ldots$ and rational $b_1, \ldots, b_N$, since $F(x)$ is continuous from the left.

By our assumption and the Lebesgue dominated convergence theorem the right hand side of (3.1) tends to zero for $n \to \infty$. So $||P^n-QP^n|| < 2$ for $n \geqq n_1$ and, since QP^n is absolutely continuous, P^n for $n \geqq n_1$ must have an absolutely continuous component.

Theorem 3. *If P is purely discrete, L_0 is the additive group generated by the difference set of the set J of all those x with $P(\{x\}) > 0$.*

Proof. Let $J = \{c_1, c_2, \ldots\}$. Then P^n is restricted to the set of all x of the form

$$x = \sum_{k=1}^{n} c_{i_k},$$

where some or all i_k may be equal. In order that $||P^n-U_aP^n|| < 2$ for some n, it is necessary that

$$a = \sum_{k=1}^{n} c_{i_k} - \sum_{k=1}^{n} c_{j_k} = \sum_{k=1}^{n} (c_{i_k}-c_{j_k})$$

for some $i_1, i_2, \ldots, i_n, j_1, j_2, \ldots, j_n$, which shows that L_0 is a subset of the additive group generated by the c_i-c_j.

On the other hand, if $x \in J$, $y \in J$, the measure P contains the measure $\frac{1}{2}U_x+\frac{1}{2}U_y$, so that $x-y \in L_0$ by lemma 6 and lemma 5. So by lemma 1 the additive group generated by the c_i-c_j is a subset of L_0.

Theorem 4. *The set* L_0 *is an* F_σ.

Proof. If P is purely discrete, L_0 is a countable set by theorem 3. Assume, then, that P has a nondiscrete component. Writing D_n, C_n for the discrete and nondiscrete component of P^n, we have

$$
\begin{aligned}
&\big|\, ||P^n - U_a P^n|| - ||C_n - U_a C_n||\, \big| \\
(3.2) \qquad &\leqq ||P^n - U_a P^n - (C_n - U_a C_n)|| \\
&= ||D_n - U_a D_n|| \leqq 2\, ||D_n|| = 2||D_1||^n,
\end{aligned}
$$

with $||D_1|| < 1$. Let

$$V_n(x) \overset{\text{df}}{=} ||C_n - U_x C_n||, \quad n = 1, 2, \ldots, \quad -\infty < x < \infty.$$

By (3.2) and theorem 1

$$L_0 = \bigcup_{n=n_0}^{\infty} \{x : V_n(x) \leqq 1\}.$$

Here n_0 is chosen so that $2\, ||D_1||^{n_0} < \frac{1}{2}$, say. Let $G_n(y)$ denote the distribution function of C_n. Then

$$(3.3) \quad V_n(x) = \sup \sum_{i=1}^{N-1} |G_n(b_{i+1}) - G_n(b_{i+1} - x) - G_n(b_i) + G_n(b_i - x)|,$$

where the supremum is taken over $N = 2, 3, \ldots$ and $b_1, b_2, \ldots, b_N$:

$$V_n(x) = \sup_{\alpha} V_{n,\alpha}(x), \quad -\infty < x < \infty, \quad n = 1, 2, \ldots,$$

where the $V_{n,\alpha}(x)$ are of the form occurring in (3.3). The $V_{n,\alpha}(x)$ are continuous functions of x. So the sets $\{x : V_{n,\alpha}(x) \leqq 1\}$ are closed, and

$$L_0 = \bigcup_{n=n_0}^{\infty} \bigcap_{\alpha} \{x : V_{n,\alpha}(x) \leqq 1\}$$

is an F_σ.

4. Examples of singular distributions

If P^n is purely singular for every n, the problem of characterizing the set L_0 is still open. Here we present two examples of purely singular P^n, $n = 1, 2, \ldots$, where L_0 is countable and where L_0 has the power of the continuum, respectively.

Example 1. For P we take the probability distribution of the random variable

$$(4.1) \qquad x \overset{\text{df}}{=} \sum_{n=1}^{\infty} x_n\, 3^{-n^2},$$

where the x_n are independent nonnegative integer valued random variables. Moreover it is assumed that there exist natural numbers n_1 and m such that the x_k for $k \geqq n_1$ have the same distribution restricted to $\{0, 1, \ldots, m\}$ with $P\{x_k = j\} > 0$, $j = 0, 1, \ldots, m$.

