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by

J. Boersma

§ 1. Introduction

The present paper deals with a number of boundary value problems in diffraction theory and wing lifting surface theory with special reference to circular disks and apertures and to slits.

The problems of the diffraction of an incident wave (scalar or electromagnetic) by a circular disk, circular aperture, strip or slit have received considerable attention during the last decades. For a survey of the various methods of solution the reader is referred to the review articles of Bouwkamp [8] and of Hönl, Maue and Westpfahl [19]. In this paper we are especially interested in low-frequency expansions for the field quantities, valid when ka or kb is small, where k is the wave number, a is the radius of the disk or aperture, $2b$ is the width of the strip or slit. These expansions will be obtained from Fredholm integral equations of the second kind, the kernels of these integral equations being small when ka or kb is small.

The reduction of the scalar diffraction problem for a circular disk or circular aperture to Fredholm integral equations has been performed by several authors. We mention Magnus [34], Jones [23], Bazer and Brown [2], Bazer and Hochstadt [3], Heins and MacCamy [17], Noble [41], Collins [9e, f] and Williams [51].

Bazer and Brown [2], Bazer and Hochstadt [3], Collins [9e, f] start from certain integral representations for the transmitted or scattered wave. These integral representations which contain an unknown function, are designed to satisfy all conditions of the problem but one. The latter condition leads to a Fredholm integral equation of the second kind for the unknown function. A related integral equation with the same kernel was derived before by Jones [23], using a very complicated method. The above-mentioned integral representations may be considered as generalizations of similar representations exploited to solve some problems in potential theory, compare Love [31], Green and Zerna [14], Collins [9]. In an appendix to [3b] Bazer and Hochstadt show their integral representations to be related to a representation for axially symmetric solutions of the wave equation due to Bateman [1].

The central feature in Heins and MacCamy's [17] method of solution is the fact that an axially symmetric solution of Helmholtz' equation is determined uniquely by its values on the axis of symmetry according to a representation due to Henrici [18]. Combining this representation with the Helmholtz representation for the total wave, Fredholm integral equations are derived with a Jones-type kernel. These integral equations are, however, different from those stated by Bazer and Hochstadt [3]. Heins [16] applied the same method of solution to the diffraction of a normally incident plane wave by a soft circular disk.

The Fredholm integral equation derived by Magnus [34] contains a kernel completely different from the kernel obtained by Jones [23]. The back-ground of the difference between the integral equations with kernels of Magnus-type and of Jones-type has been discussed by Noble [41].

Williams [51] formulates the diffraction problems for a soft or rigid circular disk in terms of Fredholm integral equations of the first kind. By means of a general method, presented in [50], these equations are reduced to Fredholm integral equations of the second kind, the kernels of these integral equations being small when ka is small. In the case of an axially symmetric incident wave, these kernels will again be of Magnus-type or of Jones-type.

The Fredholm integral equations derived by Bazer and Hochstadt [3] and by Heins and MacCamy [17] suffer from the disadvantage of containing a number of arbitrary constants which must be solved afterwards from a system of linear equations. On the contrary, the integral equations stated by Williams [51] do not contain such arbitrary constants. In § 2 of this paper we present a modification of Bazer and Hochstadt's method leading also to Fredholm integral equations, in which no arbitrary constants occur. These integral equations are however completely different from those derived by Williams. The modified method has been elaborated for the case of plane-wave excitation.

Finally, we mention some other methods of solution, leading to low-frequency expansions and presented by Bouwkamp [5], Magnus [33], de Hoop [20]. Bouwkamp [5] treats the diffraction of a plane normally incident wave by a circular aperture in a soft screen. The problem is formulated in terms of an integro-differential equation for the aperture field. This equation is expanded in powers of k leading to a recursive system of integro-differential equations which may be solved. Magnus [33], de Hoop [20] reduce the same diffraction problem to infinite systems of linear equations by means of Levine and Schwinger's variational principle. These infinite systems are investigated especially for small values of ka.

The first rigorous solutions to the problem of the diffraction of a plane electromagnetic wave by a conducting circular disk were presented by Meixner [35], Meixner and Andrejewski [37], the latter solution being the simplest. Meixner and Andrejewski [37] derived the scattered wave from an electric Hertz vector. The components of this Hertz vector were represented by series containing products of spheroidal wave functions. Low-frequency expansions for the solution of the present diffraction problem were given by Bouwkamp [6] for the special case of normal incidence. Bouwkamp derived a pair of Fredholm integral equations of the first kind for the currents induced in the disk. These equations were expanded in powers of k leading to a recursive system of integral equations, which could be solved. Similar methods were presented by Grinberg and Pimenov [15], Eggimann [11] for the case of an arbitrary incident wave. Kuritsyn [27] applied Grinberg and Pimenov's method to the special case of plane-wave excitation.

A reduction of the electromagnetic diffraction problem to Fredholm integral equations of the second kind was performed by Lebedev and Skal'skaya [28], Lur'e [32], Benkard [4], Williams [52].

Lebedev and Skal'skaya [28] consider the diffraction of a plane normally incident electromagnetic wave by a circular disk. The scattered wave is derived from a magnetic Hertz vector. The two non-zero components of this Hertz vector are expressed as Hankel transforms containing certain unknown functions. The boundary conditions in the plane of the disk lead to dual integral equations for these unknown functions. These equations are solved by expressing the unknown functions in terms of new unknown functions satisfying Fredholm integral equations of the second kind. The kernels of these integral equations are of Magnus-type and will be small when ka is small. Lur'e [32] extended Lebedev and Skal'skaya's method to the case of arbitrary plane-wave excitation.

Benkard [4] also treats the diffraction of a plane electromagnetic wave by a conducting disk. His method consists in decomposing the vector wave equation in cylindrical coordinates into a sequence of independent pairs of equations which are solved by means of the method presented by Heins and MacCamy [17]. Williams [52] reduces the electromagnetic diffraction problem to well-known scalar diffraction problems treated in [51].

In § 3 of this paper we present another method of reducing electromagnetic diffraction problems for the circular disk to Fredholm integral equations of the second kind. We only treat the case of plane-wave excitation. Starting from the formulation of the diffraction problem as presented by Meixner and Andrejewski [37], we substitute Bazer and Hochstadt [3]-type integral representations for the components of the electric Hertz vector. The boundary condition on the disk will lead to Fredholm integral equations of the second kind for the unknown functions occurring in the integral representations. When ka is small, these integral equations may be solved by iteration yielding low-frequency expansions in powers of ka for the solution. According to Meixner and Andrejewski [37] the boundary conditions on the disk will contain certain unknown coefficients which follow from the edge condition. In our method of solution a direct application of the edge condition to the Hertz vector leads to simple formulae for these unknown coefficients. The present method has been worked out in detail for the case of normal incidence. Expansions in powers of ka were derived for the scattered field on the disk and in the wave zone and for the scattering coefficient. The presented results have been checked in the following independent manner. Starting from the formulation of the problem as presented by Lebedev and Skal'skaya [28], the components of the magnetic Hertz vector are represented by suitable integrals of the Bazer and Hochstadt [3]-type. The unknown functions involved in these integral representations are derived to satisfy Fredholm integral equations of the second kind. For small values of ka these integral equations have been solved by iteration, leading ultimately to low-frequency expansions for the scattered field on the disk and in the wave zone and for the scattering coefficient.

Finally, we remark that a similar method may be presented for the case of an arbitrary incident wave. The materials for this method are the formulation of the diffraction problem as presented by Meixner [36] and the integral representations due to Bazer and Hochstadt [3].

The problems of the diffraction of a two-dimensional scalar wave by a slit in a soft and in a rigid screen may be solved rigorously in terms of Mathieu functions, compare Morse and Rubenstein [39]. Low-frequency expansions to the solutions of these problems in the case of a plane normally incident wave were presented by Sommerfeld [43], Bouwkamp [7], Müller and Westpfahl [40]. These authors formulate the diffraction problems in terms of an integro-differential equation or of a Fredholm integral equation of the first kind for the field in the slit or its normal derivative. This equation is expanded in powers of kb, leading to a recursive system of integral equations which may be solved. An extension of this method to the case of an obliquely incident plane wave, is due to Millar [38].

Another approach to the above-mentioned problems was made by de Hoop [21]. Assuming plane-wave excitation, de Hoop formulated variational principles for both types of two-dimensional diffraction problems, similar to Levine and Schwinger's principles as corrected by Bouwkamp [7]. By means of these principles both diffraction problems were reduced to infinite systems of linear equations. Expansions in powers of kb and $\log kb$ were derived for the solutions of these systems.

Jones and Noble [24] consider the scattering of a plane wave by a soft strip. A rigorous result in terms of Mathieu functions is derived for the scattering coefficient. Expanding this result for small values of kb, the uncommon character of the expansion is made clear.

In § 4 of this paper a new method is presented, which reduces the diffraction problems for a slit in a soft and in a rigid screen to Fredholm integral equations. For that purpose we state integral representations for the transmitted wave similar to the integrals presented by Bazer and Brown [2], Bazer and Hochstadt [3]. These integral representations which contain an unknown function, are designed to satisfy all conditions of the problems but one. The latter condition leads to a Fredholm integral equation of the second kind for the unknown function, the kernel of this equation being small when kb is small. In the latter case the integral equation may be solved by iteration yielding an expansion in powers of kb and $\log kb$ for the unknown function. The present method has been elaborated for the case of plane-wave excitation. Expansions were derived for the far fields, the transmission coefficients and the fields in the slit.

It may be remarked that the problem of the diffraction of a two-dimensional electromagnetic wave through a slit in a conducting screen is equivalent to the scalar diffraction problems treated above. Two cases must be distinguished according to the electric vector of the incident wave being polarized parallel or perpendicular to the edge of the slit. In the case of parallel polarization the electric field has only one component which vanishes on the screen. In the case of perpendicular polarization the magnetic field has only one component, the normal derivative of this component being zero on the screen. Hence, these two cases correspond to the scalar diffraction problems for a slit in a soft and in a rigid screen respectively.

The boundary value problems in lifting surface theory, treated in this paper, deal with aerofoils of circular and elliptic planform in incompressible and compressible flow. The problem of the determination of the pressure distribution, forces and moments on an aerofoil of circular planform in steady incompressible flow has been solved by Kinner [25], van Spiegel [44]. Both authors formulate the problem in terms of a velocity potential and an acceleration potential, these potentials being solutions of Laplace's equation. According to linearized aerofoil theory the normal velocity and the normal acceleration are prescribed on a circular disk, which replaces the aerofoil. Introducing oblate spheroidal coordinates, Laplace's equation may be solved by separation of variables.

Kinner [25] defines two kinds of solutions of Laplace's equation viz. potential functions of the first kind and of the second kind. The potential functions of the first kind are everywhere finite and will vanish at the edge of the disk. The potential functions of the second kind become infinite at this edge, while their normal derivative is zero on the disk. The acceleration potential is now represented by a combination of these potential functions in such a way, that the following conditions are satisfied. First, the normal velocity on the disk corresponding to the combination should have the prescribed value. Secondly, the acceleration potential should remain finite along the trailing edge of the disk according to the Kutta condition. The equations following from these conditions, are reduced to an infinite system of linear equations, which has been solved by truncation to a finite system. However, Kinner performed the reduction in an unsuitable way, this fact being the real cause of the discrepancies between Kinner's [25] and van Spiegel's [44] results. When the equations are reduced in the same way as a similar pair of equations occurring in § 5 of this paper, Kinner's results for lift and moment are in complete agreement with van Spiegel's values.

Van Spiegel [44] states a representation for the acceleration potential consisting of a regular term and a singular term. The regular term vanishes at the edge of the disk and yields the prescribed normal acceleration on the disk. The singular acceleration potential is expressed by an integral containing Green's function of the second kind for the boundary value problem and an unknown weight-function. This integral is designed to be zero along the trailing edge and to be infinite along the leading edge of the disk, while its normal derivative vanishes on the disk. The stated representation will satisfy all conditions of the problem with the exception of the requirement of the prescribed normal velocity on the disk. Expanding the weight-function in a Fourier series, the latter condition leads to an infinite system of linear equations for the Fourier coefficients. The infinite system, which has a dominating diagonal term, has been solved by truncation to a finite system. No investigation has been made concerning the validity of this procedure and concerning the convergence of the Fourier series for the weight-function.

In § 5 of this paper we present another method of solution to the boundary value problem stated above. First, we determine the regular part of the velocity potential. This regular part is finite everywhere and its normal derivative on the disk agrees with the prescribed normal velocity. A differentiation of this regular velocity potential with respect to the coordinate in the flow direction yields the corresponding part of the acceleration potential, which will become infinite at the edge of the disk. Secondly, we determine the singular part of the acceleration potential. This singular part becomes infinite at the edge and its normal derivative on the disk will vanish. The actual determination of these potentials is performed using suitable integral representations of the Bazer and Hochstadt [3]-type. The correct acceleration potential is now given by the sum of the derivative of the regular velocity potential and the singular acceleration potential, provided that the following conditions are satisfied. First, the acceleration potential should remain finite along the trailing edge according to the Kutta condition. Secondly, the normal velocity on the disk corresponding to the singular acceleration potential should be zero. The equations following from these conditions, have been reduced to a pair of infinite systems of linear equations. It has been shown that these infinite systems may be solved by a truncation to finite systems of linear equations. A further investigation of these infinite systems has revealed various properties for their solutions. Using these properties, the hitherto formal reduction of the boundary value problem to infinite linear systems may be performed once more in a rigorous manner. The final results presented for lift, moment and induced drag are of a simpler form than the corresponding formulae stated by van Spiegel [44]. Numerical results have been derived for a number of prescribed downwash distributions on the wing.

Recently, Levey and Wynter [29] treated the special case of a plane circular disk at a finite angle of attack in steady incompressible flow. Their method, which also leads to an infinite system of linear equations, is related to Copson's [10] method for the problem of the electrified disk.

Schade [42], Krienes and Schade [26] extended Kinner's theory to the problem of the harmonically oscillating circular wing in incompressible flow. The same problem was treated by van Spiegel [44], but only the case of low frequency was elaborated. Following the same method as for the steady case, the Fourier coefficients of the weight-function were expanded in powers of the reduced frequency, taking into account only the terms of zeroth and of first order. Both terms were derived to be solutions of infinite systems of linear equations. Similar expansions were stated for lift and moment.

In § 6 of this paper we examine the more general problem of a harmonically oscillating elliptic wing in compressible flow. The shape of the elliptic planform is connected with the Mach number. In the case of a Mach number zero, the present problem will reduce to the problem of the oscillating circular wing in incompressible flow. After a suitable transformation of coordinates the transformed velocity potential and acceleration potential will satisfy Helmholtz' equation, while the normal derivatives of these potentials are prescribed on a circular disk. Hence, the transformed problem may be solved by a similar method as presented in § 5. The final equations, following from the Kutta condition and the requirement of the prescribed normal velocity on the disk, are expanded in powers of the reduced frequency, taking into account only the first two terms of these expansions. The resulting equations are reduced to infinite systems of linear equations. Similar expansions, containing two terms, are presented for lift, moment and induced drag. The general formula which is given for the induced drag acting on an aerofoil in unsteady compressible flow, is believed to be new. Numerical results have been derived for some simple modes of oscillation.

Finally, we remark that the solutions of the above-mentioned boundary value problems in lifting surface theory are especially of interest for checking the accuracy of approximation methods, compare van Spiegel [44], Zwaan [53].

§ 2. Diffraction of a scalar wave by a circular aperture

2.1. In their papers [2], [3] Bazer and Brown, Bazer and Hochstadt treat the problem of the diffraction of a harmonic scalar wave by a circular aperture in an infinite plane screen. The screen coincides with the plane $z = 0$, the aperture is defined by $0 \leqq \rho < a$, $0 \leqq \varphi < 2\pi$, $z = 0$, where ρ, φ, z denote cylindrical coordinates. The primary wave is given by $u(\rho, \varphi, z)$, incident from $z < 0$. Two different problems can be distinguished according to the screen being perfectly soft or perfectly rigid. The corresponding diffraction problems are referred to as the first and second boundary value problem respectively. A time dependence of the form $e^{-i\omega t}$ is assumed throughout.

According to Bouwkamp [8] the diffraction problems may be formulated in the following way. In the case of the first boundary value problem the total field is given by

$$(2.1)\quad u_1(\rho, \varphi, z) = \begin{cases} u(\rho, \varphi, z) - u(\rho, \varphi, -z) + \Phi_1(\rho, \varphi, -z), & (z \leqq 0) \\ \Phi_1(\rho, \varphi, z), & (z \geqq 0) \end{cases}$$

where Φ_1, to be defined for $z \geqq 0$ only, has the following properties:

(i) Φ_1 is a solution of Helmholtz' equation, $\Delta\Phi_1 + k^2\Phi_1 = 0$, when $z > 0$;
(ii) $\Phi_1 = 0$ on the screen i.e. when $z = 0$, $\rho > a$;
(iii) Φ_1 satisfies Sommerfeld's radiation condition at infinity;
(iv) $\partial\Phi_1/\partial z = \partial u/\partial z$ in the aperture i.e. when $z = 0$, $0 \leqq \rho < a$;
(v) Φ_1 is everywhere finite;
(vi) grad Φ_1 is quadratically integrable over any domain of three-dimensional space, including the edge of the aperture.

