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by

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Introduction

The set R^X of real-valued functions on an abstract space X, with addition and order defined in the usual way, i.e. "pointwise", is a lattice ordered real vector space. We call a non-empty subset C of this space a *clan of functions* on X if C is a sublattice of R^X and satisfies the following two conditions.

(1) If f, g are in C and $f \leqq g$, then $g-f$ is in C.

(2) If f, g, h in C are such that $g-f$ and $h-g$ are in C, then $h-f$ is in C.

Lattice-ordered vector spaces and additive groups of real-valued functions on X are clans of functions on X. The functions $f \geqq 0$ in a clan C of functions on X form a subclan of C. A class $\mathscr{R}$ of subsets of X is a ring of sets, as defined in [4], if and only if the characteristic functions of the sets in $\mathscr{R}$ form a clan of functions on X. The last of these examples motivated the name "clan", since rings of subsets of X are called "clans de parties de l'ensemble X" in [3].

The examples show that the classes of functions commonly encountered in the theory of measure and integration are clans. In fact, a unified theory of measure and the Daniell integral has been developed for clans of functions. An account of this theory will be published elsewhere.

In the present paper, we develop a theory of abstract clans. An abstract clan is, by definition, a lattice in which a binary operation, called subtraction and subject to certain axioms, is defined. Boolean algebras and lattice ordered groups are abstract clans. Thus our theory of clans solves Problem 105 of Birkhoff's Lattice Theory (see [2], p. 233).

The first part of this paper (§§ 1—5) is concerned with the general theory of clans. We define clans, give some examples, and intro-

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duce addition in a clan. We show that the basic properties of lattice-ordered groups remain valid, almost without restriction, for clans.

In the second part of the paper (§§ 6—9), we develop the theory of commutative clans. We show that every commutative clan can be embedded, as a subclan, into a lattice-ordered abelian group. In fact, we define a "free" embedding functor from commutative clans to lattice-ordered abelian groups. In the last section, we discuss Boolean rings and Archimedean clans.

Two unsolved problems should be signalled. First, when is a commutative clan isomorphic to a clan of functions on an abstract space, as defined in this Introduction? Second, can every non-commutative clan be embedded, as a subclan, into a lattice-ordered group, or how can the clans be characterized which can be so embedded?

We use the notations of [2] in this paper, except that we usually write "ordered" for "partly ordered". We denote by $(m \cdot n)$ the nth proposition or theorem of § m.

1. Axioms and examples

We define an *abstract clan* as a lattice C with a binary operation σ, called *subtraction* and mapping a subset Σ of $C \times C$ into C, which satisfies the four axioms listed below and also Axiom C7 of § 5. We write $b-a$ for $\sigma(a, b)$, and we say that $b-a$ is *defined*, if $(a, b) \in \Sigma$.

The first four axioms are:

C1. For a, b in C and $a \leqq b$, $b-a$ is defined.

C2. For a, b, c in C and $a \cup b \leqq c$, we have $a \leqq b$ if and only if $c-b \leqq c-a$.

C3. For a, b, c in C if $b-a$ and $c-a$ are defined and $b \leqq c$, then $b-a \leqq c-a$ and $(c-a)-(b-a) = c-b$.

C4. For a, b, p in C, if $b \leqq p$ and $a \leqq p-b$, then there is an element c of C such that $p-c$ is defined, and $p-c = (p-b)-a$.

The following examples show that Axioms C2—C4 are essentially independent.

[C2] Let C be any lattice, and let $\sigma(a, b) = b$ for $a \leqq b$.

[C3] Let C be a lattice-ordered group, and let $\sigma(a, b) = -a$ for any a, b in C.

[C4] Let C be the set of integers $\geqq -p$, $p > 0$, and let $\sigma(a, b) = b-a$ for a, b in C and $a \leqq b$.

A sublattice S of a clan C is called a *subclan* of C if the following conditions are satisfied.

S1. For a, b in S and $a \leqq b$, $b-a$ is in S.

S2. For a, b, p in S, if $b \leqq p$ and $a \leqq p-b$, then there is an element c in S such that $p-c$ is defined and $p-c = (p-b)-a$.

A mapping $f : C \to C_1$ of a clan C into a clan C_1 is called a *clan homomorphism* from C to C_1 if f is a lattice homomorphism from C to C_1 and satisfies the following condition.

M1. For a, b in C and $a \leqq b$, $f(b)-f(a) = f(b-a)$.

These are the definitions one expects, and subclans and clan homomorphisms have the expected properties. We need not go into details.

We now are ready to give some examples of abstract clans.

(1.1) A lattice-ordered group L can be made into a clan in two ways. First, we may put $\sigma(a, b) = b-a = b+(-a)$ for $a \leqq b$ in L, or for any a, b in L. Second, we may put $\sigma(a, b) = (-a)+b$ for $a \leqq b$ in L, or for any a, b in L.

C1—C4 (and also C7) are easily verified in both cases. We shall consider the first of the two clans just defined as the clan underlying L, and the second clan as the dual clan of L (cf. § 4 below).

(1.2) Let $\mathscr{R}$ be a class of subsets of a set X, with inclusion as order relation, and with subtraction defined by $\sigma(A, B) = B-A$ for A, B in $\mathscr{R}$ and $A \subset B$. Then $\mathscr{R}$ is an abstract clan if and only if $\mathscr{R}$ is a ring of sets in the usual sense.

More generally, any Boolean ring R becomes a clan if we put $\sigma(a, b) = b-a$ for a, b in R and $a \leqq b$.

(1.3) Clans of functions on an abstract space X, as defined in the Introduction, are subclans of the clan R^X of functions from X to R. More generally, if C is any clan, then the set C^X of functions from X to C, with order and subtraction defined "pointwise", is a clan.

(1.4) As a further example, we consider clans of real numbers, with order and subtraction defined as usual.

If $C \neq \{0\}$, then C has positive elements. If C has a least positive element d, then it is easily verified that any element of C is an integral multiple of d, and that only the following three possibilities occur:

(i) $C = \{0, d, 2d, \ldots, md\}$ for some positive integer m.

(ii) C consists of all multiples kd, k a natural number.

(iii) C consists of all multiples kd, k any integer.

If C has no least positive element, and if $a > 0$ is in C, then it is easily seen that C is dense in the interval $[0, a]$. If we assume that C is a closed set of real numbers, we have again three possibilities:

(i) $C = [0, p]$ for some real number $p > 0$.
(ii) $C = R^+$ consists of all non-negative real numbers.
(iii) $C = R$ consists of all real numbers.

