COMPOSITIO MATHEMATICA

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Compositio Mathematica, tome 17 (1965-1966), p. 152-155

<http://www.numdam.org/item?id=CM_1965-1966__17__152_0>

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by

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We first prove an improved form of Steinitz' Exchange Theorem for a Dependence Space, (Hughes, 1), using the same notation as in the previous paper.

Later, we define a generalized dependence space, by allowing the directly dependent sets to become infinite, and see that the Exchange Theorem becomes invalid, though, if they are of cardinal aleph 0 at most, an invariant rank still exists.

1. Steinitz' exchange theorem

THEOREM 1. If A is a basis and B an independent subset, both well ordered, of the dependence space S, there exists an explicitly defined, one-one mapping φ of B onto A', a subset of A, such that φ is the identity map on $B \cap A$ and $B+(A-A')$ is a basis of S.

Let $C = A \cap B$, $X = A-C = (x_i)_{i \in I}$, $Y = B-C = (y_j)_{j \in J}$. It is sufficient that I and J are well ordered (by $<$).

For any $y_j \in Y$, there exists at least one relation of the form

$$(1) \qquad y_j \sim (x_i)+\sum_{r<j} y_r+\sum_{s<i} x_s+\sum C.$$

We define

$$\varphi(y_j) = x_i,$$

where $i \in I$ is the least in the well ordering such that (1) is satisfied, and put $\varphi(Y) = X'$.

LEMMA 1. φ is a one-one mapping of Y onto X'.

Suppose that $j, k \in J$, $j < k$ and $\varphi(y_k) = \varphi(y_j) = x_i$. Then

$$(2) \qquad y_k \sim (x_i)+\sum_{r<k} y_r+\sum_{s<i} x_s+\sum C.$$

From (1) and (2) we have

$$y_k \sim \sum_{r<k} y_r+\sum_{s<i} x_s+\sum C,$$

which is impossible if $\varphi(y_k) = x_i$.

LEMMA 2. $C+Y+(X-X')$ is independent.

If either $Y \cap (X-X')$ is not empty or the set above is dependent, for some $x_i \in X-X'$, the relation (1) is satisfied and we may suppose that $j \in J$ is the least possible in the well ordering. By the definition of φ, $\varphi(y_j) = x_k$, where $k < i$, so that

$$(3) \qquad y_j \sim (x_k) + \sum_{r<j} y_r + \sum_{s<k} x_s + \sum C.$$

By (1) and (3),

$$x_i \sim \sum_{r<j} y_r + \sum_{s<i} x_s + \sum C,$$

which contradicts the minimality of j in J.

If, for all $x \in C$, $x \sim \sum A$, we say that A generates or is a set of generators of S.

LEMMA 3. $C+Y+(X-X')$ generates S.

From (1), for all $x_i \in X'$,

$$x_i \sim \sum Y + \sum C + \sum_{s<i} x_s,$$

and hence, by transfinite induction, for all $x_i \in X$,

$$x_i \sim \sum Y + \sum C + \sum (X-X').$$

Since $C+X$ generates S, the Lemma follows.

Theorem 1 now follows from the lemmas, defining φ on C to be the identity mapping.

2. Generalized dependence space

We shall call the set S a generalized dependence space (with respect to Δ), if it satisfies the conditions for a dependence space (Hughes, 1) except that the members of Δ, the directly dependent sets, may be infinite subsets of S.

We carry over the notation for a dependence space. Thus, for $x \in S$, $A \subset S$, x is dependent on A, $(x \sim \sum A)$, if and only if, either $x \in A$ or there exists D, such that

$$D \in \Delta, \quad x \in D, \quad D \subset A + (x).$$

We see, by induction on the cardinal of Y, that, provided $Y = B - A$ is finite, Theorem 1 remains valid.

Hence, if S has a finite basis (or, equivalently, a finite set of generators), then any subset of S of greater cardinal is dependent and, if $x \sim \sum A$, then $x \sim \sum A'$, where A' is a finite subset of A. In fact, A' may be any maximal independent subset of A. Thus if

Δ' denotes the set of those elements of Δ, which are finite subsets of S, then S is a dependence space with respect to Δ', having exactly the same dependence relations $x \sim \sum A$.

THEOREM 2. If S is a generalized dependence space with respect to Δ and every member of Δ has cardinal aleph 0 at most, then every set of generators of S contains a set of generators of minimum cardinal (the rank of S). Any two bases of S have the same cardinal.

Let A be a set of generators of S, of minimum cardinal, which we may assume to be infinite, and B be any set of generators of S.

For every $a \in A$, there exists a set B_a, such that,

$$B_a \subset B, \quad a \sim \sum B_a, \quad \text{card}\,(B_a) \leqq \text{aleph } 0.$$

If $B' = \bigcup_{a \in A} B_a$, then $B' \subset B$ and, for any $a \in A$, $a \sim \sum B'$, so that B' generates S. Also

$$\text{card}\,(B') \leqq \text{card}\,(A) \times \text{aleph } 0 = \text{card}\,(A),$$

so that $\text{card}\,(B') = \text{card}\,(A)$.

Since a basis is a minimal set of generators, the last part follows.

3. Examples of generalized dependence space

I. If S is an infinite set and Δ consists of all subsets of S of cardinal aleph 0, then S is a generalized dependence space with no basis, for a subset of S generates S if and only if it is infinite and is independent if and only if it is finite.

II. We call the subsets A and B of the infinite set S almost equal, ($A \equiv B$), if there exists a one-one mapping φ of A onto B, such that the set of those $a \in A$, such that $\varphi(a) \neq a$, is finite.

Let S contain a system of disjoint, infinite sets P_i, ($i \in I$), and Γ be the set of all sets C, such that

$$C \equiv P_i, \text{ for some } i \in I.$$

Now let Δ denote the set of all subsets D of S, such that

$$D = C + (x), \quad C \in \Gamma, \quad x \notin C. \tag{1}$$

We see that, if $D \in \Delta$, D has the form (1) for every $x \in D$.

S is a generalized dependence space and X is a basis of S if and only if $X \in \Gamma$.

Let $I = (1, 2)$, then P_1 and P_2 are bases but may have different cardinals.

Now let I be infinite and, for all $i \in I$,

$$\text{card}\,(P_i) = \text{aleph } 0.$$

Then every $D \in \Delta$ has cardinal aleph 0. However, if, for every $i \in I$, $a_i \in P_i$, then $A = (a_i)_{i \in I}$ is independent, but is not contained in any basis. We may also have

$$\text{card}\,(A) = \text{card}\,(I) > \text{aleph } 0.$$

Thus Theorem 1 is invalid in a generalized dependence space as is Theorem 2 without the condition that the members of Δ have cardinal at most aleph 0.

REFERENCE

HUGHES

[1] Steinitz' Exchange Theorem for Infinite Bases. Compositio Mathematica. Vol. 15, Fasc. 2, pp. 113—118, (1963).

(Oblatum 19-12-63)

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