In the case of the second boundary value problem the total field is given by

$$(2.2)\quad u_2(\rho, \varphi, z) = \begin{cases} u(\rho, \varphi, z) + u(\rho, \varphi, -z) - \Phi_2(\rho, \varphi, -z), & (z \leqq 0) \\ \Phi_2(\rho, \varphi, z), & (z \geqq 0) \end{cases}$$

where Φ_2, also defined for $z \geqq 0$ only, has similar properties to Φ_1 except that (ii) and (iv) should be replaced by

(ii)′ $\partial\Phi_2/\partial z = 0$ on the screen i.e. when $z = 0$, $\rho > a$;
(iv)′ $\Phi_2 = u$ in the aperture i.e. when $z = 0$, $0 \leqq \rho < a$.

Bazer and Brown [2] assume the incident wave to be axially symmetric, whereas Bazer and Hochstadt [3] assume an excitation of the form,

$$(2.3)\quad u(\rho, \varphi, z) = u^{(m)}(\rho, z) \cos m(\varphi - \varphi^{(m)}), \quad m = 0, 1, 2, \ldots.$$

Hence, Bazer and Brown's problem corresponds to Bazer and Hochstadt's case $m = 0$. Bazer and Hochstadt state the following integral representations for the functions $\Phi_{1,2}(\rho, \varphi, z)$,

$$\Phi_{1,2}(\rho, \varphi, z) = \Phi_{1,2}^{(m)}(\rho, z) \cos m(\varphi - \varphi^{(m)})$$

$$= \rho^m \left(\frac{1}{\rho}\frac{\partial}{\partial \rho}\right)^m \int_{-1}^{1} \frac{\exp\left(ik\sqrt{\rho^2+(z+iat)^2}\right)}{\sqrt{\rho^2+(z+iat)^2}} f_{1,2}^{(m)}(t)dt \cos m(\varphi-\varphi^{(m)}), \tag{2.4}$$

valid for $z \geqq 0$. The unknown functions $f_1^{(m)}(t)$ and $f_2^{(m)}(t)$ are required to be odd and even functions of t respectively, to be regular functions of t for $|t| \leqq 1+\Delta$, Δ arbitrary small and positive, and to satisfy the conditions,

$$d^k f_{1,2}^{(m)}(1)/dt^k = 0, \quad \text{for} \quad k = 0, 1, \ldots, m-1. \tag{2.5}$$

The functions $\Phi_{1,2}^{(m)}(\rho, z)$, as given by (2.4) satisfy all the conditions of the problems with the exception of the aperture conditions (iv) and (iv)'. From these conditions Bazer and Hochstadt derive Fredholm integral equations of the second kind for the functions $f_{1,2}^{(m)}(t)$. The kernel of these integral equations is small when ka is small, in which case the equations may be solved by iteration yielding expansions in powers of ka for the functions $f_{1,2}^{(m)}(t)$. However, the integral equations and hence also the solutions contain m arbitrary constants. These constants have to be solved from a system of m linear algebraic equations following from the conditions (2.5). In practice the solution of the integral equations is therefore only possible for small values of m. We remark that in their later paper [3b] Bazer and Hochstadt avoid this difficulty and determine the functions $f_{1,2}^{(m)}(t)$ in another way without solving a system of linear equations.

In the following we present a modification of Bazer and Hochstadt's method leading to Fredholm integral equations of the second kind for the functions $f_{1,2}^{(m)}(t)/(1-t^2)^m$, in which no arbitrary constants occur. However, the kernels of these integral equations will be more complicated than the kernel of Bazer and Hochstadt's equations. In sections 2.2, 2.3 we treat the first and second boundary value problem respectively, while in section 2.4 the modified method is elaborated for the case of plane-wave excitation.

Finally, for reference we quote Bazer and Hochstadt's formulae describing the behaviour of $\Phi_{1,2}^{(m)}(\rho, z)$ and its derivatives near the edge of the aperture. In a point with coordinates $\rho = a+\delta\cos\gamma$, $z = \delta \sin\gamma$, $\delta > 0$, $0 \leqq \gamma \leqq \pi$, the following expressions hold for small values of δ,

$$
(2.6)\quad\begin{cases}
\Phi_1^{(m)} = \dfrac{2\sqrt{2}}{ia^{m+1}}\dfrac{d^m f_1^{(m)}(1)}{dt^m}\left(\dfrac{\delta}{a}\right)^{\frac{1}{2}}\sin\tfrac{1}{2}\gamma + O\left[\dfrac{\delta}{a}\right], \\
\left(\dfrac{\delta}{a}\right)^{\frac{1}{2}}\dfrac{\partial\Phi_1^{(m)}}{\partial\delta} = \dfrac{\sqrt{2}}{ia^{m+2}}\dfrac{d^m f_1^{(m)}(1)}{dt^m}\sin\tfrac{1}{2}\gamma + O\left[\left(\dfrac{\delta}{a}\right)^{\frac{1}{2}}\right], \\
\left(\dfrac{\delta}{a}\right)^{\frac{1}{2}}\dfrac{1}{\delta}\dfrac{\partial\Phi_1^{(m)}}{\partial\gamma} = \dfrac{\sqrt{2}}{ia^{m+2}}\dfrac{d^m f_1^{(m)}(1)}{dt^m}\cos\tfrac{1}{2}\gamma + O\left[\left(\dfrac{\delta}{a}\right)^{\frac{1}{2}}\right],
\end{cases}
$$

$$
(2.7)\quad\begin{cases}
\Phi_2^{(m)} = u^{(m)}(a, 0) - \dfrac{2\sqrt{2}}{a^{m+1}}\dfrac{d^m f_2^{(m)}(1)}{dt^m}\left(\dfrac{\delta}{a}\right)^{\frac{1}{2}}\cos\tfrac{1}{2}\gamma + O\left[\dfrac{\delta}{a}\right], \\
\left(\dfrac{\delta}{a}\right)^{\frac{1}{2}}\dfrac{\partial\Phi_2^{(m)}}{\partial\delta} = -\dfrac{\sqrt{2}}{a^{m+2}}\dfrac{d^m f_2^{(m)}(1)}{dt^m}\cos\tfrac{1}{2}\gamma + O\left[\left(\dfrac{\delta}{a}\right)^{\frac{1}{2}}\right], \\
\left(\dfrac{\delta}{a}\right)^{\frac{1}{2}}\dfrac{1}{\delta}\dfrac{\partial\Phi_2^{(m)}}{\partial\gamma} = \dfrac{\sqrt{2}}{a^{m+2}}\dfrac{d^m f_2^{(m)}(1)}{dt^m}\sin\tfrac{1}{2}\gamma + O\left[\left(\dfrac{\delta}{a}\right)^{\frac{1}{2}}\right].
\end{cases}
$$

2.2. According to Bazer and Hochstadt [3] the aperture condition (iv) for the first boundary value problem leads to the equation,

$$
(2.8)\quad
\begin{aligned}
&2i\left(\frac{1}{\rho}\frac{d}{d\rho}\right)^{m+1}\int_0^{\rho/a}\frac{\exp\left(i\alpha\sqrt{(\rho/a)^2-t^2}\right)}{\sqrt{(\rho/a)^2-t^2}}\,tf_1^{(m)}(t)dt \\
&-2\left(\frac{1}{\rho}\frac{d}{d\rho}\right)^{m+1}\int_{\rho/a}^{1}\frac{\sinh\left(\alpha\sqrt{t^2-(\rho/a)^2}\right)}{\sqrt{t^2-(\rho/a)^2}}\,tf_1^{(m)}(t)dt = u_1^{(m)}(\rho)
\end{aligned}
$$

where we introduced the following abbreviations,

$$
(2.9)\qquad \alpha = ka,\ u_1^{(m)}(\rho) = (1/\rho^m)\partial u^{(m)}(\rho, 0)/\partial z.
$$

Opposite to Bazer and Hochstadt we continue the first integral in the left-hand side of (2.8) over the complete interval [0, 1]. At the same time we make the substitutions,

$$
(2.10)\qquad (\rho/a)^2 = 1-\eta,\ t^2 = 1-\xi,
$$

then we obtain the relation,

$$
(2.11)\quad
\begin{aligned}
&\left(\frac{d}{d\eta}\right)^{m+1}\int_0^{\eta}\frac{\cosh\left(\alpha\sqrt{\eta-\xi}\right)}{\sqrt{\eta-\xi}}F(\xi)d\xi \\
&\qquad = i\left(\frac{d}{d\eta}\right)^{m+1}\int_0^{1}\frac{\exp\left(i\alpha\sqrt{\xi-\eta}\right)}{\sqrt{\xi-\eta}}F(\xi)d\xi \\
&\qquad\quad + \frac{(-1)^m a^{2m+2}}{2^{m+2}}u_1^{(m)}\left(a\sqrt{1-\eta}\right)
\end{aligned}
$$

where

$$F(\xi) = f_1^{(m)}(\sqrt{1-\xi})/2. \tag{2.12}$$

The square root $\sqrt{\xi-\eta}$ in the right-hand side of (2.11) is defined in the following way,

$$\sqrt{\xi-\eta} = \begin{cases} \sqrt{\xi-\eta}, & \text{when } \xi > \eta, \\ i\sqrt{\eta-\xi}, & \text{when } \xi < \eta. \end{cases} \tag{2.13}$$

In accordance with this definition we use the notation $⨍$ for the integral, denoting that the path of integration has an infinitesimal indentation passing above the point $\xi = \eta$.

Let the right-hand side of (2.11) be called $G(\eta)$, then the equation (2.11) can be solved by means of the convolution theorem for Laplace transforms. Owing to (2.5), (2.12) $F(\xi)$ will contain a factor ξ^m, hence the first m derivatives with respect to η of the integral in the left-hand side of (2.11) vanish for $\eta = 0$. A formal application of the Laplace transformation to (2.11) yields,

$$\sqrt{\pi}\, p^{m+\frac{1}{2}}\, e^{\alpha^2/(4p)}\mathfrak{L}\{F\} = \mathfrak{L}\{G\} \tag{2.14}$$

where

$$\mathfrak{L}\{F\} = \int_0^\infty e^{-pt}F(t)dt. \tag{2.14a}$$

Inversion of $\mathfrak{L}\{F\}$ from (2.14) leads to the solution,

$$F(\eta) = (1/\sqrt{\pi})(2/\alpha)^{m-\frac{1}{2}} \int_0^\eta (\eta-\mu)^{\frac{1}{2}m-\frac{1}{4}} J_{m-\frac{1}{2}}(\alpha\sqrt{\eta-\mu})G(\mu)d\mu. \tag{2.15}$$

([13], form. 4.9(39), 4.14(30) were used). The present reduction is only valid when the Laplace transforms $\mathfrak{L}\{F\}$ and $\mathfrak{L}\{G\}$ exist. However, it can easily be shown by substituting (2.15) into (2.11), that the solution (2.15) is certainly correct on the conditions stated for $f_1^{(m)}(t)$.

In (2.15) we substitute $G(\mu)$ as given by (2.11). The first term of the resulting expression may be reduced to

$$\begin{aligned} &\frac{1}{\sqrt{\pi}}\left(\frac{2}{\alpha}\right)^{m-\frac{1}{2}} \int_0^\eta (\eta-\mu)^{\frac{1}{2}m-\frac{1}{4}} J_{m-\frac{1}{2}}(\alpha\sqrt{\eta-\mu})d\mu \\ &\cdot \left[i\left(\frac{d}{d\mu}\right)^{m+1} ⨍_0^1 \frac{\exp(i\alpha\sqrt{\xi-\mu})}{\sqrt{\xi-\mu}} F(\xi)d\xi\right] \\ &= (-1)^{m+1}(i/\sqrt{\pi})(2/\alpha)^{m+\frac{1}{2}}(d/d\eta)\int_0^1 M(\alpha;\xi,\eta)F(\xi)d\xi, \end{aligned} \tag{2.16}$$

where

$$(2.17)\quad \begin{aligned} &M(\alpha;\xi,\eta)\\ &=\left(\frac{d}{d\xi}\right)^{m+1}\int_0^\eta(\eta-\mu)^{\frac{1}{2}m+\frac{1}{4}}J_{m+\frac{1}{2}}(\alpha\sqrt{\eta-\mu})\,\frac{\exp(i\alpha\sqrt{\xi-\mu})}{\sqrt{\xi-\mu}}\,d\mu.\end{aligned}$$

The function $M(\alpha;\,\xi,\,\eta)$ is determined in the following way. First, when $\xi>\eta$ we start from Lommel's expansion (cf. [47], § 5.22),

$$\begin{aligned}\frac{\exp(i\alpha\sqrt{\xi-\mu})}{\sqrt{\xi-\mu}}&=i\sqrt{\frac{\pi\alpha}{2}}\,\frac{H^{(1)}_{\frac{1}{2}}(\alpha\sqrt{\xi-\mu})}{(\xi-\mu)^{\frac{1}{4}}}\\ &=i\sqrt{\frac{\pi\alpha}{2}}\sum_{r=0}^{\infty}\frac{1}{r!}\left(\frac{\alpha\mu}{2}\right)^r\frac{H^{(1)}_{r+\frac{1}{2}}(\alpha\sqrt{\xi})}{\xi^{\frac{1}{2}r+\frac{1}{4}}},\end{aligned}$$

valid for $0\leqq\mu<\xi$. Substituting this expansion into (2.17), a term by term integration yields the result,

$$(2.18)\quad \begin{aligned} M(\alpha;\xi,\eta)&=(-1)^{m+1}\,i\sqrt{\pi}\left(\frac{\alpha}{2}\right)^{m+\frac{1}{2}}\\ &\cdot\sum_{r=0}^{\infty}\eta^{\frac{1}{2}m+\frac{1}{2}r+\frac{3}{4}}J_{m+r+\frac{3}{2}}(\alpha\sqrt{\eta})\,\frac{H^{(1)}_{m+r+\frac{3}{2}}(\alpha\sqrt{\xi})}{\xi^{\frac{1}{2}m+\frac{1}{2}r+\frac{3}{4}}},\end{aligned}$$

where we used [47], form. 12.11(1), 3.6(10). The result (2.18) may be simplified by a differentiation with respect to α. Using [47], form. 3.2(6), 3.6(9) we obtain the derivative,

$$(2.19)\quad \begin{aligned}&\frac{\partial}{\partial\alpha}\left\{\frac{M(\alpha;\,\xi,\,\eta)}{(\alpha/2)^{m+\frac{1}{2}}}\right\}\\ &=(-1)^{m+1}\,i\sqrt{\pi}\,\eta^{\frac{1}{2}m+\frac{3}{4}}J_{m+\frac{3}{2}}(\alpha\sqrt{\eta})\,\frac{H^{(1)}_{m+\frac{1}{2}}(\alpha\sqrt{\xi})}{\xi^{\frac{1}{2}m+\frac{1}{4}}}.\end{aligned}$$

From (2.18) one can easily derive,

$$(2.20)\quad \begin{aligned}\left.\frac{M(\alpha;\,\xi,\,\eta)}{(\alpha/2)^{m+\frac{1}{2}}}\right|_{\alpha=0}&=\frac{(-1)^{m+1}}{\sqrt{\pi}}\sum_{r=0}^{\infty}\frac{1}{m+r+\frac{3}{2}}\left(\frac{\eta}{\xi}\right)^{m+r+\frac{3}{2}}\\ &=\frac{(-1)^m}{\sqrt{\pi}}\left\{\log\left|\frac{\sqrt{\xi}-\sqrt{\eta}}{\sqrt{\xi}+\sqrt{\eta}}\right|+\sum_{j=0}^{m}\frac{1}{j+\frac{1}{2}}\left(\frac{\eta}{\xi}\right)^{j+\frac{1}{2}}\right\}.\end{aligned}$$

From (2.19), (2.20) we obtain the result,

$$(2.21)\quad \begin{aligned} M(\alpha;\,\xi,\,\eta)&=\frac{(-1)^m}{\sqrt{\pi}}\left(\frac{\alpha}{2}\right)^{m+\frac{1}{2}}\left\{\log\left|\frac{\sqrt{\xi}-\sqrt{\eta}}{\sqrt{\xi}+\sqrt{\eta}}\right|\right.\\ &\left.+\sum_{j=0}^{m}\frac{1}{j+\frac{1}{2}}\left(\frac{\eta}{\xi}\right)^{j+\frac{1}{2}}-\pi i\int_0^\alpha\eta^{\frac{1}{2}m+\frac{3}{4}}J_{m+\frac{3}{2}}(u\sqrt{\eta})\,\frac{H^{(1)}_{m+\frac{1}{2}}(u\sqrt{\xi})}{\xi^{\frac{1}{2}m+\frac{1}{4}}}\,du\right\}.\end{aligned}$$

The case $\xi < \eta$ may be treated in a similar manner, leading to the same result (2.21) enlarged with a term $(-1)^m i\sqrt{\pi}(\alpha/2)^{m+\frac{1}{2}}$.

Substituting (2.21) into (2.16), we are led to

$$\begin{aligned}(2.22)\qquad &(-1)^{m+1}(i/\sqrt{\pi})(2/\alpha)^{m+\frac{1}{2}}(d/d\eta)\int_0^1 M(\alpha;\,\xi,\,\eta)F(\xi)d\xi\\ &= F(\eta)+\frac{i}{\pi}\oint_0^1\left(\frac{\eta}{\xi}\right)^{m+\frac{1}{2}}\frac{F(\xi)}{\xi-\eta}d\xi-\int_0^1 N(\alpha;\,\xi,\,\eta)F(\xi)d\xi,\end{aligned}$$

where

$$(2.23)\qquad N(\alpha;\,\xi,\,\eta)=\tfrac{1}{2}\int_0^{\alpha}\eta^{\frac{1}{2}m+\frac{1}{4}}J_{m+\frac{1}{2}}(u\sqrt{\eta})\,\frac{H^{(1)}_{m+\frac{1}{2}}(u\sqrt{\xi})}{\xi^{\frac{1}{2}m+\frac{1}{4}}}\,u\,du.$$

The integral sign $\oint$ denotes that the Cauchy principal value of the corresponding integral is meant. It may be remarked that the integral (2.23) can be determined explicitly. However, the expression (2.23) is more suitable for expansion in powers of α.