2. Properties of subtraction

We assume from now on that a clan C is given. Lower case letters denote elements of C. In this and the next two sections, we use very little of the lattice properties of C. In fact, we use only the property that any two elements of the ordered set C have a common upper bound.

(2.1) If $b-a$ and $c-a$ are defined, then $b-a \leqq c-a$ if and only if $b \leqq c$.

Proof: Let $a \cup b \cup c \leqq t$. Then $(t-a)-(b-a) = t-b$, and $(t-a)-(c-a) = t-c$, by C1 and C3. Now $b-a \leqq c-a \Leftrightarrow t-c \leqq t-b \Leftrightarrow b \leqq c$ by C2.

(2.2) If $a-b$ and $a-c$ are defined, then $a-c \leqq a-b$ if and only if $b \leqq c$.

Proof: Let $a \cup b \cup c \leqq t$. Then $(t-b)-(a-b) = (t-c)-(a-c) = t-a$ by C3, and $b \leqq c \Leftrightarrow t-c \leqq t-b \Leftrightarrow (t-b)-(a-b) = (t-c)-(a-c) \leqq (t-b)-(a-c) \Leftrightarrow a-c \leqq a-b$ by C2 and (2.1).

(2.3) If $b-a$ and $b'-a$ are defined, then $b-a = b'-a$ if and only if $b = b'$. If $b-a$ and $b-a'$ are defined, then $b-a = b-a'$ if and only if $a = a'$.

This follows immediately from (2.1) and (2.2)

(2.4) The equation $x-x = x$ has a unique solution in C which we denote by 0. This *zero element* of C has the following properties. $a-a = 0$ for any $a \in C$, and $u-0 = u$ for any $u \in C$ such that $u-0$ is defined.

Proof: If $x-x = x$ and $x \leqq t$, then $t-x = (t-x)-(x-x) = (t-x)-x$ by C3, hence $t-x = t$ by (2.3). If also $y-y = y$ and $x \cup y \leqq t$, then $t-x = t = t-y$, hence $x = y$. For any $a \in C$, $(a-a)-(a-a) = a-a$ by C3, so that $a-a$ is a solution of the equation $x-x = x$. Finally, if $x-x = x$ and $u-x$ is defined, then $t-u = (t-x)-(u-x) = t-(u-x)$ for $x \cup u \leqq t$, by C3 and the results already obtained, hence $u-x = u$ by (2.3).

(2.5) Let $b \leqq p$, $a \leqq p-b$, and $(p-b)-a = p-c$. Then

$c \leqq p$. If $p \leqq r$, then $b \leqq r$ and $a \leqq r-b$. For any $q \in C$, if $b \leqq q$ and $a \leqq q-b$, then $(q-b)-a = q-c$, for the same c.

PROOF: $(p-b)-a \geqq (p-b)-(p-b) = p-p$, hence $c \leqq p$, by C3. and (2.2). The second part is obvious. For the third part, let $p \cup q \leqq r$. Then $(r-c)-(p-c) = r-p = ((r-b)-a)-((p-b)-a)$ by C3, hence $(r-b)-a = r-c$ by (2.3). In the same way, $(q-b)-a = q-c$.

3. Addition

For a, b in C such that $b \leqq p$ and $a \leqq p-b$ for some $p \in C$, we define $a+b$ in C by putting $p-(a+b) = (p-b)-a$. By C4 and (2.5), this defines $a+b$ uniquely, independently of p. If there is no $p \in C$ with $b \leqq p$ and $a \leqq p-b$, then $a+b$ is not defined in C.

In a lattice-ordered group L, we have $p-(a+b) = (p-b)-a$ for any a, b in L and $p \geqq b \cup (a+b)$. Thus addition in the clan underlying L is the same as addition in L, as it should be.

Condition S2 for subclans (p. 4) can now be reformulated as follows.

$S2_1$. For a, b in S, if $a+b$ is defined in C and majored in S, then $a+b$ is in S.

(3.1) $a+0 = 0+a = a$ for all $a \in C$. If $b-a$ is defined in C, then $(b-a)+a = b$ in C.

PROOF: $(p-0)-a = p-a$, for $a \cup 0 \leqq p$, and $(p-a)-0 = p-a$, for $a \leqq p$, prove the first part. If $b-a$ is defined in C, then $(p-a)-(b-a) = p-b$ for $a \cup b \leqq p$, by C3, proving the second part.

The second part of (3.1) shows that we could have defined clans in terms of addition, instead of using subtraction as the basic operation. However, the definition in terms of subtraction seems to be more natural and simpler.

(3.2) If $a+b$ and $a+c$ are defined in C, then $a+b \leqq a+c$ if and only if $b \leqq c$. If $b+a$ and $c+a$ are defined in C, then $b+a \leqq c+a$ if and only if $b \leqq c$.

(3.3) If $a+b$ and $a'+b$ are defined in C, then $a+b = a'+b$ if and only if $a = a'$. If $a+b$ and $a+b'$ are defined in C, then $a+b = a+b'$ if and only if $b = b'$.

PROOF: If $a+b$ and $a+c$ are defined, then we can choose $p \in C$ so that $b \cup c \leqq p$, $a \leqq p-b$, $a \leqq p-c$, by (2.5). Then

$b \leqq c \Leftrightarrow p-c \leqq p-b \Leftrightarrow (p-c)-a \leqq (p-b)-a \Leftrightarrow p-(a+c) \leqq p-(a+b) \Leftrightarrow a+b \leqq a+c$, by C2 and (2.1). The second part of (3.2) is proved similarly, and (3.3) is an immediate corollary.

(3.4) Let $a+b$ be defined in C. If $a' \leqq a$, then $a'+b$ is defined. If $b' \leqq b$, then $a+b'$ is defined.

Proof: If $b \leqq p$ and $a \leqq p-b$, then $a' \leqq p-b$ for $a' \leqq a$, and $a'+b$ is defined. The second part is proved similarly.

(3.5) If $b+c$ and $a+(b+c)$ are defined in C, then $a+b$ and $(a+b)+c$ are defined in C, and $(a+b)+c = a+(b+c)$.

Proof: We can choose $p \in C$ such that $c \leqq p$, $b \leqq p-c$, and $a \leqq p-(b+c)$. Then we have:

$$\begin{aligned} p-(a+(b+c)) &= (p-(b+c))-a = ((p-c)-b)-a \\ &= (p-c)-(a+b) = p-((a+b)+c), \end{aligned}$$

and (3.5) follows.