As shown by (4.1), the range of x is an uncountable set W and for every $c \in W$ we have $P\{x = c\} = 0$. So P cannot have a discrete component and the same then is true for all P^n. It will be shown below, from the conditions on P stated above, that $||P - U_a P|| = 2$, except for countably many a. But then this must hold also for every P^n, since, as is easily seen, P^n is of the same type as P. So L_0 is a countable set. By theorem 2 no P^n can have an absolutely continuous component, so every P^n is purely singular.

To prove our assertion on $||P - U_a P||$ we show that

$$P\{x + a \in W\} = 0,$$

which implies mutual singularity of P and $U_a P$, for all but countably many a. It is no restriction to assume $a \geqq 0$. Let

$$a = \sum_{n=1}^{\infty} a_n \, 3^{-n^2},$$

where the a_n are chosen so that

$$a_n < 3^{n^2-(n-1)^2} = 3^{2n-1}, \qquad n = 2, 3, \ldots. \tag{4.2}$$

The event $\{x + a \in W\}$ implies the existence of (random) integers $b_1, b_2, \ldots$ such that

$$\sum_{n=1}^{\infty} (a_n + x_n) 3^{-n^2} = \sum_{n=1}^{\infty} b_n 3^{-n^2}, \tag{4.3}$$

$$0 \leqq b_n \leqq m, \qquad n \geqq n_1 \tag{4.4}$$

It will be shown that (4.3) and (4.4), for all but countably many a, imply the occurrence of a sequence of events $\{x_{\nu_k} \in A\}$, with $\nu_1 < \nu_2 < \ldots$ and $P(x_{\nu_k} \in A\} < 1$, $k = 1, 2, \ldots$, from which follows, by the independence and equidistribution of the x_n for $n \geqq n_1$, that $P\{x + a \in W\} = 0$.

First we note that there is n_2 such that for $n \geqq n_2$ the carry c_n from the n^{th} to the $(n-1)^{\text{th}}$ place in the addition in (4.3) is at most 1.

We now distinguish the following cases:

a. There is an infinite sequence $\nu_1 < \nu_2 < \ldots$ such that

$$1 \leqq a_{\nu_k} \leqq 3^{2\nu_k - 1} - m - 2, \qquad k = 1, \ldots.$$

Since for $\nu_k \geqq \max(n_1, n_2)$

$$a_{\nu_k}+x_{\nu_k}+c_{1+\nu_k} \leqq 3^{2\nu_k-1}-m-2+m+1 < 3^{2\nu_k-1},$$

$c_{\nu_k} = 0$ for $\nu_k \geqq \max(n_1, n_2)$, and for (4.3) and (4.4) to hold we must have

$$x_{\nu_k}+a_{\nu_k}+c_{1+\nu_k} \leqq m,$$

implying $x_{\nu_k} \leqq m-1$ and we may take $A = \{0, 1, \ldots, m-1\}$.

b. There is n_3 such that $a_n = 0$ or $a_n \geqq 3^{2n-1}-m-1$ for $n \geqq n_3$.

*b*1. All but a finite number of the a_n are zero. The corresponding a form a countable set.

*b*2. There is n_4 with $a_n \geqq 3^{2n-1}-m-1$ for $n \geqq n_4$. To satisfy (4.3) and (4.4) we must have $c_n > 0$ for $n \geqq \max(n_1, n_4)$, so

$$x_n+a_n+c_{n+1} \geqq 3^{2n-1},$$
$$x_n \geqq 3^{2n-1}-1-a_n,$$

which by (4.2) implies $x_n \geqq 1$ for infinitely many n, except if $a_n = 3^{2n-1}-1$ for all but a finite number of n. But the set of a satisfying the latter condition is countable.

*b*3. The sets of n with $a_n = 0$ and with $a_n \geqq 3^{2n-1}-m-1$ are both infinite. Then we may select a sequence $\nu_1 < \nu_2 < \ldots$ with

$$a_{\nu_k} \geqq 3^{2n-1}-m-1, \quad a_{1+\nu_k} = 0, \quad k = 1, 2, \ldots.$$

To satisfy (4.3) and (4.4) we must have $c_{\nu_k} > 0$, $k = 1, 2, \ldots$, or, since $c_{1+\nu_k} = 0$ for $k \geqq k_1$,

$$x_{\nu_k}+a_{\nu_k} \geqq 3^{2\nu_k-1}, \qquad k \geqq k_1,$$

which by (4.2) implies the events $\{x_{\nu_k} \geqq 1\}$, $k \geqq k_1$.