Ultimately, the relation (2.15) leads to the following equation,

(2.24)

$$\begin{aligned}&\oint_0^1\left(\frac{\eta}{\xi}\right)^{m+\frac{1}{2}}\frac{F(\xi)}{\xi-\eta}d\xi\\ &=\frac{(-1)^m i\sqrt{\pi}\,a^{2m+2}}{4\sqrt{2}\,\alpha^{m-\frac{1}{2}}}\int_0^{\eta}(\eta-\mu)^{\frac{1}{2}m-\frac{1}{4}}J_{m-\frac{1}{2}}(\alpha\sqrt{\eta-\mu})\,u_1^{(m)}(a\sqrt{1-\mu})d\mu\\ &\qquad -\pi i\int_0^1 N(\alpha;\,\xi,\,\eta)F(\xi)d\xi.\end{aligned}$$

Now we make the substitutions

$$(2.25)\qquad \xi = 1-s^2,\ \eta = 1-t^2,\ \mu = 1-u^2.$$

After some elementary calculations we obtain the following singular integral equation for the function

$$(2.26)\qquad g_1^{(m)}(s) = f_1^{(m)}(s)/(1-s^2)^m,$$

(2.27)

$$\begin{aligned}&\oint_{-1}^1\frac{g_1^{(m)}(s)}{s-t}\frac{ds}{\sqrt{1-s^2}}\\ &=\frac{(-1)^{m+1}i\sqrt{\pi}\,a^{2m+2}}{\sqrt{2}\,\alpha^{m-\frac{1}{2}}(1-t^2)^{m+\frac{1}{2}}}\int_{|t|}^1(u^2-t^2)^{\frac{1}{2}m-\frac{1}{4}}J_{m-\frac{1}{2}}(\alpha\sqrt{u^2-t^2})\,u_1^{(m)}(au)\,u\,du\\ &\qquad +\pi i\int_{-1}^1 K(\alpha;\,s,\,t)\,g_1^{(m)}(s)\frac{ds}{\sqrt{1-s^2}},\end{aligned}$$

valid for $-1 < t < 1$, where

$$K(\alpha; s, t) = \tfrac{1}{2}\int_0^\alpha \frac{J_{m+\frac{1}{2}}(u\sqrt{1-t^2})}{(1-t^2)^{\frac{1}{2}m+\frac{1}{4}}}(1-s^2)^{\frac{1}{2}m+\frac{1}{4}} H^{(1)}_{m+\frac{1}{2}}(u\sqrt{1-s^2})\, us\, du. \tag{2.28}$$

The equation (2.27) is of a well-known type and is often called the aerofoil equation. Its explicit solution may be quoted from Tricomi [45] leading ultimately to the following Fredholm integral equation of the second kind for the function $g_1^{(m)}(t)$,

$$g_1^{(m)}(t) = H_1^{(m)}(t) - \frac{i}{\pi}\int_{-1}^{1} K_1^{(m)}(\alpha; s, t) g_1^{(m)}(s) \frac{ds}{\sqrt{1-s^2}}, \tag{2.29}$$

valid for $-1 \leqq t \leqq 1$, where

$$H_1^{(m)}(t) = \frac{(-1)^m i a^{2m+2}}{\pi\sqrt{2\pi}\,\alpha^{m-\frac{1}{2}}} \oint_{-1}^{1} \frac{1}{s-t}\frac{ds}{(1-s^2)^m} \cdot \left[\int_{|s|}^{1} (u^2-s^2)^{\frac{1}{2}m-\frac{1}{4}} J_{m-\frac{1}{2}}(\alpha\sqrt{u^2-s^2}) u_1^{(m)}(au) u\, du\right], \tag{2.30}$$

$$K_1^{(m)}(\alpha; s, t) = \int_0^\alpha G_1^{(m)}(u, t)(1-s^2)^{\frac{1}{2}m+\frac{1}{4}} H^{(1)}_{m+\frac{1}{2}}(u\sqrt{1-s^2}) us\, du, \tag{2.31}$$

$$G_1^{(m)}(u, t) = \tfrac{1}{2}\oint_{-1}^{1} \frac{J_{m+\frac{1}{2}}(u\sqrt{1-r^2})}{(1-r^2)^{\frac{1}{2}m-\frac{1}{4}}}\frac{dr}{r-t}. \tag{2.32}$$

The arbitrary constant C, occurring in the solution of the aerofoil equation, is equal to zero, because $g_1^{(m)}(t)$ is an odd function of t.

It can easily be shown that the function $K_1^{(m)}(\alpha; s, t)$ is a continuous function of s for $-1 \leqq s \leqq 1$ and an odd regular function of t for all values of t. When α is small, $K_1^{(m)}(\alpha; s, t)$ will be of order α^2. Assuming that the function $u_1^{(m)}(\rho)$ is an even function of ρ, regular for $|\rho| \leqq a(1+\Delta)$, $\Delta > 0$, [1]) (cf. Bazer and Hochstadt [3]) it follows, that the function $H_1^{(m)}(t)$ is an odd function of t, regular for $|t| \leqq 1+\Delta$. When α is sufficiently small, the Fredholm integral equation (2.29) will have a unique continuous solution $g_1^{(m)}(t)$, which will be an odd function of t. Moreover, because the integral occurring in (2.29) is an entire function of

[1]) In all practically important cases this assumption is fulfilled. Introducing rectangular coordinates $x = \rho\cos\varphi$, $y = \rho\sin\varphi$ the assumption is equivalent to the functions $u(x, y, 0)$, $\partial u(x, y, 0)/\partial z$ (cf. (2.3)) being regular for $x^2+y^2 \leqq a^2(1+\Delta)^2$, compare Heins and MacCamy [17].

t, the function $g_1^{(m)}(t)$ will be regular for $|t| \leqq 1+\Delta$. Hence the corresponding function $f_1^{(m)}(t)$ satisfies all the required conditions.

2.3. For the second boundary value problem, the aperture condition (iv)′ leads to the equation,

$$\begin{aligned}(2.33)\quad &\frac{2}{a}\left(\frac{1}{\rho}\frac{d}{d\rho}\right)^m \int_0^{\rho/a} \frac{\exp\left(i\alpha\sqrt{(\rho/a)^2-t^2}\right)}{\sqrt{(\rho/a)^2-t^2}} f_2^{(m)}(t)dt \\ &+\frac{2i}{a}\left(\frac{1}{\rho}\frac{d}{d\rho}\right)^m \int_{\rho/a}^{1} \frac{\sinh\left(\alpha\sqrt{t^2-(\rho/a)^2}\right)}{\sqrt{t^2-(\rho/a)^2}} f_2^{(m)}(t)dt = u_2^{(m)}(\rho),\end{aligned}$$

where

$$(2.34)\qquad u_2^{(m)}(\rho) = (1/\rho^m)u^{(m)}(\rho, 0).$$

Again we continue the first integral in (2.33) over the complete interval [0, 1] and make the substitutions (2.10),

$$\begin{aligned}(2.35)\quad &\left(\frac{d}{d\eta}\right)^m \int_0^{\eta} \frac{\cosh(\alpha\sqrt{\eta-\xi})}{\sqrt{\eta-\xi}} F(\xi)d\xi \\ &= i\left(\frac{d}{d\eta}\right)^m \int_0^{1} \frac{\exp(i\alpha\sqrt{\xi-\eta})}{\sqrt{\xi-\eta}} F(\xi)d\xi \\ &\quad+\frac{(-1)^{m+1}ia^{2m+1}}{2^{m+1}} u_2^{(m)}(a\sqrt{1-\eta}),\end{aligned}$$

where

$$(2.36)\qquad F(\xi) = f_2^{(m)}(\sqrt{1-\xi})/(2\sqrt{1-\xi}).$$

Denoting the right-hand side of (2.35) by $G(\eta)$, the equation (2.35) can be solved by means of the convolution theorem for Laplace transforms, yielding

$$\begin{aligned}(2.37)\quad &F(\eta) \\ &= (1/\sqrt{\pi})(2/\alpha)^{m-\frac{1}{2}}(d/d\eta)\int_0^{\eta} (\eta-\mu)^{\frac{1}{2}m-\frac{1}{4}} J_{m-\frac{1}{2}}(\alpha\sqrt{\eta-\mu})G(\mu)d\mu.\end{aligned}$$

Substituting $G(\mu)$ as given by (2.35) into (2.37), the reduction may be performed in a similar manner as for the first boundary value problem. Ultimately we arrive at the following singular integral equation for the function

$$(2.38)\qquad g_2^{(m)}(s) = f_2^{(m)}(s)/(1-s^2)^m,$$

$$
\begin{aligned}
&\oint_{-1}^{1} \frac{g_2^{(m)}(s)}{s-t} \sqrt{1-s^2}\, ds \\
&\quad = \frac{(-1)^{m+1}\sqrt{\pi}\, a^{2m+1} t}{\sqrt{2}\, \alpha^{m-\frac{1}{2}}(1-t^2)^{m-\frac{1}{2}}} \Big[(1-t^2)^{\frac{1}{2}m-\frac{1}{4}} J_{m-\frac{1}{2}}(\alpha\sqrt{1-t^2}) u_2^{(m)}(a) \\
&\qquad - \int_{|t|}^{1} (u^2-t^2)^{\frac{1}{2}m-\frac{1}{4}} J_{m-\frac{1}{2}}(\alpha\sqrt{u^2-t^2})(d/du) u_2^{(m)}(au)\, du \Big] \\
&\qquad + \pi i \int_{-1}^{1} \bar{K}(\alpha;\, s,\, t) g_2^{(m)}(s) \sqrt{1-s^2}\, ds,
\end{aligned} \tag{2.39}
$$

valid for $-1 < t < 1$, where

$$
\bar{K}(\alpha; s, t) = \tfrac{1}{2}\int_0^{\alpha} \frac{J_{m-\frac{1}{2}}(u\sqrt{1-t^2})}{(1-t^2)^{\frac{1}{2}m-\frac{1}{4}}} (1-s^2)^{\frac{1}{2}m-\frac{1}{4}} H^{(1)}_{m-\frac{1}{2}}(u\sqrt{1-s^2})\, ut\, du. \tag{2.40}
$$

The aerofoil equation (2.39) can be solved according to Tricomi [45], leading to the following integral equation for $g_2^{(m)}(t)$,

$$
g_2^{(m)}(t) = \frac{\bar{H}_2^{(m)}(t)}{1-t^2} + \frac{C}{1-t^2} - \frac{i}{\pi(1-t^2)} \int_{-1}^{1} \bar{K}_2^{(m)}(\alpha; s, t) g_2^{(m)}(s) \sqrt{1-s^2}\, ds, \tag{2.41}
$$

valid for $-1 < t < 1$, where

$$
\begin{aligned}
\bar{H}_2^{(m)}(t) &= \frac{(-1)^m a^{2m+1}}{\pi\sqrt{2\pi}\, \alpha^{m-\frac{1}{2}}} \oint_{-1}^{1} \frac{s}{s-t} \frac{ds}{(1-s^2)^{m-1}} \\
&\quad \cdot \Big[(1-s^2)^{\frac{1}{2}m-\frac{1}{4}} J_{m-\frac{1}{2}}(\alpha\sqrt{1-s^2}) u_2^{(m)}(a) \\
&\qquad - \int_{|s|}^{1} (u^2-s^2)^{\frac{1}{2}m-\frac{1}{4}} J_{m-\frac{1}{2}}(\alpha\sqrt{u^2-s^2})(d/du) u_2^{(m)}(au)\, du \Big],
\end{aligned} \tag{2.42}
$$

$$
\bar{K}_2^{(m)}(\alpha;\, s,\, t) = \int_0^{\alpha} \bar{G}_2^{(m)}(u,\, t)(1-s^2)^{\frac{1}{2}m-\frac{1}{4}} H^{(1)}_{m-\frac{1}{2}}(u\sqrt{1-s^2})\, u\, du, \tag{2.43}
$$

$$
\bar{G}_2^{(m)}(u, t) = \tfrac{1}{2} \oint_{-1}^{1} \frac{J_{m-\frac{1}{2}}(u\sqrt{1-r^2})}{(1-r^2)^{\frac{1}{2}m-\frac{3}{4}}} \frac{r\,dr}{r-t}. \tag{2.44}
$$

C is an arbitrary constant. According to (2.41) the function $g_2^{(m)}(t)$ will be defined for $t = \pm 1$, provided that the following condition is satisfied,

$$
C = -\bar{H}_2^{(m)}(1) + (i/\pi) \int_{-1}^{1} \bar{K}_2^{(m)}(\alpha;\, s,\, 1) g_2^{(m)}(s) \sqrt{1-s^2}\, ds. \tag{2.45}
$$

Substituting (2.45) into (2.41) we obtain a Fredholm integral equation of the second kind for the function $g_2^{(m)}(t)$, viz.

$$(2.46)\quad g_2^{(m)}(t) = H_2^{(m)}(t) - \frac{i}{\pi}\int_{-1}^{1} K_2^{(m)}(\alpha; s, t) g_2^{(m)}(s)\sqrt{1-s^2}\, ds,$$

valid for $-1 \leqq t \leqq 1$, where

$$(2.47)\quad \begin{aligned} H_2^{(m)}(t) = {} & \frac{(-1)^m a^{2m+1}}{\pi\sqrt{2\pi}\,\alpha^{m-\frac{1}{2}}} \oint_{-1}^{1} \frac{s}{s-t} \frac{ds}{(1-s^2)^m} \\ & \cdot \Big[(1-s^2)^{\frac{1}{2}m-\frac{1}{4}} J_{m-\frac{1}{2}}(\alpha\sqrt{1-s^2}) u_2^{(m)}(a) \\ & - \int_{|s|}^{1} (u^2-s^2)^{\frac{1}{2}m-\frac{1}{4}} J_{m-\frac{1}{2}}(\alpha\sqrt{u^2-s^2})(d/du) u_2^{(m)}(au)\, du \Big], \end{aligned}$$

$$(2.48)\quad K_2^{(m)}(\alpha; s, t) = \int_0^{\alpha} G_2^{(m)}(u, t)(1-s^2)^{\frac{1}{2}m-\frac{1}{4}} H^{(1)}_{m-\frac{1}{2}}(u\sqrt{1-s^2})\, u\, du,$$

$$(2.49)\quad G_2^{(m)}(u, t) = \tfrac{1}{2}\oint_{-1}^{1} \frac{J_{m-\frac{1}{2}}(u\sqrt{1-r^2})}{(1-r^2)^{\frac{1}{2}m+\frac{1}{4}}} \frac{r\,dr}{r-t}.$$

It can be shown that the function $K_2^{(m)}(\alpha; s, t)\sqrt{1-s^2}$ is a continuous function of s for $-1 \leqq s \leqq 1$ and an even regular function of t for all values of t. When α is small, $K_2^{(m)}(\alpha; s, t)$ with $m \geqq 1$ will be of order α^2, whereas $K_2^{(0)}(\alpha; s, t)$ is of order α. Assuming that the function $u_2^{(m)}(\rho)$ is an even function of ρ, regular for $|\rho| \leqq a(1+\Delta)$, $\Delta > 0$, (cf. Bazer and Hochstadt [3]) it follows, that the function $H_2^{(m)}(t)$ is an even function of t, regular for $|t| \leqq 1+\Delta$. Hence, similar to the first boundary value problem, when α is sufficiently small, the integral equation (2.46) will have a unique continuous solution $g_2^{(m)}(t)$, this function being even in t and regular for $|t| \leqq 1+\Delta$.

2.4. The present method of solution has been elaborated for the case of an obliquely incident plane wave described by the wave function

$$(2.50)\quad u(\rho, \varphi, z) = \exp\{ik(\rho \sin\gamma \cos\varphi + z\cos\gamma)\}.$$

This wave function can be expanded in a Fourier series yielding the following expression for the mth mode of the incident wave,

$$(2.51)\quad u^{(m)}(\rho, z) = i^m \varepsilon_m J_m(k\rho \sin\gamma) \exp(ikz\cos\gamma),$$

where $\varepsilon_0 = 1$, $\varepsilon_m = 2$ when $m \geqq 1$. Substituting the corresponding values of $u_1^{(m)}(\rho)$, $u_2^{(m)}(\rho)$ into (2.30), (2.47), the integral equations

(2.29), (2.46) can be solved by iteration, when α is sufficiently small. Expansions in powers of α up to relative order α^6 have been calculated for the functions $f_{1,2}^{(m)}(t)$. However, these expansions being rather lengthy, we only state the following partial results,

$$f_1^{(m)}(t) = \frac{\varepsilon_m(-i)^m a^{m+1} \sin^m \gamma \cos \gamma}{\pi(2m+1)!} \alpha^{m+1}(1-t^2)^m$$

$$(2.52) \qquad \cdot \left[t-\alpha^2 \frac{2mt-(2m+1)t^3}{(4m+2)(2m+3)} - \alpha^2 \sin^2 \gamma \frac{mt+t^3}{(2m+2)(2m+3)} \right.$$

$$\left. + \frac{2i}{9\pi} \alpha^3 \delta_{m,0} t + O(\alpha^4)\right], \qquad \text{for} \quad m = 0, 1, 2, \ldots,$$

$$f_2^{(m)}(t) = \frac{\varepsilon_m(-i)^m a^{m+1} \sin^m \gamma}{\pi(2m)!} \alpha^m(1-t^2)^m$$

$$(2.53) \qquad \cdot \left[1-\alpha^2 \frac{2m-(2m-1)t^2}{(4m-2)(2m+1)} - \alpha^2 \sin^2 \gamma \frac{m+t^2}{(2m+1)(2m+2)} \right.$$

$$\left. - \frac{4i}{9\pi} \alpha^3 \delta_{m,1} + O(\alpha^4)\right], \qquad \text{for } m = 1, 2, 3, \ldots.$$

Kronecker's symbol $\delta_{m,n}$ is defined by $\delta_{m,n} = 1$ when $m = n$, $\delta_{m,n} = 0$ when $m \neq n$. The function $f_2^{(0)}(t)$ can better be determined by means of the original method of Bazer and Hochstadt [3].