In view of (3.1), it seems natural to extend subtraction in C by putting $b-a = x$ whenever $x+a = b$ in C. We show that axioms C1—C4, and hence all results proved so far, remain valid, and that addition in C is not extended, if we extend subtraction in this way.

First, if $x+a = b \geqq a$, then $(b-a)+a = b$ by C1 and (3.1), so that $x = b-a$. In other words, no new positive differences $b-a$, $a \leqq b$, are obtained by extending subtraction.

Now it is clear that addition is not extended, and that C1, C2, C4 remain valid, since only positive differences are used in these axioms, and in the definition of addition. We must prove, however, that C3 remains valid.

Suppose that $x+a = b$, $y+a = c$, and $b \leqq c$. Then $x \leqq y$ by (3.2), and $y+a = c = (c-b)+b = (c-b)+(x+a) = ((c-b)+x)+a$ by (3.1) and (3.5), so that $y = (c-b)+x$, and $(c-a)-(b-a) = y-x = c-b$. This proves C3 for extended subtraction.

From now on, we shall use subtraction in C in the extended sense. Then we have the following useful result.

(3.6) If $b-a$ and $c-b$ are defined in C, then $c-a$ is defined in C, and $c-a = (c-b)+(b-a)$.

Proof: If $x+a = b$ and $y+b = c$, then $y+(x+a) = (y+x)+a = c$ by (3.5), so that $c-a = (c-b)+(b-a)$.

4. Symmetric and commutative clans

We call a clan C *symmetric* if C satisfies:

C5. For a, b in C and $a \leqq b$, there is an element t of C such that $a+t = b$.

A clan C is called *commutative* if C satisfies:

C6. For a, b in C, if $a+b$ is defined in C, then $b+a$ is defined in C, and $a+b = b+a$.

In a symmetric clan C, we define *dual subtraction* σ^* by putting $\sigma^*(a, b) = b * a = x$ if $a+x = b$, and leaving $\sigma^*(a, b)$ undefined if the equation $a+x = b$ does not have a solution in C.

For a lattice-ordered group L, dual substraction is defined for any a, b in L by $b * a = (-a)+b$. This example shows that C6 is independent of the other axioms, including C5. On the other hand, we have:

(4.1) A commutative clan C is symmetric, and $b * a = b-a$ whenever $b-a$ is defined.

Proof: $a+x = b \Leftrightarrow x+a = b$ if C is commutative, and $(b-a)+b = a+(b-a) = b$ if $a \leqq b$.

The following example shows that C5 is independent of C1—C4 and C7.

[C5] Let L be the group generated by an element e and a doubly infinite sequence of elements a_k, all of infinite order, with the relations

$$a_k+e = e+a_{k-1}, \quad a_h+a_k = a_k+a_h,$$

for all integers h, k. Then L consists of all formal sums $me+\sum_{k=-\infty}^{\infty} n_k a_k$, with only a finite number of coefficients $n_k \neq 0$. We put

$$me+\sum n_k a_k < m'e+\sum n'_k a_k$$

if either $m < m'$, or $m = m'$ and $n_k < n'_k$ for the smallest integer k such that $n_k \neq n'_k$. This defines a linear order relation in L which is easily seen to be compatible with addition. Thus L is a linearly ordered group, and all the more a lattice-ordered group. Let now C consist of all elements $me+\sum n_k a_k \geqq 0$ of L for which $m = 0$ or $m = 1$, and $n_k = 0$ for all integers $k < 0$. It is easily verified that C is a subclan of L (use $S2_1$ (p. 8) instead of S2!). However, a_0 and e are in C, and $a_0 < e$, but $(-a_0)+e = e-a_{-1}$ is not in C.

The following result is trivial, but useful.

(4.2) In a symmetric clan C, $a+b = c \Leftrightarrow c-b = a \Leftrightarrow c * a = b$.

If a statement $\mathfrak{P}$ about a symmetric clan C is formulated in terms of lattice operations and subtraction, we obtain a *dual* *statement* $\mathfrak{P}^*$ by replacing subtraction by dual subtraction throughout. From the formulas

$$(p-b)-a = p-(a+b), \quad b-(b*a) = a,$$
$$(p*b)*a = p*(b+a), \quad b*(b-a) = a,$$

it follows that dual subtraction must be replaced by subtraction, and the order of terms in a sum reversed, when changing $\mathfrak{P}$ into $\mathfrak{P}^*$. From this, it follows that the dual statement of $\mathfrak{P}^*$ is $\mathfrak{P}$.

(4.3) A statement $\mathfrak{P}$ about a symmetric clan C is valid if and only if the dual statement $\mathfrak{P}^*$ is valid.

PROOF: We need only show that the dual statements C1*—C5* of C1—C5 are valid. Then any proof of a statement $\mathfrak{P}$ becomes a proof of $\mathfrak{P}^*$ if every step of the proof is dualized.

Obviously, C1* $\Leftrightarrow$ C5 and C5* $\Leftrightarrow$ C1.

If $a \cup b \leqq c$, and if $c = a+x = b+y$, then $a \leqq b \Leftrightarrow a+y \leqq b+y = a+x \Leftrightarrow y \leqq x$. This proves C2*.

If $b = a+x$, $c = a+y$, and $b \leqq c$, then $x \leqq y$ by (3.2). If $y = x+u$, then $c = a+(x+u) = (a+x)+u = b+u$ by (3.5). Thus $u = c*b = y*x = (c*a)*(b*a)$. This proves C3*.

If $b \leqq p$ and $a \leqq p*b$, let $p = b+x$ and $p*b = x = a+y$. Then $p = b+(a+y) = (b+a)+y$, so that $(p*b)*a = p*(b+a)$. This proves C4*, and one of the formulas displayed above.

Combining (3.5) and its dual statement, we obtain the following strong associative law of addition for symmetric clans.

(4.4) For a, b, c in a symmetric clan C, $a+b$ and $(a+b)+c$ are defined in C if and only if $b+c$ and $a+(b+c)$ are defined in C, and then $(a+b)+c = a+(b+c)$.

We define a *symmetric subclan* of a symmetric clan C as a subclan of C which satisfies the dual condition S1* of S1 as well as S1 and S2 (see p. 174). Example [C5] of p. 178 shows that not every subclan of a symmetric clan C is symmetric.

The following result shows that condition S2, for a symmetric subclan S of a symmetric clan C, can be replaced by:

S3. For a, b, c in S, if $b-a$ and $c-b$ are defined in C and elements of S, then $c-a$ is in S.