Example 2. This example is taken from a paper by Wiener and Young [4], section 7. Let $n_1, n_2, \ldots$ be an increasing sequence of natural numbers, such that

$$\sum_{k=1}^{\infty} n_k^{-1} < \infty, \tag{4.5}$$

and consider the expansion of $x \in (0, 1)$:

$$x = \frac{m_1}{n_1}+\frac{m_2}{n_1 n_2}+\frac{m_3}{n_1 n_2 n_3}+\cdots, \tag{4.6}$$

the m_i being nonnegative integers with $m_i < n_i$, ambiguity being removed by taking the terminating expansion whenever possible. The n_k are assumed even, $n_k = 2r_k$, $k = 1, 2, \ldots$. Let $F(x)$ be defined by

$$F(x) = 0, \quad x \leqq 0, \quad F(x) = 1, \quad x \geqq 1,$$

$$F(x) = \frac{m_1/2}{r_1} + \frac{m_2/2}{r_1 r_2} + \frac{m_3/2}{r_1 r_2 r_3} + \ldots,$$

if every m_k in (4.6) is even, and

$$F(x) = \frac{m_1/2}{r_1} + \frac{m_2/2}{r_1 r_2} + \ldots + \frac{m_{n-1}/2}{r_1 r_2 \ldots r_{n-1}} + \frac{[m_n/2]+1}{r_1 r_2 \ldots r_n},$$

if m_n is the first odd m_k in (4.6).

It was shown by Wiener and Young, that $F(x)$ is the distribution function of a purely singular probability measure P and that the set of a with $||P - U_a P|| < 2$ has the power of the continuum. So by our lemma 2 and theorem 1 the set L_0 for this P has the power of the continuum. For the sake of our example we only have to show that P^n for every n is purely singular. To this end we note that F is the distribution function of the random variable

$$x = \sum_{k=1}^{\infty} x_k \, (n_1 n_2 \ldots n_k)^{-1}, \tag{4.7}$$

where the x_k are independent and

$$P\{x_k = j\} = r_k^{-1}, \quad j = 0, 2, \ldots, n_k - 2, \quad k = 1, 2, \ldots. \tag{4.8}$$

Clearly P^n for every n is a convergent infinite convolution of discrete distributions. By a theorem of Wintner, [5], p. 89, no. 148, such a distribution is of pure type. Since P is not discrete, it is sufficient to show that P^n is not purely absolutely continuous. This will follow from the fact that

$$\limsup_{u \to \infty} |\varphi(u)| > 0, \tag{4.9}$$

where $\varphi(u)$ denotes the characteristic function of P, since, if P^n were absolutely continuous, its characteristic function $\varphi^n(u)$ would tend to zero for $|u| \to \infty$ by the Riemann-Lebesgue lemma.

From (4.7) and (4.8) we have

$$\varphi(u) = \prod_{k=1}^{\infty} \varphi_k(u), \qquad -\infty < u < \infty, \tag{4.10}$$

with

$$\varphi_k(u) = \frac{1}{r_k} \sum_{h=0}^{r_k - 1} \exp\left(\frac{2hiu}{n_1 n_2 \ldots n_k}\right), \tag{4.11}$$

so that

$$\varphi_k(n_1 n_2 \ldots n_H \pi) = 1, \qquad k = 1, 2, \ldots, H, \tag{4.12}$$

(4.13) $\varphi_{H+1}(n_1 n_2 \dots n_H \pi) = 2r_{H+1}^{-1} \{1-\exp(2\pi i/n_{H+1})\}^{-1}$,

$$\varphi_{H+m}(n_1 n_2 \dots n_H \pi) = \frac{1}{r_{H+m}} \sum_{h=0}^{r_{H+m}-1} \exp\left(\frac{2\pi i h}{n_{H+1} \dots n_{H+m}}\right), \quad m = 2, 3, \dots,$$

$$\prod_{m=2}^{M} \varphi_{H+m}(n_1 n_2 \dots n_H \pi) = \frac{1}{r_{H+2} \dots r_{H+M}} \sum_{h_2=0}^{r_{H+2}-1} \dots \sum_{h_M=0}^{r_{H+M}-1} A(h_2, h_3, \dots, h_M),$$

with

$$A(h_2, h_3, \dots, h_M) = \exp\left(2\pi i \sum_{m=2}^{M} \frac{h_m}{n_{H+1} \dots n_{H+m}}\right).$$

Now

$$|1-A(h_2, \dots, h_M)| \leqq 2\pi \sum_{m=2}^{M} \frac{h_m}{n_{H+1} \dots n_{H+m}} \leqq \pi \sum_{m=2}^{M} \frac{1}{n_{H+1} \dots n_{H+m-1}},$$

so that

(4.14) $$|1 - \prod_{m=2}^{M} \varphi_{H+m}(n_1 n_2 \dots n_H \pi)| \leqq \pi \sum_{m=2}^{M} \frac{1}{n_{H+1} \dots n_{H+m-1}}.$$

From (4.10)—(4.14) and $\lim_{H\to\infty} n_H = +\infty$ it follows that

$$\lim_{H\to\infty} \varphi(n_1 n_2 \dots n_H \pi) = -2/\pi i,$$

which proves (4.9).