According to Bazer and Hochstadt [3] the following formulae hold for the transmitted wave at a large distance from the aperture and in the aperture:

$$(2.54) \qquad \Phi_{1,2}^{(m)}(R \sin \theta, R \cos \theta) \sim A_{1,2}^{(m)}(\theta) e^{ikR}/R,$$

where

$$(2.55) \quad A_1^{(m)}(\theta) = -2(ik \sin \theta)^m \int_0^1 \sinh (\alpha t \cos \theta) f_1^{(m)}(t) dt,$$

$$(2.56) \quad A_2^{(m)}(\theta) = 2(ik \sin \theta)^m \int_0^1 \cosh (\alpha t \cos \theta) f_2^{(m)}(t) dt,$$

valid for $0 \leqq \theta \leqq \pi/2$ and large values of R;

$$(2.57) \quad \Phi_1^{(m)}(\rho, 0) = \frac{2\rho^m}{ia} \left(\frac{1}{\rho} \frac{d}{d\rho}\right)^m \int_{\rho/a}^1 \frac{\cosh (\alpha\sqrt{t^2-(\rho/a)^2})}{\sqrt{t^2-(\rho/a)^2}} f_1^{(m)}(t) dt,$$

$$(2.58)\quad \frac{\partial \Phi_2^{(m)}(\rho,0)}{\partial z} = 2\rho^m \left(\frac{1}{\rho}\frac{d}{d\rho}\right)^{m+1} \int_{\rho/a}^{1} \frac{\cosh\left(\alpha\sqrt{t^2-(\rho/a)^2}\right)}{\sqrt{t^2-(\rho/a)^2}}\, t f_2^{(m)}(t)dt,$$

valid for $0 \leqq \rho < a$. Substituting the expansions (2.52), (2.53) into these formulae we obtain the following results,

(2.59)

$$A_1^{(m)}(\theta) = -\frac{2^{2m+2}\varepsilon_m m!(m+1)!a(\sin\gamma\sin\theta)^m\cos\gamma\cos\theta}{\pi(2m+1)!(2m+3)!}\alpha^{2m+2}$$
$$\cdot\left[1-\alpha^2\frac{(2m-1)+(2m+1)(\sin^2\gamma-\cos^2\theta)}{(2m+1)(4m+10)}+\frac{2i}{9\pi}\alpha^3\delta_{m,0}+O(\alpha^4)\right],$$

(2.60)

$$A_2^{(m)}(\theta) = \frac{2^{2m+2}\varepsilon_m m!(m+1)!a(\sin\gamma\sin\theta)^m}{\pi(2m)!(2m+2)!}\alpha^{2m}$$
$$\cdot\left[1-\alpha^2\frac{(2m+1)+(2m-1)(\sin^2\gamma-\cos^2\theta)}{(2m-1)(4m+6)}-\frac{4i}{9\pi}\alpha^3\delta_{m,1}+O(\alpha^4)\right],$$

(2.61)

$$\Phi_1^{(m)}(\rho,0) = -\frac{2^{m+1}\varepsilon_m i^{m+1}m!\sin^m\gamma\cos\gamma}{\pi(2m+1)!}\alpha^{m+1}\left(\frac{\rho}{a}\right)^m\varepsilon$$
$$\cdot\left[1+\alpha^2\frac{3+(2m+1)\varepsilon^2}{(4m+2)(6m+9)}-\alpha^2\sin^2\gamma\frac{3-2\varepsilon^2}{12m+18}+\frac{2i}{9\pi}\alpha^3\delta_{m,0}+O(\alpha^4)\right],$$

(2.62)

$$\frac{\partial \Phi_2^{(m)}(\rho,0)}{\partial z} = -\frac{2^{m+1}\varepsilon_m i^m m!\sin^m\gamma}{\pi a(2m)!}\alpha^m\left(\frac{\rho}{a}\right)^m\frac{1}{\varepsilon}$$
$$\cdot\left[1-\alpha^2\frac{1+(2m-1)\varepsilon^2}{(2m-1)(4m+2)}-\alpha^2\sin^2\gamma\frac{1-2\varepsilon^2}{4m+2}-\frac{4i}{9\pi}\alpha^3\delta_{m,1}+O(\alpha^4)\right],$$

where $\varepsilon = \sqrt{1-(\rho/a)^2}$. The results (2.59), (2.61) hold for $m = 0, 1, 2, \ldots$, while (2.60), (2.62) are valid for $m = 1, 2, 3, \ldots$.

The transmission coefficients $t_{1,2}^{(m)}$ are given by (cf. Bazer and Hochstadt [3]),

$$(2.63)\qquad t_{1,2}^{(m)} = 2\int_0^{\pi/2} |A_{1,2}^{(m)}(\theta)|^2 \sin\theta\, d\theta/(\varepsilon_m a^2\cos\gamma).$$

Starting from the complete expansions for $f_{1,2}^{(m)}(t)$ up to relative order α^6, the following results were derived with respect to $t_{1,2}^{(m)}$,

$$t_1^{(m)} = \frac{\varepsilon_m\{2^{2m+2}m!\,(m+1)!\}^3 \sin^{2m}\gamma \cos\gamma}{\pi^2\{(2m+1)!\}^2\{(2m+3)!\}^3}\,\alpha^{4m+4}$$

$$\cdot\left[1-\alpha^2\left\{\frac{4m^2+2m-8}{(2m+1)(2m+5)^2}+\frac{\sin^2\gamma}{2m+5}\right\}\right.$$

$$(2.64)\quad +\alpha^4\left\{\frac{32m^6+144m^5+24m^4-156m^3+1236m^2+1758m-311}{(2m-1)(2m+1)^2(2m+5)^3(2m+7)^2}\right.$$

$$+\frac{8m^3+32m^2+4m-53}{(2m+1)(2m+5)^3(2m+7)}\sin^2\gamma$$

$$\left.\left.+\frac{m+3}{(2m+5)^2(2m+7)}\sin^4\gamma\right\}+O(\alpha^6)\right],$$

$$t_2^{(m)} = \frac{\varepsilon_m\{2^{2m+2}m!\,(m+1)!\}^3 \sin^{2m}\gamma}{\pi^2\cos\gamma\{(2m)!\}^2\{(2m+2)!\}^3}\,\alpha^{4m}$$

$$\cdot\left[1-\alpha^2\left\{\frac{4m^2+6m+4}{(2m-1)(2m+3)^2}+\frac{\sin^2\gamma}{2m+3}\right\}\right.$$

$$(2.65)\quad +\alpha^4\left\{\frac{32m^6+144m^5+136m^4-364m^3-960m^2-758m-213}{(2m-3)(2m-1)^2(2m+3)^3(2m+5)^2}\right.$$

$$+\frac{8m^3+32m^2+40m+19}{(2m-1)(2m+3)^3(2m+5)}\sin^2\gamma$$

$$\left.\left.+\frac{m+2}{(2m+3)^2(2m+5)}\sin^4\gamma\right\}+O(\alpha^6)\right],$$

these expansions being valid for $m = 0, 1, 2, \ldots$ and $m = 1, 2, 3, \ldots$ respectively.

The presented results constitute an extension of the solutions given by Bazer and Hochstadt [3], Williams [51]. Williams treats the complementary problem of the diffraction of a scalar wave by a perfectly soft or a perfectly rigid circular disk. Following quite a different method Williams reduces both boundary value problems to Fredholm integral equations for a pair of unknown functions. These functions can be shown to be proportional to our functions

$$t^{m+1}(d/t\,dt)^m\{f_1^{(m)}(t)/t\} \quad\text{and}\quad t^m(d/t\,dt)^m f_2^{(m)}(t).$$

§ 3. Diffraction of a plane electromagnetic wave by a circular disk

3.1. In the present paragraph we treat the diffraction of a plane electromagnetic wave by a perfectly conducting circular disk. Introducing rectangular coordinates x, y, z and cylindrical coordinates ρ, φ, z connected by $x = \rho \cos \varphi$, $y = \rho \sin \varphi$, the circular disk will be determined by $0 \leqq \rho \leqq a$, $0 \leqq \varphi < 2\pi$, $z = 0$.

We start from the formulation of the diffraction problem as given by Meixner and Andrejewski [37]. Let a monochromatic plane-polarized electromagnetic wave, denoted by $(\underline{E}^i, \underline{H}^i)$ impinge upon the disk. The vectors $\underline{E}^i$, $\underline{H}^i$ will show a time dependence $e^{-i\omega t}$, this factor being omitted in what follows. The wave number k is given by $k = \omega\sqrt{\varepsilon\mu}$, where ε and μ are the dielectric constant and the magnetic permeability of the medium surrounding the disk.

Both the incident wave $(\underline{E}^i, \underline{H}^i)$ and the scattered wave $(\underline{E}^s, \underline{H}^s)$ are derived from electric Hertz vectors $\underline{\Pi}^i$ and $\underline{\Pi}^s$ respectively viz.

$$\underline{E} = (1/\varepsilon) \operatorname{rot} \operatorname{rot} \underline{\Pi}, \quad \underline{H} = -i\omega \operatorname{rot} \underline{\Pi}, \tag{3.1}$$

where rationalized Giorgi units have been used. These Hertz vectors may be chosen in such a way that their z-components vanish. The Hertz vector for the total wave will be denoted by $\underline{\Pi} = \underline{\Pi}^i + \underline{\Pi}^s$.

According to Meixner and Andrejewski [37] the boundary condition

$$E^i_{\text{tang}} + E^s_{\text{tang}} = 0 \tag{3.2}$$

on the disk, leads to the following conditions for the components of $\underline{\Pi}$,

$$\Pi_x = \partial U/\partial x, \ \Pi_y = \partial U/\partial y, \text{ when } z = 0, \ 0 \leqq \rho \leqq a \tag{3.3}$$

where

$$U = \sum_{m=-\infty}^{\infty} U_m J_m(k\rho) e^{im\varphi}. \tag{3.4}$$

The unknown coefficients U_m will follow from the edge condition. From (3.3), (3.4) the following Fourier series can be derived for $\Pi_{x,y}$ on the disk,

$$\Pi_x = (k/2) \sum_{m=-\infty}^{\infty} (U_{m+1} - U_{m-1}) J_m(k\rho) e^{im\varphi}, \tag{3.5}$$

$$\Pi_y = (ik/2) \sum_{m=-\infty}^{\infty} (U_{m+1} + U_{m-1}) J_m(k\rho) e^{im\varphi}. \tag{3.6}$$

Next we expand the Hertz vectors $\underline{\Pi}^i$ and $\underline{\Pi}^s$ in Fourier series with respect to φ,

$$(3.7) \qquad \underline{\Pi}^{i,s}(\rho, \varphi, z) = \sum_{m=-\infty}^{\infty} \underline{\Pi}^{i,s}_m(\rho, z)e^{im\varphi}$$

where the vectors $\underline{\Pi}^{i,s}_m$ will have the components $(\Pi^{i,s}_{mx}, \Pi^{i,s}_{my}, 0)$. The functions $\Pi^s_{mx}(\rho, z)$, $\Pi^s_{my}(\rho, z)$ are now required to satisfy the following conditions:

(i) Π^s_{mx}, Π^s_{my} are solutions of Helmholtz' equation, viz.

$$\left(\frac{\partial^2}{\partial\rho^2} + \frac{1}{\rho}\frac{\partial}{\partial\rho} + \frac{\partial^2}{\partial z^2} + k^2 - \frac{m^2}{\rho^2}\right)\Pi^s_{mx,y} = 0;$$

(ii) $\partial\Pi^s_{mx}/\partial z = \partial\Pi^s_{my}/\partial z = 0$ when $z = 0$, $\rho > a$; this condition follows from the functions Π^s_{mx}, Π^s_{my} being even in z;

(iii) Π^s_{mx}, Π^s_{my} satisfy Sommerfeld's radiation condition at infinity;

(iv) $$\left.\begin{aligned}\Pi^s_{mx}+\Pi^i_{mx} &= (k/2)(U_{m+1}-U_{m-1})J_m(k\rho)\\ \Pi^s_{my}+\Pi^i_{my} &= (ik/2)(U_{m+1}+U_{m-1})J_m(k\rho)\end{aligned}\right\} \begin{aligned}&\text{when } z = 0,\\ &0 \leqq \rho \leqq a;\end{aligned}$$

(v) Meixner and Andrejewski [37] present the following edge condition for the Hertz vector $\underline{\Pi}^s$: Near the edge of the disk the components Π^s_x and Π^s_y remain finite. Further, in a point at distance δ from the edge the expansion of $\Pi^s_x \cos\varphi + \Pi^s_y \sin\varphi$ in powers of δ does not contain a term with $\delta^{\frac{1}{2}}$. Hence, Π^s_{mx}, Π^s_{my} remain also finite near the edge of the disk, while the sum

$$(3.8) \qquad \Pi^s_{m-1,x}+\Pi^s_{m+1,x}-i\Pi^s_{m-1,y}+i\Pi^s_{m+1,y},$$

considered in a point at distance δ from the edge, has an expansion in powers of δ which does not contain a term with $\delta^{\frac{1}{2}}$.

In analogy with Bazer and Hochstadt's [3] second boundary value problem, we introduce the integral representations,

$$(3.9) \quad F^{(1,2)}_m(\rho, z) = \rho^{|m|}\left(\frac{1}{\rho}\frac{\partial}{\partial\rho}\right)^{|m|}\int_{-1}^{1}\frac{\exp\left(ik\sqrt{\rho^2+(z+iat)^2}\right)}{\sqrt{\rho^2+(z+iat)^2}}f^{(1,2)}_m(t)dt,$$

$$(3.10) \quad G_m(\rho, z) = \rho^{|m|}\left(\frac{1}{\rho}\frac{\partial}{\partial\rho}\right)^{|m|}\int_{-1}^{1}\frac{\exp\left(ik\sqrt{\rho^2+(z+iat)^2}\right)}{\sqrt{\rho^2+(z+iat)^2}}g_m(t)dt,$$

valid for $z \geqq 0$, where m is an integer. For $z \leqq 0$ we define $F^{(1,2)}_m(\rho, z) = F^{(1,2)}_m(\rho, -z)$, $G_m(\rho, z) = G_m(\rho, -z)$. The unknown functions $f^{(1,2)}_m(t)$, $g_m(t)$ are required to be even functions of t, to be regular for $|t| \leqq 1+\Delta$, $\Delta > 0$ and to satisfy the conditions,

$$(3.11) \quad d^k f_m^{(1,2)}(1)/dt^k = d^k g_m(1)/dt^k = 0, \text{ for } k = 0, 1, \ldots, |m|-1.$$

The functions $F_m^{(1,2)}(\rho, z)$, $G_m(\rho, z)$ have to assume the following boundary values,

$$(3.12) \qquad F_m^{(1,2)}(\rho, 0) = -\Pi^i_{mx,y}(\rho, 0), \; G_m(\rho, 0) = J_m(k\rho),$$

valid for $0 \leqq \rho \leqq a$. Using the method described in § 2, section 2.3, the conditions (3.12) lead to Fredholm integral equations of the second kind for the functions $f_m^{(1,2)}(t)/(1-t^2)^m$, $g_m(t)/(1-t^2)^m$. When $\alpha = ka$ is sufficiently small, these integral equations may be solved by iteration yielding expansions in powers of α for the above-mentioned functions. We remark that $g_m(t) = (-1)^m g_{-m}(t)$ as follows easily from (3.12).

From (3.12) and the condition (iv) it is obvious that Π^s_{mx}, Π^s_{my} are represented by

$$(3.13) \quad \Pi^s_{mx}(\rho, z) = F_m^{(1)}(\rho, z) + (k/2)(U_{m+1} - U_{m-1})G_m(\rho, z),$$

$$(3.14) \quad \Pi^s_{my}(\rho, z) = F_m^{(2)}(\rho, z) + (ik/2)(U_{m+1} + U_{m-1})G_m(\rho, z).$$

It can easily be shown that the functions Π^s_{mx}, Π^s_{my} as given by (3.13), (3.14) satisfy the conditions (i) to (iv) and the first part of (v). Using formula (2.7), expansions in powers of δ can be derived for $\Pi^s_{mx,y}$ considered in a point with coordinates $\rho = a+\delta\cos\gamma$, $z = \delta\sin\gamma$, $\delta > 0$, $0 \leqq \gamma \leqq \pi$ near the edge of the disk. When these expansions are substituted into (3.8), the term with $\delta^{\frac{1}{2}}$ may be set equal to zero in agreement with the condition (v), yielding the following explicit results for the coefficients U_m,

$$(3.15) \begin{cases} 2kU_0 Dg_1 = D\{f_{-1}^{(1)} - if_{-1}^{(2)}\} + D\{f_1^{(1)} + if_1^{(2)}\}, \\ kU_m\{a^2 D^{m-1} g_{m-1} - D^{m+1} g_{m+1}\} \\ \quad = -a^2 D^{m-1}\{f_{m-1}^{(1)} - if_{m-1}^{(2)}\} - D^{m+1}\{f_{m+1}^{(1)} + if_{m+1}^{(2)}\}, \quad (m \geqq 1) \\ kU_{-m}\{a^2 D^{m-1} g_{-m+1} - D^{m+1} g_{-m-1}\} \\ \quad = a^2 D^{m-1}\{f_{-m+1}^{(1)} + if_{-m+1}^{(2)}\} + D^{m+1}\{f_{-m-1}^{(1)} - if_{-m-1}^{(2)}\}, \; (m \geqq 1) \end{cases}$$

where $D = d/dt$ and the omitted argument of the various derivatives must be taken equal to 1. Hence, the diffraction problem is essentially solved.