(4.5) For a sublattice S of a symmetric clan C which satisfies S1 and S1*, the four conditions S2, S3, S2*, S3*, are logically equivalent.

Proof: If $b-a$ and $c-b$ are defined and in S, with a, b, c in S, let $a \cup b \cup c \leqq t$, $t \in S$. Then $(t-a)-(b-a) = t-b$ and $(t-b)-(c-b) = t-c = (t-a)-(c-a)$ are in S. Using S2 with $p = t-a$, we conclude that $c-a$ is in S. Thus S2 $\Rightarrow$ S3.

For a, b, p in S, with $b \leqq p$ and $a \leqq p * b$, we have $b = p-(p * b)$ and $a = (p * b)-((p * b) * a)$. If S3 is valid for S, then $p-((p * b) * a) = b+a$ is in S. Thus S3 $\Rightarrow$ S2*.

Dually, S2* $\Rightarrow$ S3* and S3* $\Rightarrow$ S2.

5. Lattice properties of addition

From now on, we shall use the lattice properties of the clan C to full extent. (5.1)—(5.3) use only axioms C1—C4. From (5.4) on, we use also C5 and the following axiom.

C7. For u, v in C, if $u \cap v = 0$, then $u+v$ is defined in C.

This is obviously self-dual. The following example shows that C7 is independent of the other axioms, including C6.

[C7] Let $C = \{0, a, b, c\}$, with $0 \leqq a \leqq c$ and $0 \leqq b \leqq c$ defining the order relation. Let $x-x = 0$ and $x-0 = x$ for any $x \in C$, and let $c-a = a$ and $c-b = b$. This satisfies C1—C4 and C6, hence also C5, but $a \cap b = 0$ and $a+b$ is not defined in C.

(5.1) If $u = a-(a \cap b)$ and $v = b-(a \cap b)$, then $u \cap v = 0$.

Proof: $u \geqq a-a = 0$ and $v \geqq 0$, hence $u \cap v \geqq 0$. On the other hand, if $x \leqq u \cap v$, then $x+(a \cap b) \leqq u+(a \cap b) = a$, and $x+(a \cap b) \leqq b$, hence $x+(a \cap b) \leqq a \cap b$, and $x \leqq 0$.

(5.2) If $a+c$ and $b+c$ are defined in C, then $(a+c) \cap (b+c) = (a \cap b)+c$.

Proof: Obviously, $(a \cap b)+c \leqq (a+c) \cap (b+c)$. On the other hand, let $a = u+(a \cap b)$, $b = v+(a \cap b)$, $(a+c) \cap (b+c) = z+(a \cap b)+c$. Then $z+(a \cap b)+c \leqq u+(a \cap b)+c$, hence $z \leqq u$. Similarly, $z \leqq v$. Thus $z \leqq 0$ by (5.1). But $z \geqq 0$, hence $z = 0$.

(5.3) If $a+c$ and $b+c$ are defined in C, then $(a \cup b)+c$ is defined in C, and $(a+c) \cup (b+c) = (a \cup b)+c$.

Proof: If $a+c$ and $b+c$ are defined, then we can choose $p \in C$ so that $p \geqq c$ and $p-c \geqq a$, $p-c \geqq b$. It follows that $(a \cup b)+c$ is defined. Clearly $(a+c) \cup (b+c) \leqq (a \cup b)+c$. On the other hand, let $x \geqq a+c$, $x \geqq b+c$. Then $x = u+(a+c) = (u+a)+c$, with $u \geqq 0$ and $u+a = x-c \geqq a$. Similarly, $x-c \geqq b$. Thus $x-c \geqq a \cup b$, and $x = (x-c)+c \geqq (a \cup b)+c$.

From now on, we assume C5 and C7.

(5.4) If $u \cap v = 0$, then $u+v = v+u = u \cup v$.

Proof: Since $u \cup v \leqq u+v$, we have $u \cup v = u'+v = u+v'$, with $0 \leqq u' \leqq u$, $0 \leqq v' \leqq v$, by C1, C5, and (3.2). If $v = x+v'$, then $u \cup v = u'+(x+v') = (u'+x)+v' = u+v'$, hence $u = u'+x$. Thus $x \leqq u$, $x \leqq v$, hence $x \leqq 0$. But $x \geqq 0$, hence $x = 0$.

(5.5) $a-(a \cap b) = (a \cup b)-b$.

Proof: Let $u = a-(a \cap b)$ and $v = b-(a \cap b)$. Then $u \cap v = 0$ by (5.1), and $a \cup b = (u+(a \cap b)) \cup (v+(a \cap b)) = (u \cup v)+(a \cap b) = u+v+(a \cap b) = u+b$, by (5.3) and (5.4). Thus $u = (a \cup b)-b$.

We denote by C^+ the set of elements $a \geqq 0$ of C, by C^0 the set of all $a \in C$ such that $-a = 0-a = 0 * a$ is defined in C, and we put $C^- = C^+ \cap C^0$. For any $a \in C$, we put $a^+ = a \cup 0$ and $a^- = 0-(a \cap 0) = 0 * (a \cap 0)$. These definitions are self-dual (in the sense of § 4).

(5.7) C^0 is a symmetric subclan of C and a lattice-ordered group. C^+ and C^- are symmetric subclans of C and closed under addition in C. For $p \in C^0$ and any $a \in C$, $a+p$ and $p+a$ are defined in C. For $p \in C^0$ and $q \leqq p$ in C, $q \in C^0$.

Proof: For $a \in C$, $p \in C^0$, $a-0$ and $0-p$ are defined, so that $a-p$ is defined by (3.6). Replacing p by $-p$, $a+p$ is defined. Dually, $a * p$ and $p+a$ are defined. If also $a \in C^0$, then $p * a = -(a-p)$ is defined, hence $a-p$ in C^0. Thus C^0 is a group. If $p \in C^0$, $q \leqq p$, then $0-p$ and $p-q$ are defined, so that $0-q$ is defined, and $q \in C^0$. For p, q in C^0, $p \cup q = p-(p \cap q)+q$ is in C^0 by (5.5) and the preceding results. Thus C^0 is lattice-ordered, and a symmetric subclan of C. The statement about C^+ is obvious, and that about C^- follows.

(5.8) $a = a^+-a^- = a^+ * a^-$, $a^+ \in C^+$, $a^- \in C^-$, and $a^+ \cap a^- = 0$.