5. The relation (1.4)

For fixed probability measure P let

(5.1) $$D_n(x, Q) \overset{\mathrm{df}}{=} ||P^n Q - U_x P^n Q||, \qquad n = 1, 2, \dots,$$

with Q absolutely continuous,

(5.2) $$D(x, Q) \overset{\mathrm{df}}{=} \lim_{n\to\infty} D_n(x, Q),$$

the limit existing by lemma 2, and

(5.3) $$D(x) \overset{\mathrm{df}}{=} \sup D(x, Q),$$

the supremum being taken over all absolutely continuous probability measures Q.

LEMMA 7. *$D_n(x, Q)$ and $D(x, Q)$ are continuous functionals of Q, uniformly in x and n, in fact*

$$|D_n(x, Q_1) - D_n(x, Q_2)| \leqq 2\, ||Q_1 - Q_2||,$$
$$|D(x, Q_1) - D(x, Q_2)| \leqq 2\, ||Q_1 - Q_2||.$$

Proof. By (1.2) and (1.3)

$$\begin{aligned}|D_n(x, Q_1) - D_n(x, Q_2)| &\leqq ||P^n Q_1 - U_x P^n Q_1 - (P^n Q_2 - U_x P^n Q_2)|| \\ &\leqq ||P^n(Q_1 - Q_2)|| + ||U_x P^n (Q_1 - Q_2)|| \leqq 2\, ||Q_1 - Q_2||.\end{aligned}$$

Lemma 8. *Let Q_k, $k = 1, 2, \ldots$, be a sequence of probability measures with densities $q_k(y)$, $k = 1, 2, \ldots$, such that*

$$q_k(y) = kq_1(ky), \quad -\infty < y < \infty, \quad k = 1, 2, \ldots.$$

Then

$$D(x) = \sup_k D(x, Q_k).$$

Proof. By definition of $D(x)$

(5.4) $$S(x) \overset{\text{df}}{=} \sup_k D(x, Q_k) \leqq D(x), \quad -\infty < x < \infty.$$

For any Q we have by (1.3)

$$D_n(x, QQ_k) \leqq D_n(x, Q_k),$$

so, for $n \to \infty$,

(5.5) $$D(x, QQ_k) \leqq D(x, Q_k) \leqq S(x), \quad k = 1, 2, \ldots.$$

Since Q is absolutely continuous, $||Q - QQ_k||$ tends to zero for $k \to \infty$, so from (5.5) and lemma 7 it follows that $D(x, Q) \leqq S(x)$ for every absolutely continuous Q, implying

(5.6) $$D(x) \leqq S(x),$$

and the lemma follows from (5.4) and (5.6).

Theorem 5. *If P is not a lattice distribution,*

(5.7) $$D(x) = 0, \quad -\infty < x < \infty.$$

If P is a lattice distribution with span c,

(5.8) $$\begin{aligned} D(x) &= 0, \quad x = nc, \quad n \text{ integer}, \\ D(x) &= 2, \quad \text{elsewhere}. \end{aligned}$$

By a lattice distribution is meant here a discrete distribution concentrated in a subset of $\{a + nd, n \text{ integer}\}$ for some a and d, the span being the largest value that may be taken for d.

Proof of (5.7). By lemma 8 it is sufficient to prove that

(5.9) $$\lim_{n \to \infty} D_n(x, Q_k) = 0, \quad k = 1, 2, \ldots,$$

for a suitable sequence Q_k, $k = 1, 2, \ldots$, of the form considered in lemma 8. We choose Q_1 in such a way that it is symmetric about zero and has finite second moment, that its density $q_1(y)$ belongs to L_2 and its characteristic function $\vartheta_1(u)$ satisfies

$$\vartheta_1(u) = 0, \qquad |u| \geqq 1. \tag{5.10}$$

This may be accomplished by taking

$$q_1(y) = \alpha(4y^{-1} \sin \tfrac{1}{4}y)^4, \qquad -\infty < y < \infty,$$

with α a norming constant, as will be seen by applying the Fourier inversion formula to the characteristic function of the fourfold convolution of the uniform distribution on $[-\frac{1}{4}, \frac{1}{4}]$.