According to Bazer and Hochstadt [3] the functions $F_m^{(1,2)}(\rho, z)$ and $G_m(\rho, z)$ assume the following asymptotic values at large distances from the disk,

$$(3.16)\qquad \begin{cases} F_m^{(1,2)}(R\sin\theta,\ R\cos\theta) \sim A_m^{(1,2)}(\theta)\, e^{ikR}/R, \\ G_m(R\sin\theta,\ R\cos\theta) \quad \sim B_m(\theta)e^{ikR}/R, \end{cases}$$

where

$$(3.17)\qquad \begin{cases} A_m^{(1,2)}(\theta) = 2(ik\sin\theta)^{|m|}\int_0^1 \cosh(\alpha t\cos\theta) f_m^{(1,2)}(t)dt, \\ B_m(\theta) \quad = 2(ik\sin\theta)^{|m|}\int_0^1 \cosh(\alpha t\cos\theta) g_m(t)dt, \end{cases}$$

valid for $0 \leqq \theta \leqq \pi$ and large values of R. From (3.17) expansions in powers of α can be derived for the functions $A_m^{(1,2)}(\theta)$, $B_m(\theta)$.

Using (3.7), (3.13), (3.14), (3.16) similar asymptotic values can be stated for Π_x^s, Π_y^s valid at a large distance from the disk. Now we introduce spherical coordinates r, θ, φ, defined by

$$(3.18)\qquad x = r\sin\theta\cos\varphi,\ y = r\sin\theta\sin\varphi,\ z = r\cos\theta.$$

Then the components of $\underline{\Pi}^s$ in spherical coordinates considered in a point $(R,\ \theta,\ \varphi)$ are represented by the following asymptotic values,

$$(3.19)\qquad \Pi_r^s \sim \tfrac{1}{2}\sum_{m=-\infty}^{\infty} [A_{m-1}^{(1)} + A_{m+1}^{(1)} - iA_{m-1}^{(2)} + iA_{m+1}^{(2)} + kU_m\{B_{m-1} - B_{m+1}\}]e^{im\varphi}(e^{ikR}/R)\sin\theta,$$

$$(3.20)\qquad \Pi_\theta^s \sim \tfrac{1}{2}\sum_{m=-\infty}^{\infty} [A_{m-1}^{(1)} + A_{m+1}^{(1)} - iA_{m-1}^{(2)} + iA_{m+1}^{(2)} + kU_m\{B_{m-1} - B_{m+1}\}]e^{im\varphi}(e^{ikR}/R)\cos\theta,$$

$$(3.21)\qquad \Pi_\varphi^s \sim \tfrac{1}{2}\sum_{m=-\infty}^{\infty} [iA_{m-1}^{(1)} - iA_{m+1}^{(1)} + A_{m-1}^{(2)} + A_{m+1}^{(2)} + ikU_m\{B_{m-1} + B_{m+1}\}]e^{im\varphi}(e^{ikR}/R),$$

where the argument θ in $A_m^{(1,2)}(\theta)$ and $B_m(\theta)$ has been omitted.

From (3.1) the leading terms of the components of the scattered field in the wave zone can be derived, viz.

$$(3.22)\qquad \begin{cases} E_r^s \sim H_r^s \sim 0, \\ E_\theta^s \sim \sqrt{\mu/\varepsilon}\, H_\varphi^s \sim (k^2/\varepsilon)\Pi_\theta^s,\ E_\varphi^s \sim -\sqrt{\mu/\varepsilon}\, H_\theta^s \sim (k^2/\varepsilon)\Pi_\varphi^s. \end{cases}$$

Hence in the wave zone the scattered field behaves as an outgoing transverse spherical wave. Low-frequency expansions in powers of α follow from the corresponding expansions for $A_m^{(1,2)}(\theta)$, $B_m(\theta)$.

The scattered energy E_{sc} is found by integration of Poynting's vector over a sphere with radius R and taking the limit for $R \to \infty$. Using (3.20), (3.21), (3.22) we obtained

$$E_{sc} = \lim_{R\to\infty} \tfrac{1}{2}\sqrt{\varepsilon/\mu} \int_0^{\pi}\int_0^{2\pi} \{|E^s_\theta|^2+|E^s_\varphi|^2\}R^2 \sin\theta\, d\varphi\, d\theta$$

$$(3.23)\qquad = \frac{\pi k^4}{4\varepsilon\sqrt{\varepsilon\mu}}\int_0^{\pi} [\sum_{m=-\infty}^{\infty} |A^{(1)}_{m-1}+A^{(1)}_{m+1}-iA^{(2)}_{m-1}+iA^{(2)}_{m+1} + kU_m\{B_{m-1}-B_{m+1}\}|^2 \cos^2\theta + \sum_{m=-\infty}^{\infty} |iA^{(1)}_{m-1}-iA^{(1)}_{m+1} + A^{(2)}_{m-1}+A^{(2)}_{m+1}+ikU_m\{B_{m-1}+B_{m+1}\}|^2] \sin\theta\, d\theta.$$

The scattering coefficient, τ, is defined as the ratio of the scattered energy and the incident energy with the disk's area as basis. By means of (3.23) a low-frequency expansion in powers of α may be derived for τ.

Finally we state the following formulae for the scattered field on the disk. On the positive side of the disk the components of $\underline{E}^s$, $\underline{H}^s$ in cylindrical coordinates are given by

$$(3.24)\qquad \begin{aligned} H^s_\rho &= (ik/\sqrt{\mu\varepsilon})(\partial/\partial z)\{-\Pi^s_x \sin\varphi+\Pi^s_y\cos\varphi\} \\ &= -\frac{k}{\sqrt{\mu\varepsilon}}\sum_{m=-\infty}^{\infty}\left[\frac{\partial F^{(1)}_{m-1}(\rho,0)}{\partial z}-\frac{\partial F^{(1)}_{m+1}(\rho,0)}{\partial z} - i\frac{\partial F^{(2)}_{m-1}(\rho,0)}{\partial z} - i\frac{\partial F^{(2)}_{m+1}(\rho,0)}{\partial z} + kU_m\left\{\frac{\partial G_{m-1}(\rho,0)}{\partial z}+\frac{\partial G_{m+1}(\rho,0)}{\partial z}\right\}\right]e^{im\varphi}, \end{aligned}$$

$$(3.25)\qquad \begin{aligned} H^s_\varphi &= (-ik/\sqrt{\mu\varepsilon})(\partial/\partial z)\{\Pi^s_x \cos\varphi+\Pi^s_y\sin\varphi\} \\ &= -\frac{ik}{\sqrt{\mu\varepsilon}}\sum_{m=-\infty}^{\infty}\left[\frac{\partial F^{(1)}_{m-1}(\rho,0)}{\partial z}+\frac{\partial F^{(1)}_{m+1}(\rho,0)}{\partial z} - i\frac{\partial F^{(2)}_{m-1}(\rho,0)}{\partial z} + i\frac{\partial F^{(2)}_{m+1}(\rho,0)}{\partial z} + kU_m\left\{\frac{\partial G_{m-1}(\rho,0)}{\partial z}-\frac{\partial G_{m+1}(\rho,0)}{\partial z}\right\}\right]e^{im\varphi}, \end{aligned}$$

$$(3.26)\qquad \begin{aligned} H^s_z = 0,\ E^s_\rho &= -E^i_\rho = -E^i_x\cos\varphi - E^i_y\sin\varphi, \\ E^s_\varphi &= -E^i_\varphi = E^i_x\sin\varphi - E^i_y\cos\varphi, \end{aligned}$$

$$(3.27)\qquad E^s_z = (i/k)\sqrt{\mu/\varepsilon}[(1/\rho)(\partial/\partial\rho)(\rho H^s_\varphi)-(1/\rho)(\partial/\partial\varphi)H^s_\rho],$$

where we used (3.1), (3.2), (3.7), (3.13), (3.14).

The current density $\underline{I}$ and the surface-charge density σ induced in the disk are related to the scattered field on the disk, viz.

$$(3.28) \qquad I_\rho = -2H^s_\varphi,\; I_\varphi = 2H^s_\rho,\; \sigma = 2\varepsilon E^s_z,$$

where H^s_ρ, H^s_φ, E^s_z are given by (3.24), (3.25), (3.27).

Low-frequency expansions in powers of α to the scattered field on the disk can be derived from the following relations due to Bazer and Hochstadt [3],

$$(3.29) \qquad \begin{cases} \dfrac{\partial F^{(1,2)}_m(\rho,0)}{\partial z} = 2\rho^{|m|}\left(\dfrac{1}{\rho}\dfrac{d}{d\rho}\right)^{|m|+1} \\ \qquad \cdot \displaystyle\int_{\rho/a}^{1} \dfrac{\cosh\left(\alpha\sqrt{t^2-(\rho/a)^2}\right)}{\sqrt{t^2-(\rho/a)^2}}\, t f^{(1,2)}_m(t)dt, \\ \dfrac{\partial G_m(\rho,0)}{\partial z} = 2\rho^{|m|}\left(\dfrac{1}{\rho}\dfrac{d}{d\rho}\right)^{|m|+1} \\ \qquad \cdot \displaystyle\int_{\rho/a}^{1} \dfrac{\cosh\left(\alpha\sqrt{t^2-(\rho/a)^2}\right)}{\sqrt{t^2-(\rho/a)^2}}\, t g_m(t)dt, \end{cases}$$

valid for $0 \leqq \rho < a$.

Concerning the plane incident wave $(\underline{E}^i, \underline{H}^i)$, Meixner and Andrejewski [37] distinguish two cases according to the electric vector $\underline{E}^i$ being polarized perpendicular or parallel to the plane of incidence of the wave. For these two cases the foregoing results may be simplified considerably.

3.2. In this section the presented method is worked out in detail for the special case of a normally incident plane wave with rectangular components,

$$(3.30) \quad \underline{E}^i = \{-E, 0, 0\}e^{ikz},\; \underline{H}^i = \sqrt{\varepsilon/\mu}\{0, -E, 0\}e^{ikz}.$$

The corresponding Hertz vector $\underline{\Pi}^i$ will have the components,

$$(3.31) \qquad \Pi^i_x = (-\varepsilon E/k^2)e^{ikz},\; \Pi^i_y = 0.$$

Hence it follows from (3.7), (3.9), (3.12) that all functions $f^{(1,2)}_m(t)$ are equal to zero except the function $f^{(1)}_0(t)$. According to (3.15) the only coefficients U_m, which do not vanish, are given by

$$(3.32) \qquad kU_1 = -kU_{-1} = -\frac{a^2 f^{(1)}_0(1)}{a^2 g_0(1) - D^2 g_2(1)}.$$

So the only functions to be determined are $f^{(1)}_0(t)$, $g_0(t)$, $g_2(t)$. By means of the method of Bazer and Brown [2] the following Fredholm integral equations can be derived for the functions $f^{(1)}_0(t)$, $g_0(t)$,

$$(3.33)\quad f_0^{(1)}(t) = \frac{\varepsilon E a}{\pi k^2}\cosh \alpha t + \frac{1}{\pi i}\int_{-1}^{1} \frac{\sinh\big(\alpha(t-s)\big)}{t-s} f_0^{(1)}(s)ds,$$

$$(3.34)\quad g_0(t) = \frac{a}{\pi} + \frac{1}{\pi i}\int_{-1}^{1} \frac{\sinh\big(\alpha(t-s)\big)}{t-s} g_0(s)ds,$$

valid for $-1 \leqq t \leqq 1$. The integral equation (3.33) was solved by Hurd [22] yielding an expansion in powers of α for $f_0^{(1)}(t)$ up to and including terms of relative order α^{12}. The integral equation (3.34) is identical with Bazer and Hochstadt's [3] equation for their function $f_2^{(0)}(t)$ in the case of plane-wave excitation with $\gamma = \pi/2$. These authors derived an expansion in powers of α for the function $f_2^{(0)}(t)$ up to relative order α^5. We extended this expansion up to relative order α^8. A Fredholm integral equation for the function $g_2(t)/(1-t^2)^2$ follows from (2.46) with $m = 2$ and $u_2^{(2)}(\rho) = J_2(k\rho)/\rho^2$ to be substituted into (2.47). The same integral equation holds for the function $f_2^{(2)}(t)/(1-t^2)^2$, occurring in section 2.4, when $\gamma = \pi/2$. An expansion in powers of α was derived for the function $f_2^{(2)}(t)$ up to relative order α^6.

From the expansions for the functions $f_0^{(1)}(t)$, $g_0(t)$, $g_2(t)$ up to relative orders α^8, α^8 and α^6 respectively, corresponding results were found for the scattered field at a large distance from the disk and on the disk and for the scattering coefficient. These results were in complete agreement with the expansions stated by Bouwkamp [6]. Later on we extended the previous results by calculating one further term in the expansions of $f_0^{(1)}(t)$, $g_0(t)$, $g_2(t)$. Adding this term we obtained expansions for the scattered wave in the wave zone and the scattering coefficient, containing one extra term with respect to Bouwkamp's results. However, the latter expansions will not be presented here, because in section 3.3 still more extensive results will be derived.

3.3. In this section we present another solution to the diffraction problem in the case of normal incidence. Following the formulation of the problem as given by Lebedev and Skal'skaya [28] the incident wave ($\underline{E}^i$, $\underline{H}^i$) and the scattered wave ($\underline{E}^s$, $\underline{H}^s$) are derived from magnetic Hertz vectors $^*\underline{\Pi}^i$ and $^*\underline{\Pi}^s$ respectively, viz.

$$(3.35)\quad \underline{E} = i\omega \operatorname{rot} {}^*\underline{\Pi},\ \underline{H} = (1/\mu)\operatorname{rot}\operatorname{rot} {}^*\underline{\Pi}.$$

The Hertz vector $^*\underline{\Pi}^i$ corresponding to the incident wave (3.30) will have the rectangular components,

$$(3.36)\quad {}^*\Pi_x^i = 0,\ {}^*\Pi_y^i = (-E\sqrt{\varepsilon\mu}/k^2)e^{ikz},\ {}^*\Pi_z^i = 0.$$

According to Lebedev and Skal'skaya the components of the Hertz vector $^*\underline{\Pi}^s$ may be represented by

$$(3.37)\qquad {}^*\Pi^s_x = 0,\ {}^*\Pi^s_y = \Phi(\rho, z),\ {}^*\Pi^s_z = \Psi(\rho, z)\sin\varphi$$

where the functions $\Phi(\rho, z)$ and $\Psi(\rho, z)$ are required to satisfy the following conditions:

(i) $\Phi(\rho, z)$ and $\Psi(\rho, z)\sin\varphi$ are solutions of Helmholtz' equation, hence,

$$\left(\frac{\partial^2}{\partial\rho^2}+\frac{1}{\rho}\frac{\partial}{\partial\rho}+\frac{\partial^2}{\partial z^2}+k^2\right)\Phi = 0,\ \left(\frac{\partial^2}{\partial\rho^2}+\frac{1}{\rho}\frac{\partial}{\partial\rho}+\frac{\partial^2}{\partial z^2}+k^2-\frac{1}{\rho^2}\right)\Psi = 0;$$

(ii) $\Phi = 0$, $\partial\Psi/\partial z = 0$ when $z = 0$, $\rho > a$; this condition follows from the functions Φ and Ψ being odd and even in z respectively;
(iii) Φ, Ψ satisfy Sommerfeld's radiation condition at infinity;
(iv) $\partial\Phi/\partial z = C-E/(i\omega)$, $\Psi = C\rho$, when $z = 0$, $0 \leqq \rho < a$; C is an arbitrary constant which will follow from the edge condition;
(v) the edge condition prescribing the behaviour of the magnetic Hertz vector $^*\underline{\Pi}^s$ in the neighbourhood of the edge of the disk has the same form as the edge condition for the electric Hertz vector; therefore, according to Meixner [36] the edge condition can be formulated in the following way: Near the edge of the disk the components ${}^*\Pi^s_x$, ${}^*\Pi^s_y$, ${}^*\Pi^s_z$ remain finite. Further, in a point with coordinates (ρ, φ, z), where $\rho = a+\delta\cos\gamma$, $z = \delta\sin\gamma$, $\delta > 0$, $-\pi \leqq \gamma \leqq \pi$, the expressions

$$\frac{\partial}{\partial\gamma}({}^*\Pi^s_x\cos\varphi+{}^*\Pi^s_y\sin\varphi)-\tfrac{1}{2}{}^*\Pi^s_z,\ \frac{\partial}{\partial\gamma}{}^*\Pi^s_z+\tfrac{1}{2}({}^*\Pi^s_x\cos\varphi+{}^*\Pi^s_y\sin\varphi),$$

will have an expansion in powers of δ, in which no term with $\delta^{\frac{1}{2}}$ occurs. Hence, for the functions Φ and Ψ the edge condition will read: Φ and Ψ remain finite near the edge of the disk. In a point with coordinates (ρ, φ, z) as stated above, the expansions of

$$(3.38)\qquad \partial\Phi/\partial\gamma-\tfrac{1}{2}\Psi,\ \partial\Psi/\partial\gamma+\tfrac{1}{2}\Phi$$

in powers of δ do not contain a term with $\delta^{\frac{1}{2}}$.