Proof: $a^+-a = 0-(a \cap 0) = a^-$ and $a^+-0 = a-(a \cap 0) = a+a^-$ by (5.5), hence $a^+ * a^- = a^+-a^- = a$. Obviously, $a^+ \in C^+$ and $a^- \in C^-$. Finally, $(a^+ \cap a^-)+(a \cap 0) = (a^+ + (a \cap 0)) \cap (a^-+(a \cap 0)) = a \cap 0$ by (5.2), and hence $a^+ \cap a^- = 0$.

(5.9) If $a+b$ is defined in C and $c \leqq a+b$, then $c = a'+b'$ for elements $a' \leqq a$ and $b' \leqq b$ of C.

Proof: Let $u = c-(b \cap c)$ and $a' = a \cap u$, so that $a' \leqq a$.

Since $c * u = b \cap c$ and $u * a'$ are defined, $b' = c * a'$ is defined by the dual of (3.6), and $c = a' + b'$. We must show that $b' \leqq b$. Now

$$a' + b' = c \leqq (a+b) \cap (b \cup c) = (a+b) \cap (u+b)$$
$$= (a \cap u) + b = a' + b,$$

and thus $b' \leqq b$.

6. The group EC

We assume from now on that C is a commutative clan. We discuss in this section the canonical embedding of C into an ordered abelian group EC. This group is constructed in two stages. The first stage consists of constructing a semigroup $E'C$ into which C is embedded. In this, we follow [1]. The second stage consists of embedding $E'C$ into a group EC.

We form words $(a_1, \ldots, a_r)$ with entries in the commutative clan C, and we add these words by the usual formula

$$(a_1, \ldots, a_r) + (a_{r+1}, \ldots, a_{r+s}) = (a_1, \ldots, a_{r+s}).$$

Words then form an additive semigroup which we denote by WC. We call two words $A + (a, b) + B$ and $A + (c) + B$ *directly similar* in WC if $a + b = c$ in C. Here A or B or both may be the "empty word". Two words in WC are called *similar* if they are related by a finite chain of direct similarities. This defines a congruence relation in WC. We denote by $E'C$ the quotient semigroup of WC with respect to this congruence relation, or a semigroup isomorphic to this quotient semigroup, by $\langle a_1, \ldots, a_r \rangle$ the image in $E'C$ of a word $(a_1, \ldots, a_r)$ of WC, and we use lower case german letters to denote elements of $E'C$, and of the group EC into which we shall embed $E'C$. We note that always $\langle a_1, \ldots, a_r \rangle = \langle a_1 \rangle + \ldots + \langle a_r \rangle$ in $E'C$.

(6.1) The mapping $a \to \langle a \rangle$ of C into $E'C$ is one-to-one, and $\langle a \rangle + \langle b \rangle = \langle c \rangle$ in $E'C$ if and only if $a + b = c$ in C. More generally, $\langle a_1, \ldots, a_r \rangle = \langle a \rangle$ in $E'C$ if and only if $a_1 + \ldots + a_r = a$ in C.

This follows from the strong associative law (4.4). We refer to [7] for a detailed discussion.

(6.2) $\mathfrak{a} + \mathfrak{b} = \mathfrak{b} + \mathfrak{a}$ for any $\mathfrak{a}$, $\mathfrak{b}$ in $E'C$.

PROOF: It is enough to show that $\langle a, b \rangle = \langle b, a \rangle$ for any a, b in C. Let $p = a \cap b$, $q = a \cup b$, $u = a - p = q - b$, $v = b - p = q - a$. Then $\langle a, b \rangle = \langle p, u, b \rangle = \langle p, q \rangle = \langle p, v, a \rangle = \langle b, a \rangle$.

Before proceeding further, we need two lemmas.

(6.3) If $a+b = c'+c''$ in C, then there are decompositions $a'+a'' = a$ and $b'+b'' = b$ in C such that $a'+b' = c'$ and $a''+b'' = c''$.

PROOF: $a+b = (a-b^-)+b^+$ by (5.7) and (5.8), hence $a-b^- \leqq c'+c''$, and $a-b^- = a_1+a''$, with $a_1 \leqq c'$, $a'' \leqq c''$, by (5.9). If $a' = a_1+b^-$, then $b' = c'-a' = (c'-a_1)-b^-$ and $b'' = c''-a''$ are defined in C, and we have the desired relations.

(6.4) If $\mathfrak{a}+\mathfrak{b} = \langle c_1, \ldots, c_n\rangle$ in $E'C$, then there are decompositions $c_i = c_i'+c_i''$ in C, for $i = 1, \ldots, n$, such that $\mathfrak{a} = \langle c_1', \ldots, c_n'\rangle$ and $\mathfrak{b} = \langle c_1'', \ldots, c_n''\rangle$ in $E'C$.

PROOF: For $\mathfrak{a}+\mathfrak{b} = \langle a_1, \ldots, a_r, b_1, \ldots, b_s\rangle$, with $\langle a_1, \ldots, a_r\rangle = \mathfrak{a}$ and $\langle b_1, \ldots, b_s\rangle = \mathfrak{b}$, we put $a_i = a_i+0$ and $b_j = 0+b_j$, for $i = 1, \ldots, r$, $b = 1, \ldots, s$, to obtain the desired decomposition. Now we must show only that "decomposability" of $(c_1, \ldots, c_n)$ is preserved under direct similarity of words.

Let $c_i = c_i'+c_i''$, $i = 1, \ldots, n$, be a "good" decomposition of the word $(c_1, \ldots, c_n)$. If we replace two letters c_j, c_{j+1} by a single letter $d = c_j+c_{j+1}$, then we decompose d by putting $d = d'+d''$, with $d' = c_j'+c_{j+1}'$, $d'' = c_j''+c_{j+1}''$. If we replace one letter c_j by two letters p, q, with $p+q = c_j$ in C, then we have, by (6.3), decompositions $p = p'+p''$, $q = q'+q''$ in C, with $p'+q' = c_j'$, $p''+q'' = c_j''$. In both cases, we do not change decompositions of unaffected letters c_i, and we obtain the desired decomposition of the word directly similar to $(c_1, \ldots, c_n)$.

(6.5) If $\mathfrak{a}+\mathfrak{b} \doteq \mathfrak{a}+\mathfrak{c}$ in $E'C$, then $\mathfrak{b} = \mathfrak{c}$.