By lemma 5 it is no restriction to assume that P has finite second moment. We also center P at its first moment.

For fixed k let $p_n(x)$ and $r_n(x)$ be the densities of $P^n Q_k$ and $U_a P^n Q_k$, respectively. Then

$$D_n(a, Q_k) = \int |p_n(x) - r_n(x)| dx,$$

$$D_n(a, Q_k) \leqq \int_{-A}^{A} |p_n(x) - r_n(x)| dx + 2 \int_{|x| \geqq A-a} p_n(x) dx.$$

Here A is allowed to depend on n. By the inequality between arithmetic and quadratic mean and by Chebychev's inequality

$$D_n(a, Q_k) \leqq \left[2A \int_{-\infty}^{+\infty} \{p_n(x) - r_n(x)\}^2 dx \right]^{\frac{1}{2}} + 2 \frac{nd^2 + v^2 k^{-2}}{(A-a)^2},$$

where d^2 is the variance of P and v^2 the variance of Q_1. Since $q_1 \in L_2$, also $q_k \in L_2$ and therefore $p_n \in L_2$, $r_n \in L_2$. So by Parseval's formula

$$D_n(a, Q_k) \leqq \left[\frac{A}{\pi} \int_{-\infty}^{+\infty} |(1 - e^{iua}) \varphi^n(u) \vartheta_k(u)|^2 du \right]^{\frac{1}{2}} + 2 \frac{nd^2 + v^2 k^{-2}}{(A-a)^2},$$

where $\varphi(u)$ denotes the characteristic function of P and $\vartheta_k(u) = \vartheta_1(u/k)$ the characteristic function of Q_k. Making use of (5.10) and the inequality $|\vartheta_k(u)| \leqq 1$, and putting $A = Cn^{\frac{1}{2}}$, we find for n suitably large

$$D_n(a, Q_k) \leqq \left[\frac{Cn^{\frac{1}{2}}}{\pi} \int_{-k}^{k} |\varphi(u)|^{2n} \, |1 - e^{iua}|^2 du \right]^{\frac{1}{2}} + 3d^2 C^{-2}.$$

Since P is not degenerate and has finite second moment, there are $\varepsilon \in (0, 1)$ and $\beta \in (0, 1)$ such that

$$|\varphi(u)|^2 \leqq 1 - \beta u^2, \qquad |u| \leqq \varepsilon.$$

Moreover, P being not a lattice distribution, there is a constant $\gamma \in [0, 1)$ so that

$$|\varphi(u)| \leqq \gamma, \qquad \varepsilon \leqq |u| \leqq k.$$

(See Lukacs [3], theorem 2.1.4). So, since also

$|1-e^{iua}|^2 \leqq a^2u^2 \leqq a^2|u|$ for $|u| \leqq \varepsilon$,

$$\limsup_n D_n(a, Q_k) \leqq 3d^2C^{-2}+\limsup_n \left[\frac{2a^2Cn^{\frac{1}{2}}}{\pi}\int_0^\varepsilon (1-\beta u^2)^n u du\right]^{\frac{1}{2}}$$

$$= 3d^2C^{-2}+\limsup_n \left[\frac{aCn^{\frac{1}{2}}}{\beta\pi(n+1)}\{1-(1-\beta\varepsilon^2)^{n+1}\}\right]^{\frac{1}{2}} = 3d^2C^{-2}.$$

Since this holds for every $C > 0$, the proof of (5.9) is concluded.

PROOF OF (5.8) That $D(x) = 0$ for $x = nc$, follows from theorem 3. That $D(x) = 2$ for $x \neq nc$, is seen by taking for Q the uniform distribution on $[-h, h]$, with h so small that no intervals $[nc-h, nc+h]$ and $[mc+x-h, mc+x+h]$, m, n integer, overlap.

REFERENCES

HALMOS, P. R.,

[1] Measure Theory, Sec. ed. Van Nostrand, 1950.

LOÈVE, M.,

[2] Probability Theory, Sec. ed. Van Nostrand, 1960.

LUKACS, E.,

[3] Characteristic Functions. Griffin, 1960.

WIENER, N. and R. C. YOUNG,

[4] The Total Variation of $g(x+h)-g(x)$. Trans. Amer. Math. Soc. 35 (1933), 327—340.

WINTNER, A.,

[5] The Fourier Transforms of Probability Distributions. Baltimore, 1947.

(Oblatum 4-10-65).