Similar to Bazer and Hochstadt [3] we introduce the following integral representations for Φ and Ψ,

$$(3.39)\qquad \Phi(\rho, z) = \left(C-\frac{E}{i\omega}\right)\int_{-1}^{1}\frac{\exp\left(ik\sqrt{\rho^2+(z+iat)^2}\right)}{\sqrt{\rho^2+(z+iat)^2}}f(t)dt,$$

$$(3.40)\qquad \Psi(\rho, z) = C\rho\left(\frac{1}{\rho}\frac{\partial}{\partial\rho}\right)\int_{-1}^{1}\frac{\exp\left(ik\sqrt{\rho^2+(z+iat)^2}\right)}{\sqrt{\rho^2+(z+iat)^2}}g(t)dt,$$

valid for $z \geqq 0$. For $z \leqq 0$ we state $\Phi(\rho, z) = -\Phi(\rho, -z)$, $\Psi(\rho, z) = \Psi(\rho, -z)$. The unknown functions $f(t)$ and $g(t)$ are required to be odd and even functions of t respectively and to be regular in t for $|t| \leqq 1+\Delta$, $\Delta > 0$. Moreover $g(t)$ must satisfy the condition $g(1) = 0$.

It can easily be shown that the representations (3.39), (3.40) satisfy the conditions (i) to (iii) and the first part of condition (v). The conditions (iv) will lead to Fredholm integral equations for the functions $f(t)$ and $g(t)/(1-t^2)$. Using the method of Bazer and Brown [2] the following integral equation will hold for the function $f(t)$,

$$f(t) = \frac{a}{\pi i k} \sinh \alpha t + \frac{1}{\pi i} \int_{-1}^{1} \frac{\sinh\left(\alpha(t-s)\right)}{t-s} f(s) ds. \tag{3.41}$$

The same integral equation holds for Bazer and Brown's [2] function $f_1(t)$ in the case of normal plane-wave incidence. These authors derived an expansion in powers of α for their function $f_1(t)$ up to relative order α^{10}. A Fredholm integral equation for the function $g(t)/(1-t^2)$ follows from (2.46) with $m = 1$ and $u_2^{(1)}(\rho) = 1$ to be substituted into (2.47). Solving this integral equation by iteration, we obtained the following expansion for $g(t)$,

(3.42)
$$\begin{aligned} g(t) = -\frac{a^3}{\pi}(1-t^2)\Big[1 &- \alpha^2\left\{\frac{1}{3}-\frac{t^2}{6}\right\} - \frac{4i}{9\pi}\alpha^3 + \alpha^4\left\{\frac{2}{15}-\frac{t^2}{30}+\frac{t^4}{120}\right\} \\ &+ \frac{i}{\pi}\alpha^5\left\{\frac{68}{225}-\frac{2t^2}{45}\right\} - \alpha^6\left\{\left(\frac{17}{315}+\frac{16}{81\pi^2}\right) - \frac{4t^2}{315}+\frac{t^4}{840}-\frac{t^6}{5040}\right\} \\ &- \frac{i}{\pi}\alpha^7\left\{\frac{216}{1225}-\frac{46t^2}{1575}+\frac{t^4}{630}\right\} + \alpha^8\left\{\left(\frac{62}{2835}+\frac{416}{2025\pi^2}\right)\right. \\ &\left. - \left(\frac{29}{5670}+\frac{8}{405\pi^2}\right)t^2 + \frac{t^4}{2268} - \frac{t^6}{45360} + \frac{t^8}{362880}\right\} + O(\alpha^9)\Big]. \end{aligned}$$

Using formulae (2.6), (2.7), expansions in powers of δ may be derived for Φ, Ψ and their derivatives in a point with coordinates $\rho = a+\delta\cos\gamma$, $z = \delta \sin\gamma$, $\delta > 0$, $0 \leqq \gamma \leqq \pi$. When these expansions are substituted into (3.38), the term with $\delta^{\frac{1}{2}}$ must be set equal to zero in agreement with the condition (v), leading to the relation,

$$C = \frac{E}{i\omega} \frac{af(1)}{af(1)+ig'(1)} \tag{3.43}$$

where a prime denotes differentiation. Substituting the expansions for $f(t)$ and $g(t)$ into (3.43), we obtain the value of the coefficient C, viz.

$$
\begin{aligned}
C = -\frac{E}{i\omega}\Big[1 &+ \frac{2}{3}\alpha^2 + \frac{4i}{3\pi}\alpha^3 + \frac{2}{15}\alpha^4 + \frac{136i}{135\pi}\alpha^5 \\
&+ \left(\frac{4}{315} - \frac{32}{27\pi^2}\right)\alpha^6 + \frac{8i}{21\pi}\alpha^7 \\
&+ \left(\frac{2}{2835} - \frac{32}{25\pi^2}\right)\alpha^8 + O(\alpha^9)\Big].
\end{aligned}
\tag{3.44}
$$

According to Bazer and Hochstadt [3] the functions Φ and Ψ as given by (3.39), (3.40) assume the following asymptotic values at large distances from the disk,

$$
\begin{cases}
\Phi(R\sin\theta,\ R\cos\theta) \sim A(\theta)e^{ikR}/R, \\
\Psi(R\sin\theta,\ R\cos\theta) \sim B(\theta)e^{ikR}/R,
\end{cases}
\tag{3.45}
$$

where

$$
\begin{cases}
A(\theta) = -2\{C - E/(i\omega)\}\int_0^1 \sinh(\alpha t\cos\theta) f(t)dt, \\
B(\theta) = 2Cik\sin\theta\int_0^1 \cosh(\alpha t\cos\theta) g(t)dt,
\end{cases}
\tag{3.46}
$$

valid for $0 \leqq \theta \leqq \pi$ and large values of R. Low-frequency expansions in powers of α for the functions $A(\theta)$, $B(\theta)$ can be derived from (3.46).

Using spherical coordinates in accordance with (3.18), the components of $^*\underline{\Pi}^s$ considered in a point (R, θ, φ) are represented by the following asymptotic values,

$$
^*\Pi_r^s \sim \{A(\theta)\sin\theta + B(\theta)\cos\theta\}\sin\varphi\, e^{ikR}/R,
\tag{3.47}
$$

$$
^*\Pi_\theta^s \sim \{A(\theta)\cos\theta - B(\theta)\sin\theta\}\sin\varphi\, e^{ikR}/R,
\tag{3.48}
$$

$$
^*\Pi_\varphi^s \sim A(\theta)\cos\varphi\, e^{ikR}/R.
\tag{3.49}
$$

From (3.35) the leading terms of the components of the scattered field in the wave zone can be expressed in these asymptotic values, yielding ultimately the following low-frequency expansions,

$$
H_r^s \sim E_r^s \sim 0,
\tag{3.50}
$$

(3.51)

$$H_\theta^s \sim -\sqrt{\varepsilon/\mu}E_\varphi^s \sim (k^2/\mu)\,{}^*\Pi_\theta^s$$

$$\sim -\frac{4}{3\pi}\sqrt{\frac{\varepsilon}{\mu}}\,Ea\alpha^2\Bigg[1+\frac{8}{15}\left(1-\frac{5s^2}{16}\right)\alpha^2+\frac{16}{105}\left(1-\frac{21s^2}{32}+\frac{7s^4}{128}\right)\alpha^4$$
$$+\frac{128}{4725}\left\{\left(1-\frac{175}{6\pi^2}\right)-\frac{123s^2}{128}+\frac{99s^4}{512}-\frac{15s^6}{2048}\right\}\alpha^6$$
$$+\frac{512}{155925}\left\{\left(1-\frac{9317}{32\pi^2}\right)-\left(\frac{1265}{1024}-\frac{385}{8\pi^2}\right)s^2+\frac{1639s^4}{4096}-\frac{319s^6}{8192}\right.$$
$$\left.+\frac{55s^8}{65536}\right\}\alpha^8+\frac{8i}{9\pi}\alpha^3\left\{1+\frac{22}{25}\left(1-\frac{5s^2}{22}\right)\alpha^2\right.$$
$$\left.+\frac{7312}{18375}\left(1-\frac{385s^2}{914}+\frac{1575s^4}{58496}\right)\alpha^4\right\}+O(\alpha^9)\Bigg]\sin\varphi\, e^{ikR}/R,$$

(3.52)

$$H_\varphi^s \sim \sqrt{\varepsilon/\mu}\;E_\theta^s \sim (k^2/\mu)\,{}^*\Pi_\varphi^s$$

$$\sim -\frac{4}{3\pi}\sqrt{\frac{\varepsilon}{\mu}}\,Ea\alpha^2\Bigg[1+\frac{8}{15}\left(1-\frac{3s^2}{16}\right)\alpha^2+\frac{16}{105}\left(1-\frac{11s^2}{32}+\frac{3s^4}{128}\right)\alpha^4$$
$$+\frac{128}{4725}\left\{\left(1-\frac{175}{6\pi^2}\right)-\frac{69s^2}{128}+\frac{35s^4}{512}-\frac{5s^6}{2048}\right\}\alpha^6$$
$$+\frac{512}{155925}\left\{\left(1-\frac{9317}{32\pi^2}\right)-\left(\frac{783}{1024}-\frac{385}{16\pi^2}\right)s^2+\frac{635s^4}{4096}-\frac{85s^6}{8192}\right.$$
$$\left.+\frac{15s^8}{65536}\right\}\alpha^8+\frac{8i}{9\pi}\alpha^3\left\{1+\frac{22}{25}\left(1-\frac{5s^2}{44}\right)\alpha^2\right.$$
$$\left.+\frac{7312}{18375}\left(1-\frac{399s^2}{1828}+\frac{525s^4}{58496}\right)\alpha^4\right\}+O(\alpha^9)\Bigg]\cos\theta\cos\varphi\, e^{ikR}/R,$$

where $s=\sin\theta$. Compared with Bouwkamp's [6] results the expansions (3.51), (3.52) contain three new terms.

Finally we present the corresponding expansion for the scattering coefficient τ. According to (3.30) the energy incident on the disk will be given by $\frac{1}{2}\sqrt{\varepsilon/\mu}\,\pi a^2E^2$. Hence, using (3.23) the following formula holds for τ,

$$\tau=\lim_{R\to\infty}\int_0^{\pi}\int_0^{2\pi}\{|E_\theta^s|^2+|E_\varphi^s|^2\}R^2\sin\theta\; d\varphi\, d\theta/\pi a^2E^2. \tag{3.53}$$

Substituting the expansions (3.51), (3.52) we obtain the result,

$$\tau = \frac{128}{27\pi^2}\alpha^4 \left[1 + \frac{22}{25}\alpha^2 + \frac{7312}{18375}\alpha^4 + \left(\frac{60224}{496125} - \frac{64}{81\pi^2}\right)\alpha^6 + \left(\frac{35048192}{1260653625} - \frac{2464}{2025\pi^2}\right)\alpha^8 + O(\alpha^{10})\right]. \tag{3.54}$$

The first three terms of (3.54) are given by Bouwkamp [6], the last two terms of (3.54) are believed to be new.

§ 4. Diffraction of a scalar wave by a slit

4.1. Let x, y, z be rectangular coordinates. An infinite screen coinciding with the plane $z = 0$, contains a slit defined by $-\infty < x < \infty$, $-b < y < b$, $z = 0$. The screen will be perfectly soft or perfectly rigid. We are now concerned with the diffraction of a scalar wave $u(y, z)$, incident from $z < 0$, through the slit. Because the incident wave is independent of x, both the soft screen and the rigid screen diffraction problems will be two-dimensional. In the sequel these diffraction problems will be referred to as the first and second boundary value problem respectively. A time dependence of the form $e^{-i\omega t}$ is assumed throughout.

According to Bouwkamp [8] the diffraction problems may be formulated in the following way. In the case of the first boundary value problem, we have for the total field,

$$u_1(y, z) = \begin{cases} u(y, z) - u(y, -z) + \Phi_1(y, -z), & (z \leqq 0) \\ \Phi_1(y, z), & (z \geqq 0) \end{cases} \tag{4.1}$$

where Φ_1, to be defined for $z \geqq 0$ only, has the following properties:
(i) Φ_1 is a solution of Helmholtz' equation, $\Delta\Phi_1 + k^2\Phi_1 = 0$, when $z > 0$;
(ii) $\Phi_1 = 0$ on the screen i.e. when $z = 0$, $|y| > b$;
(iii) Φ_1 satisfies Sommerfeld's radiation condition at infinity;
(iv) $\partial\Phi_1/\partial z = \partial u/\partial z$ in the slit i.e. when $z = 0$, $|y| < b$;
(v) Φ_1 is everywhere finite;
(vi) grad Φ_1 is quadratically integrable over any domain of three-dimensional space, including the edge of the slit.

In the case of the second boundary value problem, the total field is given by

$$u_2(y, z) = \begin{cases} u(y, z) + u(y, -z) - \Phi_2(y, -z), & (z \leqq 0) \\ \Phi_2(y, z), & (z \geqq 0) \end{cases} \tag{4.2}$$

where Φ_2, also defined for $z \geqq 0$ only, has similar properties to Φ_1 except that (ii) and (iv) should be replaced by
(ii)′ $\partial \Phi_2/\partial z = 0$ on the screen i.e. when $z = 0$, $|y| > b$;
(iv)′ $\Phi_2 = u$ in the slit i.e. when $z = 0$, $|y| < b$.

We now present integral representations for the functions Φ_1, Φ_2 similar to those stated in the case of a circular opening by Bazer and Brown [2], Bazer and Hochstadt [3]. For that purpose we split the incident wave $u(y, z)$ and the functions $\Phi_{1,2}(y, z)$ in an even and an odd part with respect to the variable y, viz.

$$u(y, z) = u^e(y, z) + u^o(y, z), \tag{4.3}$$

$$\Phi_{1,2}(y, z) = \Phi^e_{1,2}(y, z) + \Phi^o_{1,2}(y, z), \tag{4.4}$$

where the superscripts e, o stand for even, odd respectively. Then the functions $\Phi^e_{1,2}(y, z)$, $\Phi^o_{1,2}(y, z)$ will be represented by the following integrals,

$$\Phi^e_{1,2}(y, z) = \int_{-1}^{1} H_0^{(1)}(k\sqrt{y^2+(z+ibt)^2})\, \frac{f^e_{1,2}(t)}{\sqrt{1-t^2}}\, dt, \tag{4.5}$$

$$\Phi^o_{1,2}(y, z) = \frac{\partial}{\partial y}\int_{-1}^{1} H_0^{(1)}(k\sqrt{y^2+(z+ibt)^2})\, \frac{f^o_{1,2}(t)}{\sqrt{1-t^2}}\, dt, \tag{4.6}$$

valid for $z \geqq 0$. The unknown functions $f^e_{1,2}(t)$, $f^o_{1,2}(t)$ are required to satisfy the following conditions:
(i) $f^e_1(t)$, $f^o_1(t)$ are odd functions of t; $f^e_2(t)$, $f^o_2(t)$ are even functions of t;
(ii) all functions are regular in t for $|t| \leqq 1+\Delta$, Δ arbitrary small and positive;
(iii)

$$f^o_1(1) = f^o_2(1) = 0. \tag{4.7}$$

The square root $\sqrt{y^2+(z+ibt)^2}$ is fixed by requiring that,

$$\lim_{z\to+0} \sqrt{y^2+(z+ibt)^2} = \begin{cases} e^{-\pi i/2}\sqrt{b^2t^2-y^2}, & \text{when } bt < -|y|, \\ \sqrt{y^2-b^2t^2}, & \text{when } -|y| < bt < |y|, \\ e^{\pi i/2}\sqrt{b^2t^2-y^2}, & \text{when } bt > |y|. \end{cases} \tag{4.8}$$

It can be shown that the functions $\Phi^e_{1,2}$, $\Phi^o_{1,2}$ as given by (4.5), (4.6) are indeed even and odd functions of y respectively and that they satisfy all the conditions (i) to (vi) listed above with the exception of the conditions (iv) and (iv)′. In order to verify the conditions (v) and (vi), the behaviour of the integrals (4.5), (4.6) near the edge of the slit may be investigated in a similar manner as presented in appendix I of Bazer and Brown's paper [2a]. It can be proved that in a point with coordinates $y = b+\delta \cos \gamma$, $z = \delta \sin \gamma$, $\delta > 0$, $0 \leqq \gamma \leqq \pi$, the following expansions hold for small values of δ,

$$
(4.9) \quad \begin{cases} \Phi_1^e = -2\sqrt{2}\, f_1^e(1) \left(\dfrac{\delta}{b}\right)^{\frac{1}{2}} \sin \tfrac{1}{2}\gamma + O\left[\dfrac{\delta}{b}\right], \\ \left(\dfrac{\delta}{b}\right)^{\frac{1}{2}} \dfrac{\partial \Phi_1^e}{\partial \delta} = -\dfrac{\sqrt{2}}{b} f_1^e(1) \sin \tfrac{1}{2}\gamma + O\left[\left(\dfrac{\delta}{b}\right)^{\frac{1}{2}}\right], \\ \left(\dfrac{\delta}{b}\right)^{\frac{1}{2}} \dfrac{1}{\delta} \dfrac{\partial \Phi_1^e}{\partial \gamma} = -\dfrac{\sqrt{2}}{b} f_1^e(1) \cos \tfrac{1}{2}\gamma + O\left[\left(\dfrac{\delta}{b}\right)^{\frac{1}{2}}\right], \end{cases}
$$

while similar results can be stated for the functions Φ_2^e, $\Phi_{1,2}^o$ and their derivatives.