PROOF: It is obviously sufficient to consider the case $\mathfrak{a} = \langle a\rangle$ only. Let $\mathfrak{c} = \langle c_1, \ldots, c_n\rangle$. By (6.4), we have decompositions $a = a'+a''$, $c_i = c_i'+c_i''$ in C such that $\langle a\rangle = \langle a', c_i', \ldots, c_n'\rangle$ and $\mathfrak{b} = \langle a'', c_1'', \ldots, c_n''\rangle$. Then $a = a'+c_1'+ \ldots +c_n'$ in C by (6.1), and it follows that $c_1'+ \ldots +c_n' = a''$. Thus $\mathfrak{b} = \langle c_1', \ldots, c_n', c_1'', \ldots, c_n''\rangle = \langle c_1, \ldots, c_n\rangle = \mathfrak{c}$.

We have shown that $E'C$ is a commutative cancellation semigroup. Thus $E'C$ can be embedded, by the usual procedure, into an abelian group of "formal differences" of elements of $E'C$. We denote this group by EC and we consider $E'C$ as a subsemigroup of EC. The group EC is determined up to isomorphism.

We denote by EC^+ the set of all elements $\mathfrak{u}$ of EC of the form $\mathfrak{u} = \langle u_1, \ldots, u_r\rangle$, with $u_1, \ldots, u_r$ in C^+. This obviously is a subsemigroup of EC. We put $\mathfrak{a} \leqq \mathfrak{b}$ for elements $\mathfrak{a}$, $\mathfrak{b}$ of EC if $\mathfrak{b}-\mathfrak{a}$ is in EC^+.

(6.6) With the order relation $\mathfrak{a} \leqq \mathfrak{b}$ just defined, EC is an ordered abelian group.

Proof: Since $E'C$ is a subsemigroup of EC, the relation $\mathfrak{a} \leqq \mathfrak{b}$ is reflexive, transitive and compatible with addition. If $\mathfrak{a} \leqq \mathfrak{b}$ and $\mathfrak{b} \leqq \mathfrak{a}$, then $\mathfrak{a} = \mathfrak{a}+\mathfrak{u}+\mathfrak{v}$ for elements $\mathfrak{u}$, $\mathfrak{v}$ of EC^+, and it follows that $\langle u_1, \ldots, u_r, v_1, \ldots, v_s\rangle = 0$, and hence $u_1+ \ldots +u_r +v_1+ \ldots +v_s = 0$ by (6.1), for elements u_i, v_j of C^+. But then $u_i = v_j = 0$ for all i and j, and hence $\mathfrak{u} = \mathfrak{v} = 0$ and $\mathfrak{a} = \mathfrak{b}$. This completes the proof.

Let now $f : C \to C_1$ be a homomorphism of commutative clans. Then

$$(Wf)(a_1, \ldots, a_r) = (f(a_1), \ldots, f(a_r))$$

defines an additive mapping $Wf : WC \to WC_1$. Similar words are mapped into similar words by Wf (see [1] for details), and thus

$$(E'f)\langle a_1, \ldots, a_r\rangle = \langle f(a_1), \cdots, f(a_r)\rangle$$

defines an additive quotient mapping $E'f : E'C \to E'C_1$. This mapping has a unique extension to a group homomorphism $Ef : EC \to EC_1$. It is easily verified that this homomorphism is order preserving.

The induced maps Wf, $E'f$ and Ef have the properties one expects, and thus W, E' and E can be considered as functors. We shall discuss the functor E further in § 8.

The *canonical embedding* $\alpha_C : C \to EC$ is defined by $\alpha_C(a) = \langle a\rangle$, for $a \in C$. By (6.1) and the preceding discussion, α_C is additive, one-to-one and order preserving, and the group EC is generated by the image $\alpha_C(C)$ of C. Moreover, α_C is a universal mapping in the following sense.

(6.7) If $h : C \to A$ is an additive and order preserving mapping of the commutative clan C into an ordered abelian group A, then $h = h^*\alpha_C$ for a uniquely determined homomorphism $h^* : EC \to A$ of ordered abelian groups. If $f : C \to C_1$ is a clan homomorphism, then $\alpha_{C_1} f = (Ef)\alpha_C$.

Proof: We must have $h^*(\langle a\rangle) = h(a)$ for $a \in C$, and by [1, Thm. 1], this defines a unique additive mapping from $E'C$ to A. This mapping has a unique extension to a group homomorphism $h^* : EC \to A$, and it is easily verified that h^* is order preserving. The second part of (6.7) follows immediately from the definition of Ef.

7. *EC* is a lattice-ordered group

(7.1) $k(a-b) = \alpha_{C^+}(a)-\alpha_{C^+}(b)$, for $a \in C^+$ and $b \in C^-$, defines an additive and order preserving mapping $k : C \to E(C^+)$.

PROOF: Let a, c be in C^+ and b, d in C^-. Then $\langle a\rangle-\langle b\rangle = \langle c\rangle-\langle d\rangle$ in $E(C^+)$ iff $\langle a\rangle+\langle d\rangle = \langle a+d\rangle = \langle b\rangle+\langle c\rangle = \langle b+c\rangle$, hence iff $a+d = b+c$ in C^+, and also iff $a-b = c-d$ in C. With (5.8), this shows that $k : C \to E(C^+)$ is well defined.

The mapping k obviously is order presering. By (4.4) and (5.7), $(a-b)+(c-d) = (a+c)-(b+d)$ is defined in C iff $a+c$ is defined in C^+, and it follows that k is additive.

(7.2) If $j : C^+ \to C$ is the inclusion mapping, then $Ej : E(C^+) \to EC$ is an isomorphism of ordered abelian groups.

In view of this result, we shall identify EC and $E(C^+)$, and the set EC^+ of elements $\mathfrak{u} \geqq 0$ of EC with $(E(C^+))^+ = E'(C^+)$.

$$\begin{tikzcd}
C^+ \arrow[r, "\alpha_{C^+}"] \arrow[d, "j"'] & E(C^+) \arrow[d, shift right, "Ej"'] \\
C \arrow[ur, "k"] \arrow[r, "\alpha_C"'] & EC \arrow[u, shift right, "k^*"']
\end{tikzcd}$$

PROOF: For the mapping k of (7.1), obviously $kj = \alpha_{C^+}$, and hence $(Ej)k^*\alpha_C j = (Ej)kj = (Ej)\alpha_{C^+} = \alpha_C j$, and $k^*(Ej)\alpha_{C^+} = k^*\alpha_C j = kj = \alpha_{C^+}$, with (6.7). Since EC and $E(C^+)$ are generated by $\alpha_C(j(C^+))$ and $\alpha_{C^+}(C^+)$ respectively, we conclude that $(Ej)k^* = 1_{EC}$ and $k^*(Ej) = 1_{E(C^+)}$. This proves (7.2).

(7.3) $(\mathfrak{a}+\mathfrak{c}) \cap (\mathfrak{b}+\mathfrak{c})$ is defined in EC if and only if $\mathfrak{a} \cap \mathfrak{b}$ is defined, and then $(\mathfrak{a}+\mathfrak{c}) \cap (\mathfrak{b}+\mathfrak{c}) = (\mathfrak{a} \cap \mathfrak{b})+\mathfrak{c}$.

(7.4) $\mathfrak{a} \cup \mathfrak{b}$ is defined in EC if and only if $\mathfrak{a} \cap \mathfrak{b}$ is defined, and then $(\mathfrak{a} \cup \mathfrak{b})+(\mathfrak{a} \cap \mathfrak{b}) = \mathfrak{a}+\mathfrak{b}$.

These results are valid in any ordered abelian group. We omit the straightforward proofs.

(7.5) $\langle a\rangle \cap \langle b\rangle$ and $\langle a\rangle \cup \langle b\rangle$ are defined in EC for any a, b in C, and $\langle a\rangle \cap \langle b\rangle = \langle a \cap b\rangle$, $\langle a\rangle \cup \langle b\rangle = \langle a \cup b\rangle$.

PROOF: Let $a-(a \cap b) = u$, $b-(a \cap b) = v$. If $\langle a\rangle \leqq \mathfrak{x}$, $\langle b\rangle \leqq \mathfrak{x}$, then $\mathfrak{x}-\langle a \cap b\rangle = \langle u\rangle+\mathfrak{v} = \langle v\rangle+\mathfrak{u}$, with $\mathfrak{u} \geqq 0$, $\mathfrak{v} \geqq 0$. We apply (6.4) to $EC^+ = E'(C^+)$ to obtain decompositions

$v = v' + v''$ in C^+, $\mathfrak{u} = \mathfrak{u}' + \mathfrak{u}''$ in EC^+, such that $\langle v' \rangle + \mathfrak{u}' = \langle u \rangle$, $\langle v'' \rangle + \mathfrak{u}'' = \mathfrak{v}$. By (6.1) and (4.4), $\mathfrak{u}' = \langle u' \rangle$, with $u' \in C^+$ and $u' + v' = u$. Now $v' \leqq u \cap v = 0$, and hence $v' = 0$. Thus $\mathfrak{x} = \langle a \cap b \rangle + \langle u \rangle + \langle v \rangle + \mathfrak{u}'' = \langle a \cup b \rangle + \mathfrak{u}''$, and $\mathfrak{x} \geqq \langle a \cup b \rangle$. On the other hand, $\langle a \cup b \rangle \geqq \langle a \rangle$ and $\langle a \cup b \rangle \geqq \langle b \rangle$. Thus $\langle a \cup b \rangle = \langle a \rangle \cup \langle b \rangle$ in EC. By (5.5) and (7.4), $\langle a \rangle \cap \langle b \rangle = \langle a \cap b \rangle$ follows.

We prove now that EC is lattice-ordered, by proving that $\mathfrak{a} \cap \mathfrak{b}$ is defined in EC for all $\mathfrak{a}$, $\mathfrak{b}$ in EC. This requires several steps.

(7.6) Let $\mathfrak{a} \cap \mathfrak{b}$ be defined and $\mathfrak{u} \geqq 0$. If $\mathfrak{u} \cap (\mathfrak{b} - (\mathfrak{a} \cap \mathfrak{b}))$ is defined, then $(\mathfrak{a} + \mathfrak{u}) \cap \mathfrak{b}$ is defined.

Proof: $(\mathfrak{a} \cap \mathfrak{b}) + [\mathfrak{u} \cap (\mathfrak{b} - (\mathfrak{a} \cap \mathfrak{b}))] = ((\mathfrak{a} \cap \mathfrak{b}) + \mathfrak{u}) \cap \mathfrak{b} = (\mathfrak{a} + \mathfrak{u}) \cap (\mathfrak{b} + \mathfrak{u}) \cap \mathfrak{b} = (\mathfrak{a} + \mathfrak{u}) \cap \mathfrak{b}$ by (7.3).

(7.7) $\mathfrak{a} \cap \mathfrak{b}$ is defined for $\mathfrak{a}$ and $\mathfrak{b}$ in EC^+.

Proof: Since $\mathfrak{a}$ in EC^+ is of the form $\mathfrak{a} = \sum \langle u_i \rangle$, with all u_i in C^+, it is enough to show that $(\mathfrak{a} + \langle u \rangle) \cap \mathfrak{b}$ is defined if $\mathfrak{a} \cap \mathfrak{b}$ is defined.

Let first $\mathfrak{b} = \langle b \rangle$ with $b \in C^+$. If $\langle b \rangle = \mathfrak{v} + (\mathfrak{a} \cap \langle b \rangle)$, then $\mathfrak{v} = \langle v \rangle$, with $v \in C^+$, by (6.1). Thus $\langle u \rangle \cap \mathfrak{v}$ is defined by (7.5). But then $(\mathfrak{a} + \langle u \rangle) \cap \mathfrak{b}$ is defined if $\mathfrak{a} \cap \mathfrak{b}$ is defined, by (7.6).

For arbitrary $\mathfrak{b}$ in EC^+, $\langle u \rangle \cap (\mathfrak{b} - (\mathfrak{a} \cap \mathfrak{b}))$ is defined by the preceding paragraph, and thus $(\mathfrak{a} + \langle u \rangle) \cap \mathfrak{b}$ is defined by (7.6) if $\mathfrak{a} \cap \mathfrak{b}$ is defined. This proves (7.7).

(7.8) $\mathfrak{a} \cap \mathfrak{b}$ is defined for any $\mathfrak{a}$, $\mathfrak{b}$ in EC.

Proof: Since $EC = E(C^+)$, we have $\mathfrak{a} = \mathfrak{a}' - \mathfrak{a}''$ and $\mathfrak{b} = \mathfrak{b}' - \mathfrak{b}''$ for elements of $E'(C^+) = EC^+$. Now $(\mathfrak{a}' + \mathfrak{b}'') \cap (\mathfrak{b}' + \mathfrak{a}'')$ is defined by (7.7), and it follows from (7.3) that $\mathfrak{a} \cap \mathfrak{b}$ is defined.