In the following sections, starting from the conditions (iv) and (iv)′ Fredholm integral equations of the second kind will be derived for the functions $f_{1,2}^e(t)$, $f_{1,2}^o(t)$. Successively we treat the first and second boundary value problem for the case of an incident wave even in y and odd in y.

Similar to Bazer and Hochstadt [3] we introduce the following abbreviations,

$$
(4.10) \qquad u_1^{e,o}(y) = \partial u^{e,o}(y, 0)/\partial z, \quad u_2^{e,o}(y) = u^{e,o}(y, 0).
$$

It is assumed that the functions $u_{1,2}^e(y)$ and $u_{1,2}^o(y)$ are even and odd functions of y respectively and that these functions are regular in y for $|y| \leqq b(1+\Delta)$, $\Delta > 0$.

4.2. For the first boundary value problem with an incident wave even in y, the condition (iv) becomes,

$$
(4.11) \qquad \partial \Phi_1^e/\partial z = u_1^e(y), \text{ when } z = 0, \ |y| < b.
$$

Starting from (4.5), the derivative $\partial \Phi_1^e/\partial z$ may be represented by

$$
(4.12) \quad \frac{\partial \Phi_1^e}{\partial z} = \frac{1}{y} \frac{\partial}{\partial y} \int_{-1}^{1} H_0^{(1)}\left(k\sqrt{y^2+(z+ibt)^2}\right) \frac{(z+ibt) f_1^e(t)}{\sqrt{1-t^2}}\, dt,
$$

valid for $z > 0$. Assuming $0 \leqq y < b$, we let z approach zero and make use of (4.8) and [47], form. 3.7(8), 3.71(18). Then the condition (4.11) leads to the following equation,

$$
\begin{aligned}
&\frac{2ib}{y}\frac{d}{dy}\int_0^{y/b} H_0^{(1)}(\beta\sqrt{(y/b)^2-t^2})\,\frac{tf_1^e(t)}{\sqrt{1-t^2}}\,dt \\
(4.13)\quad &+\frac{2ib}{y}\frac{d}{dy}\int_{y/b}^{1}\left\{-\frac{2i}{\pi}K_0(\beta\sqrt{t^2-(y/b)^2})+I_0(\beta\sqrt{t^2-(y/b)^2})\right\} \\
&\qquad\cdot\frac{tf_1^e(t)}{\sqrt{1-t^2}}\,dt = u_1^e(y),
\end{aligned}
$$

valid for $0 \leqq y < b$, where $\beta = kb$. The second integral in the left-hand side of (4.13) is continued over the complete interval [0, 1]. At the same time we make the substitutions,

$$(4.14)\qquad (y/b)^2 = \eta,\quad t^2 = \xi,$$

then we obtain the relation,

$$
\begin{aligned}
&(d/d\eta)\int_0^{\eta} J_0(\beta\sqrt{\eta-\xi})F(\xi)d\xi \\
(4.15)\qquad &= (d/d\eta)\int_0^1 \{(-2i/\pi)K_0(\beta\sqrt{\xi-\eta})+I_0(\beta\sqrt{\xi-\eta})\}F(\xi)d\xi \\
&\qquad + (ib/4)u_1^e(b\sqrt{\eta})
\end{aligned}
$$

where

$$(4.16)\qquad F(\xi) = f_1^e(\sqrt{\xi})/(2\sqrt{1-\xi}).$$

The square root $\sqrt{\xi-\eta}$ in the right-hand side of (4.15) is defined by

$$(4.17)\qquad \sqrt{\xi-\eta} = \left\{\begin{array}{ll}\sqrt{\xi-\eta}, & \text{when } \xi > \eta, \\ e^{-\pi i/2}\sqrt{\eta-\xi}, & \text{when } \xi < \eta.\end{array}\right\}$$

In accordance with this definition the notation $\int$ is used, denoting that the path of integration has an infinitesimal indentation passing below the point $\xi = \eta$.

Let the right-hand side of (4.15) be called $G(\eta)$, then the equation (4.15) can be solved by means of the convolution theorem for Laplace transforms. A formal application of the Laplace transformation to (4.15) yields,

$$(4.18)\qquad e^{-\beta^2/(4p)}\mathfrak{L}\{F\} = \mathfrak{L}\{G\}$$

where $\mathfrak{L}\{F\}$ is defined by (2.14a). Inverting $\mathfrak{L}\{F\}$ from (4.18), we obtain the solution of (4.15) viz.

$$(4.19)\qquad F(\eta) = (d/d\eta)\int_0^{\eta} I_0(\beta\sqrt{\eta-\mu})G(\mu)d\mu.$$

([13], form. 4.14(25), 4.16(14) were used). The derivation of (4.19) is only valid when the Laplace transforms $\mathfrak{L}\{F\}$, $\mathfrak{L}\{G\}$

exist. However, by substituting (4.19) into the left-hand side of (4.15), it can easily be shown that the solution (4.19) is correct on the conditions assumed for the function $f_1^e(t)$.

Substituting $G(\eta)$ as given by (4.15) into (4.19) the resulting expression will consist of two terms. The first term may be reduced to

$$(4.20)\quad \begin{aligned}(d/d\eta)\int_0^\eta I_0(\beta\sqrt{\eta-\mu})d\mu \Big[(d/d\mu)\int_0^1 \Big\{(-2i/\pi)K_0(\beta\sqrt{\xi-\mu})\\ +I_0(\beta\sqrt{\xi-\mu})\Big\} F(\xi)d\xi\Big] = (d/d\eta)\int_0^1 M(\beta;\, \xi, \eta)F(\xi)d\xi\end{aligned}$$

where

$$(4.21)\quad \begin{aligned} M(\beta;\xi,\eta) &= (-2i/\pi)K_0(\beta\sqrt{\xi-\eta})+I_0(\beta\sqrt{\xi-\eta})\\ &+I_0(\beta\sqrt{\eta})\{(2i/\pi)K_0(\beta\sqrt{\xi})-I_0(\beta\sqrt{\xi})\}\\ &+\frac{\beta}{2}\int_0^\eta \frac{I_1(\beta\sqrt{\eta-\mu})}{\sqrt{\eta-\mu}}\Big\{-\frac{2i}{\pi}K_0(\beta\sqrt{\xi-\mu})+I_0(\beta\sqrt{\xi-\mu})\Big\}\, d\mu.\end{aligned}$$

First we determine $M(\beta;\, \xi, \eta)$ in the case $\xi > \eta$. We make use of Lommel's expansions (cf. [47], § 5.22),

$$(4.22)\quad \begin{cases} K_0(\beta\sqrt{\xi-\mu}) = \displaystyle\sum_{r=0}^{\infty} \frac{1}{r!}\left(\frac{\beta\mu}{2}\right)^r \frac{K_r(\beta\sqrt{\xi})}{\xi^{\frac{1}{2}r}},\\ I_0(\beta\sqrt{\xi-\mu}) = \displaystyle\sum_{r=0}^{\infty} \frac{(-1)^r}{r!}\left(\frac{\beta\mu}{2}\right)^r \frac{I_r(\beta\sqrt{\xi})}{\xi^{\frac{1}{2}r}},\end{cases}$$

valid for $0 \leqq \mu < \xi$, and the integral,

$$\int_0^\eta \frac{I_1(\beta\sqrt{\eta-\mu})}{\sqrt{\eta-\mu}}\, \mu^r d\mu = r!\left(\frac{2}{\beta}\right)^{r+1} \eta^{\frac{1}{2}r} I_r(\beta\sqrt{\eta}) - \frac{2}{\beta}\eta^r, \quad r = 0, 1, 2, \ldots,$$

which follows by an expansion of the Bessel functions I_1, I_r in power series. Another application of the expansions (4.22) will then yield the result,

$$(4.23)\quad \begin{aligned} M(\beta;\xi,\eta)\\ = \sum_{r=1}^{\infty} \eta^{\frac{1}{2}r} I_r(\beta\sqrt{\eta})\Big\{-\frac{2i}{\pi}\frac{K_r(\beta\sqrt{\xi})}{\xi^{\frac{1}{2}r}} + (-1)^r \frac{I_r(\beta\sqrt{\xi})}{\xi^{\frac{1}{2}r}}\Big\}.\end{aligned}$$

Differentiating this expression with respect to β, using [47], form. 3.71(5), (6), we obtain, (4.24) $(\partial/\partial\beta)M(\beta;\xi,\eta) = \eta^{\frac{1}{2}}I_1(\beta\sqrt{\eta})\{(2i/\pi)K_0(\beta\sqrt{\xi})-I_0(\beta\sqrt{\xi})\}.$

From (4.23) it follows easily that

$$M(0;\xi,\eta) = (-i/\pi)\sum_{r=1}^{\infty}(\eta/\xi)^r = (i/\pi)\log(1-\eta/\xi). \tag{4.25}$$

Hence we finally obtain the representation,

$$\begin{aligned} M(\beta;\xi,\eta) &= (i/\pi)\log|1-\eta/\xi| \\ &\quad + \int_0^{\beta}\eta^{\frac{1}{2}}I_1(u\sqrt{\eta})\{(2i/\pi)K_0(u\sqrt{\xi})-I_0(u\sqrt{\xi})\}du. \end{aligned} \tag{4.26}$$

In a similar manner it can be shown that in the case $\xi < \eta$, $M(\beta;\xi,\eta)$ is given by (4.26) enlarged with a term $+1$. Substituting the result for $M(\beta;\xi,\eta)$ into (4.20), we obtain

$$\begin{aligned} &\frac{d}{d\eta}\int_0^1 M(\beta;\xi,\eta)F(\xi)d\xi \\ &\quad = F(\eta)+\frac{i}{\pi}\oint_0^1\frac{F(\xi)}{\eta-\xi}d\xi + \int_0^1 N(\beta;\xi,\eta)F(\xi)d\xi \end{aligned} \tag{4.27}$$

where

$$N(\beta;\xi,\eta) = \tfrac{1}{2}\int_0^{\beta}I_0(u\sqrt{\eta})\{(2i/\pi)K_0(u\sqrt{\xi})-I_0(u\sqrt{\xi})\}u\,du. \tag{4.28}$$

The integral sign $\oint$ denotes that the Cauchy principal value of the corresponding integral is meant.

Ultimately, the solution (4.19) leads to the following equation,

$$\begin{aligned} \oint_0^1\frac{F(\xi)}{\xi-\eta}d\xi &= \frac{\pi b}{4}\Big[I_0(\beta\sqrt{\eta})u_1^e(0) \\ &\quad + \int_0^{\eta}I_0(\beta\sqrt{\eta-\mu})(d/d\mu)u_1^e(b\sqrt{\mu})d\mu\Big]-\pi i\int_0^1 N(\beta;\xi,\eta)F(\xi)d\xi. \end{aligned} \tag{4.29}$$

In the equation (4.29) we make the substitutions

$$\xi = s^2,\ \eta = t^2,\ \mu = u^2. \tag{4.30}$$

After some elementary calculations we obtain a singular integral equation for the function $f_1^e(t)$, viz.

(4.31)

$$\begin{aligned} \oint_{-1}^1\frac{f_1^e(s)}{s-t}\frac{ds}{\sqrt{1-s^2}} &= \frac{\pi b}{2}\Big[I_0(\beta t)u_1^e(0) \\ &\quad + \int_0^t I_0(\beta\sqrt{t^2-u^2})(d/du)u_1^e(bu)du\Big] + \int_{-1}^1 K(\beta;s,t)\frac{sf_1^e(s)}{\sqrt{1-s^2}}ds, \end{aligned}$$

valid for $-1 < t < 1$, where

$$K(\beta;\, s,\, t) = \int_0^\beta I_0(ut)K_0(us)u\,du. \tag{4.32}$$

The integral sign $⨏$ in (4.31) denotes that arg s in the corresponding integral has to be chosen according to

$$\arg s = 0 \text{ when } s > 0,\ \arg s = -\pi \text{ when } s < 0. \tag{4.33}$$

Further, we used [47], form. 3.71(18). The integral (4.32) may be evaluated explicitly, however the expression (4.32) is more suitable to expand in powers of β.

The integral equation (4.31) is again the aerofoil equation. Using the explicit solution of the latter equation as given by Tricomi [45], we are led to the following Fredholm integral equation of the second kind for the function $f_1^e(t)$,

$$f_1^e(t) = H_1^e(t) - \frac{1}{\pi^2} ⨏_{-1}^{1} K_1(\beta;\, s,\, t) \frac{s f_1^e(s)}{\sqrt{1-s^2}}\, ds, \tag{4.34}$$

valid for $-1 \leqq t \leqq 1$, where

$$\begin{aligned} H_1^e(t) = -\frac{b}{2\pi} ⨏_{-1}^{1} \frac{\sqrt{1-s^2}}{s-t}\, ds \Big[& I_0(\beta s)u_1^e(0) \\ & + \int_0^s I_0(\beta\sqrt{s^2-u^2})(d/du)u_1^e(bu)du \Big], \end{aligned} \tag{4.35}$$

$$K_1(\beta;\, s,\, t) = \int_0^\beta G_0(u,\, t)K_0(us)u\,du, \tag{4.36}$$

$$G_0(u,\, t) = ⨏_{-1}^{1} \frac{I_0(ur)}{r-t} \sqrt{1-r^2}\, dr. \tag{4.37}$$

Arg s in the right-hand side of (4.34) must be chosen as stated in (4.33). The arbitrary constant C occurring in the solution of the aerofoil equation vanishes, because $f_1^e(t)$ is odd in t.

For practical calculations the function $K_1(\beta;\, s,\, t)$ is replaced by

$$\bar{K}_1(\beta;\, s,\, t) = \tfrac{1}{2}\{K_1(\beta;\, |s|,\, t) + K_1(\beta;\, |s|e^{-\pi i},\, t)\}, \tag{4.38}$$

owing to the function $f_1^e(s)$ being odd in s. It can easily be shown that the function $s\bar{K}_1(\beta;\, s,\, t)$ is a continuous function of s for $-1 \leqq s \leqq 1$ and an odd regular function of t for all values of t. When β is small, $\bar{K}_1(\beta;\, s,\, t)$ will be of order $\beta^2 \log \beta$. Similarly, the function $u_1^e(y)$ being an even regular function of y for $|y| \leqq b(1+\Delta)$, it follows that the function $H_1^e(t)$ is an odd function of t, regular for $|t| \leqq 1+\Delta$. Finally, when β is sufficiently small, the integral equation (4.34) will have a unique continuous solution $f_1^e(t)$, this solution being odd in t. Substituting this function into the integral in the right-hand side of (4.34), this integral will be an entire function of t. Hence, the solution $f_1^e(t)$ will be regular in t for $|t| \leqq 1+\Delta$ and all conditions, assumed about the function $f_1^e(t)$, are satisfied.

4.3. Next we treat the second boundary value problem. In the case of an incident wave even in y, the condition (iv)′ becomes

$$\Phi_2^e = u_2^e(y), \text{ when } z = 0, \ |y| < b. \tag{4.39}$$

Assuming $0 \leqq y < b$, we let z approach zero in the integral representation (4.5) for Φ_2^e. Using (4.8) and [47], form. 3.7(8), 3.71(18), the condition (4.39) leads to the equation,

(4.40)

$$2\int_0^{y/b} H_0^{(1)}(\beta\sqrt{(y/b)^2-t^2})\,\frac{f_2^e(t)}{\sqrt{1-t^2}}\,dt$$
$$+2\int_{y/b}^1 \left\{-\frac{2i}{\pi}K_0(\beta\sqrt{t^2-(y/b)^2})+I_0(\beta\sqrt{t^2-(y/b)^2})\right\}\frac{f_2^e(t)}{\sqrt{1-t^2}}\,dt = u_2^e(y),$$

valid for $0 \leqq y < b$. The second integral in the left-hand side of (4.40) is again continued over the complete interval [0, 1]. At the same time we make the substitutions (4.14), then we obtain the relation,

(4.41)

$$\int_0^\eta J_0(\beta\sqrt{\eta-\xi})F(\xi)d\xi$$
$$= \int_0^1 \{(-2i/\pi)K_0(\beta\sqrt{\xi-\eta})+I_0(\beta\sqrt{\xi-\eta})\}F(\xi)d\xi - \tfrac{1}{2}u_2^e(b\sqrt{\eta})$$

where

$$F(\xi) = f_2^e(\sqrt{\xi})/(2\sqrt{\xi(1-\xi)}) \tag{4.42}$$

and the square root $\sqrt{\xi-\eta}$ in the right-hand side of (4.41) is defined as stated in (4.17).

Denoting the right-hand side of (4.41) by $G(\eta)$, it is obvious that the integral equation (4.41) only has a solution if $G(0) = 0$, viz.

(4.43) $\int_0^1 \{(-2i/\pi)K_0(\beta\sqrt{\xi})+I_0(\beta\sqrt{\xi})\}F(\xi)d\xi = \frac{1}{2}u_2^e(0).$

We substitute $\xi = s^2$ and use [47], form. 3.71(18), then (4.43) reduces to

$$\int_{-1}^{1} K_0(\beta s)\frac{f_2^e(s)}{\sqrt{1-s^2}}ds = \frac{\pi i}{2}u_2^e(0), \tag{4.44}$$

where arg s must be chosen in accordance with (4.33).