Consider now a homomorphism $f : C \to C_1$ of commutative clans. It follows easily from the construction of meets in EC, and from (7.5), that Ef preserves meets, and hence joins. This completes the proof of the following theorem.

(7.9) For any commutative clan C, EC is a lattice-ordered abelian group, and the embedding mapping $\alpha_C : C \to EC$ is a clan homomorphism. If $f : C \to C_1$ is a homomorphism of commutative clans, then $Ef : EC \to EC_1$ is a homomorphism of lattice-ordered abelian groups.

8. The functor E

We denote the category of commutative clans by $\mathscr{C}$ and the category of lattice-ordered abelian groups by $\mathscr{L}$. By (7.9), the embeddings $C \to EC$ define a functor E from $\mathscr{C}$ to $\mathscr{L}$. Since EC, for a commutative clan C, is only determined up to isomorphism, the functor E is determined up to a natural equivalence.

The clan underlying a lattice-ordered abelian group L is commutative. We denote this clan by FL. A homomorphism $g : L \to L_1$ in $\mathscr{L}$ determines a homomorphism $Fg : FL \to FL_1$ of the underlying clans. Thus we have a "forgetful" functor F from $\mathscr{L}$ to $\mathscr{C}$.

For a lattice-ordered abelian group L, the group $E(FL)$ obviously is isomorphic to L. Since $E(FL)$ is only determined up to an isomorphism, we identify $E(FL)$ with L. It follows that $EF = 1_{\mathscr{L}}$.

In the present setting, a canonical embedding α_C must be regarded as a homomorphism $\alpha_C : C \to FEC$ in $\mathscr{C}$. If $f : C \to C_1$ is a homomorphism in $\mathscr{C}$, it follows from (6.7) that $\alpha_{C_1} f = (FEf)\alpha_C$. Thus the canonical embeddings α_C define a natural transformation $\alpha : 1_{\mathscr{C}} \to FE$. Moreover, we have:

(8.1) If L is a lattice-ordered abelian group, then $\alpha_{FL} = 1_{FL}$. If C is a commutative clan, then $E(\alpha_C) = 1_{EC}$.

We omit the straightforward proof.

With these notations, we may strengthen (7.9) as follows.

(8.2) The embedding functor E from $\mathscr{C}$ to $\mathscr{L}$ is left adjoint to the "forgetful" functor F from $\mathscr{L}$ to $\mathscr{C}$.

We refer to [5] for the definition of a left adjoint functor. (8.2) follows already from the remarks preceding (8.1), see [5; sec. 6]. We give a direct proof of (8.2) which uses only the definition of a left adjoint functor.

Proof: Let C be a commutative clan and L a lattice-ordered abelian group. Inverse natural equivalences $\alpha_{C,L} : \hom_{\mathscr{C}}(C, FL) \to \hom_{\mathscr{L}}(EC, L)$ and $\beta_{C,L} : \hom_{\mathscr{L}}(EC, L) \to \hom_{\mathscr{C}}(C, FL)$ are obtained as follows. For a map $f : C \to FL$, we put $\alpha_{C,L}(f) = Ef$, and for a map $g : EC \to L$, we put $\beta_{C,L}(g) = (Fg)\alpha_C$. With (8.1), it follows immediately that $\alpha_{C,L}$ and $\beta_{C,L}$ are inverse mappings. The proof that these mappings define natural transformations is straightforward; we omit it.

9. Miscellaneous results

As an interesting application of (7.9), we have the following theorem.

(9.1) Any Boolean ring R can be embedded into a lattice-ordered abelian group in such a way that meets $a \cap b$, joins $a \cup b$, and relative complements $b-a$, $a \leqq b$, are preserved.

In this connection, we note the following:

(9.2) The following properties of an abstract clan C are logically equivalent.

(i) If $a \leqq b$, then $a \cap (b-a) = 0$.

(ii) If $a+b$ is defined in C, then $a \cap b = 0$.

(iii) C is isomorphic to a clan underlying a Boolean ring R.

We omit the straightforward proof. The clan underlying a Boolean ring has been defined in (1.2). By (9.2), (ii), this clan is commutative.

A commutative clan C is called *Archimedean* if C satisfies the following condition. If a in C is such that $n\langle a\rangle \leqq \langle b\rangle$ in EC, for all natural numbers n and a fixed element b of C, then $a \leqq 0$.

It follows easily from (6.1) that na is defined in C, and $na \leqq b$ in C, if $n\langle a\rangle \leqq \langle b\rangle$ in EC. Examples of Archimedean clans are clans of functions (see Introduction) and clans underlying Boolean rings.

The Archimedean property of a clan C is closely connected with the possibility of constructing a completion of C, see [2; ch. XIV, § 9]. This is a topic for further research. We prove only one theorem.

(9.3) A clan C is Archimedean if and only if EC is Archimedean.

PROOF: The "if" part is trivial. Let now C be Archimedean, and let $\mathfrak{a}$, $\mathfrak{b}$ in EC be such that $n\mathfrak{a} \leqq \mathfrak{b}$ for all natural numbers n. It follows that $n\mathfrak{a}^+ \leqq \mathfrak{b}^+$ for all natural numbers n, see [2; p. 225, proof of Lemma 1]. We put $\mathfrak{b}^+ = \langle u_1\rangle + \dots + \langle u_r\rangle$, with $u_1, \dots, u_r$ in C^+, and we proceed by induction with respect to r. If $n\mathfrak{a}^+ \leqq \langle u\rangle$ for all n, then $\mathfrak{a}^+ = \langle v\rangle$, with $v \in C^+$, by (6.1), and $nv \leqq u$ for all n. It follows that $v = 0$ and hence $\mathfrak{a} \leqq 0$. Suppose now that $\mathfrak{c} \leqq 0$ if $n\mathfrak{c} \leqq \mathfrak{u}$ for all natural numbers n, where $\mathfrak{u} \geqq 0$, and let $n\mathfrak{a}^+ \leqq \mathfrak{u} + \langle u\rangle$ for all natural numbers n, with $u \in C^+$. Then $m(n\mathfrak{a}^+ - \langle u\rangle) \leqq mn\mathfrak{a}^+ - \langle u\rangle \leqq \mathfrak{u}$ for all $m \geqq 0$ and $n \geqq 0$, so that $n\mathfrak{a}^+ - \langle u\rangle \leqq 0$ for all n. But then $\mathfrak{a} \leqq 0$. This shows that EC is Archimedean.

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