By means of the convolution theorem for Laplace transforms we derived the following solution to (4.41),

$$\begin{aligned} F(\eta) &= (d^2/d\eta^2)\int_0^\eta I_0(\beta\sqrt{\eta-\mu})G(\mu)d\mu \\ &= \frac{d}{d\eta}\left[G(\eta)+\frac{\beta}{2}\int_0^\eta \frac{I_1(\beta\sqrt{\eta-\mu})}{\sqrt{\eta-\mu}}G(\mu)d\mu\right], \end{aligned} \tag{4.45}$$

valid provided that the condition (4.44) is satisfied. Substituting $G(\mu)$ as given by (4.41) into (4.45), the first term of $G(\mu)$ will yield a contribution,

(4.46)

$$\begin{aligned} &(d/d\eta)\left[\int_0^1 \{(-2i/\pi)K_0(\beta\sqrt{\xi-\eta})+I_0(\beta\sqrt{\xi-\eta})\}F(\xi)d\xi\right. \\ &\left.+\frac{\beta}{2}\int_0^\eta \frac{I_1(\beta\sqrt{\eta-\mu})}{\sqrt{\eta-\mu}}d\mu\left[\int_0^1\left\{-\frac{2i}{\pi}K_0(\beta\sqrt{\xi-\mu})+I_0(\beta\sqrt{\xi-\mu})\right\}F(\xi)d\xi\right]\right] \\ &\qquad = (d/d\eta)\int_0^1 \bar{M}(\beta;\,\xi,\,\eta)F(\xi)d\xi \end{aligned}$$

where

$$\begin{aligned} \bar{M}(\beta;\,\xi,\,\eta) &= (-2i/\pi)K_0(\beta\sqrt{\xi-\eta})+I_0(\beta\sqrt{\xi-\eta}) \\ &+\frac{\beta}{2}\int_0^\eta \frac{I_1(\beta\sqrt{\eta-\mu})}{\sqrt{\eta-\mu}}\left\{-\frac{2i}{\pi}K_0(\beta\sqrt{\xi-\mu})+I_0(\beta\sqrt{\xi-\mu})\right\}d\mu. \end{aligned} \tag{4.47}$$

The function $\bar{M}(\beta;\,\xi,\,\eta)$ is closely related to $M(\beta;\,\xi,\,\eta)$ as given by (4.21).

The further reduction of the solution (4.45) runs along the same lines as described in section 4.2 for the first boundary value problem. Ultimately, we obtain the following singular integral equation for the function $f_2^e(t)$,

(4.48)

$$\oint_{-1}^{1} \frac{f_2^e(s)}{s-t} \frac{ds}{\sqrt{1-s^2}} = \frac{\pi i}{2} \Big[\beta I_1(\beta t) u_2^e(0) + (d/dt) u_2^e(bt) + \beta t \int_0^t \frac{I_1(\beta\sqrt{t^2-u^2})}{\sqrt{t^2-u^2}} (d/du) u_2^e(bu) du \Big] + \int_{-1}^{1} \bar{K}(\beta; s, t) \frac{s f_2^e(s)}{\sqrt{1-s^2}} ds,$$

valid for $-1 < t < 1$, where

$$\bar{K}(\beta; s, t) = \int_0^\beta I_1(ut) K_1(us) u du \tag{4.49}$$

and arg s is chosen according to (4.33). The aerofoil equation (4.48) may be solved, using Tricomi [45] and we are led to the following Fredholm integral equation of the second kind for the function $f_2^e(t)$,

$$f_2^e(t) = H_2^e(t) + C - \frac{1}{\pi^2} \int_{-1}^{1} K_2(\beta; s, t) \frac{s f_2^e(s)}{\sqrt{1-s^2}} ds, \tag{4.50}$$

valid for $-1 \leqq t \leqq 1$, where

$$H_2^e(t) = -\frac{i}{2\pi} \oint_{-1}^{1} \frac{\sqrt{1-s^2}}{s-t} ds \Big[\beta I_1(\beta s) u_2^e(0) + (d/ds) u_2^e(bs) + \beta s \int_0^s \frac{I_1(\beta\sqrt{s^2-u^2})}{\sqrt{s^2-u^2}} (d/du) u_2^e(bu) du \Big], \tag{4.51}$$

$$K_2(\beta; s, t) = \int_0^\beta G_1(u, t) K_1(us) u du, \tag{4.52}$$

$$G_1(u, t) = \oint_{-1}^{1} \frac{I_1(ur)}{r-t} \sqrt{1-r^2} dr. \tag{4.53}$$

C is an arbitrary constant which is to be determined by means of (4.44).

For practical calculations the function $K_2(\beta; s, t)$ is replaced by

$$\bar{K}_2(\beta; s, t) = \tfrac{1}{2}\{K_2(\beta; |s|, t) - K_2(\beta; |s|e^{-\pi i}, t)\} \operatorname{sign} s, \tag{4.54}$$

owing to the function $f_2^e(s)$ being even in s. It can easily be shown that the function $s\bar{K}_2(\beta; s, t)$ is a continuous function of s for $-1 \leqq s \leqq 1$ and an even regular function of t for all values of t. When β is small, $\bar{K}_2(\beta; s, t)$ will be of order β^2. From the function $u_2^e(y)$ being an even regular function of y for $|y| \leqq b(1+\Delta)$, it follows easily that the function $H_2^e(t)$ is an even function of t, regular for $|t| \leqq 1+\Delta$. When β is sufficiently small, the integral equation (4.50) will have a unique continuous solution $f_2^e(t)$, this solution being even in t. Substituting this function in the right-hand side of (4.50), it is obvious that $f_2^e(t)$ is regular in t for $|t| \leqq 1+\Delta$. Finally, when β is sufficiently small, the condition (4.44) determines the constant C uniquely, as can easily be shown.

4.4. For the first boundary value problem with an incident wave odd in y, the condition (iv) reads,

$$\partial\Phi_1^o/\partial z = u_1^o(y), \text{ when } z = 0,\ |y| < b. \tag{4.55}$$

Starting from the integral representation (4.6) for Φ_1^o, the condition (4.55) may be reduced to an equation similar to the equation (4.13), viz.

$$\begin{aligned} &\frac{d}{dy}\left(\frac{2ib}{y}\frac{d}{dy}\right)\int_0^{y/b} H_0^{(1)}(\beta\sqrt{(y/b)^2-t^2})\,\frac{tf_1^o(t)}{\sqrt{1-t^2}}\,dt \\ &\quad+\frac{d}{dy}\left(\frac{2ib}{y}\frac{d}{dy}\right)\int_{y/b}^1\left\{-\frac{2i}{\pi}K_0(\beta\sqrt{t^2-(y/b)^2}) \right.\\ &\quad\left.+I_0((\beta\sqrt{t^2-(y/b)^2})\right\}\frac{tf_1^o(t)}{\sqrt{1-t^2}}\,dt = u_1^o(y), \end{aligned} \tag{4.56}$$

valid for $0 \leqq y < b$. Integration of (4.56) with respect to y yields an equation which is of exactly the same form as (4.13). Hence, the method of section 4.2 can be applied leading ultimately to the following Fredholm integral equation of the second kind for the function $f_1^o(t)$,

$$f_1^o(t) = H_1^o(t)+CG_0(\beta, t)-\frac{1}{\pi^2}\oint_{-1}^1 K_1(\beta; s, t)\,\frac{sf_1^o(s)}{\sqrt{1-s^2}}\,ds, \tag{4.57}$$

valid for $-1 \leqq t \leqq 1$, where

$$H_1^o(t) = -\frac{b^2}{2\pi}\oint_{-1}^1 \frac{\sqrt{1-s^2}}{s-t}\,ds\left[\int_0^s I_0(\beta\sqrt{s^2-u^2})u_1^o(bu)du\right]. \tag{4.58}$$

$K_1(\beta; s, t)$, $G_0(\beta, t)$ are given by (4.36), (4.37). C is an arbitrary constant, which is to be determined by means of the condition (4.7).

The equation (4.56) can also be reduced in a direct manner similar to the procedures followed in sections 4.2, 4.3. Omitting the details of the derivation, we only state the resulting Fredholm integral equation of the second kind for the function $f_1^o(t)$, (4.59) $$f_1^o(t) = H_1^o(t)+Ct-\frac{1}{\pi^2}\int_{-1}^{1} K_2(\beta;\, s,\, t)\,\frac{tf_1^o(s)}{\sqrt{1-s^2}}\,ds,$$

valid for $-1 \leqq t \leqq 1$, where $H_1^o(t)$, $K_2(\beta; s, t)$ are given by (4.58), (4.52). The arbitrary constant C is again determined from the condition (4.7).

Strictly speaking the integral equation (4.59) is not a common Fredholm integral equation, because $K_2(\beta; s, t)$ becomes infinite of order $1/s$ when $s \to 0$ (cf. form. (4.52)). However, $f_1^o(t)$ being odd and regular in t, this function may be replaced by $t\bar{f}_1^o(t)$, leading to a correct Fredholm integral equation for the function $\bar{f}_1^o(t)$.

It can be shown that, if β is sufficiently small, the integral equations (4.57), (4.59) have a unique solution $f_1^o(t)$, satisfying all the conditions stated in section 4.1.

4.5. Finally we treat the second boundary value problem with an incident wave odd in y. Then, the condition (iv)′ reads,

(4.60) $$\Phi_2^o = u_2^o(y), \text{ when } z = 0,\ |y| < b.$$

Similar to section 4.4 the following pair of Fredholm integral equations of the second kind can be derived for the function $f_2^o(t)$,

(4.61) $$f_2^o(t) = H_2^o(t)+C+\frac{\beta}{\pi}\,C^*G_1(\beta, t)-\frac{1}{\pi^2}\int_{-1}^{1} K_2(\beta; s, t)\,\frac{sf_2^o(s)}{\sqrt{1-s^2}}\,ds,$$

(4.62) $$f_2^o(t) = H_2^o(t)+C-\frac{1}{\pi^2}\int_{-1}^{1} K_1(\beta; s, t)\,\frac{tf_2^o(s)}{\sqrt{1-s^2}}\,ds,$$

both valid for $-1 \leqq t \leqq 1$, where

(4.63)
$$H_2^o(t) = -\frac{ib}{2\pi}\oint_{-1}^{1}\frac{\sqrt{1-s^2}}{s-t}\,ds\left[u_2^o(bs)+\beta s\int_0^s\frac{I_1(\beta\sqrt{s^2-u^2})}{\sqrt{s^2-u^2}}\,u_2^o(bu)du\right].$$

$K_1(\beta; s, t)$, $K_2(\beta; s, t)$, $G_1(\beta, t)$ are defined by (4.36), (4.52), (4.53). The arbitrary constants C and C^* occurring in (4.61), are determined from the condition (4.7) and a condition related to (4.44), viz.

(4.64) $$\int_{-1}^{1} K_0(\beta s)\,\frac{f_2^o(s)}{\sqrt{1-s^2}}\,ds = -\pi C^*.$$

The arbitrary constant C in (4.62) is determined from the condition (4.7).

Again it can be shown, that for sufficiently small values of β the integral equations (4.61), (4.62) have a unique solution $f_2^o(t)$, satisfying all the conditions assumed in section 4.1.

4.6. In this section we present special integral representations for the transmitted field at a large distance from the slit and in the slit.

Consider a point with coordinates $y = R\cos\theta$, $z = R\sin\theta$ where $0 \leqq \theta \leqq \pi$. For large values of R the following asymptotic expansions hold (cf. [47], form. 7.2(1)),

$$\sqrt{y^2+(z+ibt)^2} \sim R+ibt\sin\theta+O(1/R), \tag{4.65}$$

$$H_0^{(1)}(k\sqrt{y^2+(z+ibt)^2}) \sim \sqrt{2/(\pi kR)} \cdot \exp(ikR-\beta t\sin\theta-\pi i/4)+O(1/R^{\frac{3}{2}}). \tag{4.66}$$

Hence, at a large distance from the slit, the functions $\Phi_{1,2}^e$, $\Phi_{1,2}^o$ as given by (4.5), (4.6) will assume the following asymptotic values,

$$\begin{aligned} \Phi_{1,2}^e(R\cos\theta,\ R\sin\theta) &\sim A_{1,2}^e(\theta)e^{ikR}/\sqrt{R}, \\ \Phi_{1,2}^o(R\cos\theta,\ R\sin\theta) &\sim A_{1,2}^o(\theta)e^{ikR}/\sqrt{R}, \end{aligned} \tag{4.67}$$

where

$$A_1^e(\theta) = -2\sqrt{2/(\pi k)}\,e^{-\pi i/4}\int_0^1 \sinh(\beta t\sin\theta)\frac{f_1^e(t)}{\sqrt{1-t^2}}\,dt, \tag{4.68}$$

$$A_2^e(\theta) = 2\sqrt{2/(\pi k)}\,e^{-\pi i/4}\int_0^1 \cosh(\beta t\sin\theta)\frac{f_2^e(t)}{\sqrt{1-t^2}}\,dt, \tag{4.69}$$

$$A_1^o(\theta) = -2\sqrt{2k/\pi}\,e^{\pi i/4}\cos\theta\int_0^1 \sinh(\beta t\sin\theta)\frac{f_1^o(t)}{\sqrt{1-t^2}}\,dt, \tag{4.70}$$

$$A_2^o(\theta) = 2\sqrt{2k/\pi}\,e^{\pi i/4}\cos\theta\int_0^1 \cosh(\beta t\sin\theta)\frac{f_2^o(t)}{\sqrt{1-t^2}}\,dt. \tag{4.71}$$

The following integral representations can be derived for the transmitted field or its normal derivative in the slit,

$$\Phi_1^e(y,0) = -2\int_{y/b}^1 I_0(\beta\sqrt{t^2-(y/b)^2})\frac{f_1^e(t)}{\sqrt{1-t^2}}\,dt, \tag{4.72}$$

$$(4.73)\quad \partial\Phi_2^e(y,0)/\partial z = -\frac{2ib}{y}\frac{d}{dy}\int_{y/b}^{1} I_0(\beta\sqrt{t^2-(y/b)^2})\,\frac{tf_2^e(t)}{\sqrt{1-t^2}}\,dt,$$

$$(4.74)\quad \Phi_1^o(y,0) = -2\frac{d}{dy}\int_{y/b}^{1} I_0(\beta\sqrt{t^2-(y/b)^2})\,\frac{f_1^o(t)}{\sqrt{1-t^2}}\,dt,$$

$$(4.75)\quad \partial\Phi_2^o(y,0)/\partial z = -\frac{d}{dy}\left(\frac{2ib}{y}\frac{d}{dy}\right)\int_{y/b}^{1} I_0(\beta\sqrt{t^2-(y/b)^2})\,\frac{tf_2^o(t)}{\sqrt{1-t^2}}\,dt,$$

valid for $-b < y < b$.

4.7. The present method has been worked out for the case of an obliquely incident plane wave, described by the wave function

$$(4.76)\qquad u(y, z) = \exp\{ik(\alpha_0 y+\sqrt{1-\alpha_0^2}z)\}$$

where $\alpha_0 = \cos\theta_0$, $0 \leqq \theta_0 \leqq \pi$. According to (4.3) the even and odd part of $u(y, z)$ are given by

$$(4.77)\qquad \begin{aligned} u^e(y, z) &= \cos(k\alpha_0 y)\exp(ik\sqrt{1-\alpha_0^2}z),\\ u^o(y, z) &= i\sin(k\alpha_0 y)\exp(ik\sqrt{1-\alpha_0^2}z). \end{aligned}$$

Substituting the corresponding functions $u_{1,2}^e(y)$, $u_{1,2}^o(y)$ into (4.35), (4.51), (4.58), (4.63) the Fredholm integral equations for the functions $f_{1,2}^e(t)$, $f_{1,2}^o(t)$ may be solved by iteration yielding expansions in powers of β for these functions, the coefficients of these powers being dependent on $\log\beta$. In this manner we calculated expansions up to orders β^9 and β^8 for the functions $f_1^e(t)$, $f_1^o(t)$ and $f_2^e(t)$, $f_2^o(t)$ respectively. By means of (4.68) to (4.75) similar expansions have been derived for the transmitted field at a large distance from the slit and in the slit. The various results, which will not be presented here, are, apart from some slight deviations, in agreement with Millar's [38] values.

The transmission coefficients t_1, t_2 for the first and second boundary value problem are defined to be the ratio of the energy transmitted through the slit to the incident energy with the slit's area as basis. The following formulae hold for $t_{1,2}$,

$$(4.78)\qquad \begin{aligned} t_{1,2} &= \int_0^{\pi/2}\{|A_{1,2}^e(\theta)|^2+|A_{1,2}^o(\theta)|^2\}d\theta/(b\sin\theta_0)\\ &= \mathrm{Re}\,[\sqrt{\pi i/(2k)}\{A_{1,2}^e(\theta_0)+A_{1,2}^o(\theta_0)\}]/(b\sin\theta_0). \end{aligned}$$

Substituting the expansions for $A_{1,2}^e(\theta)$, $A_{1,2}^o(\theta)$, we derived the following expansions for t_1, t_2,

$$t_1 = \frac{\pi^2 \beta^3}{32} \sin \theta_0 \Bigg[1+\beta^2 \left(-\frac{q}{2} + \frac{5}{16} - \frac{\alpha_0^2}{4} \right)$$

$$+ \beta^4 \left\{ \left(\frac{3q^2}{16} - \frac{7q}{32} + \frac{109}{1536} - \frac{\pi^2}{64} \right) + \left(\frac{q}{8} - \frac{17}{256} \right) \alpha_0^2 + \frac{5\alpha_0^4}{192} \right\}$$

$$+ \beta^6 \left\{ \left(-\frac{q^3}{16} + \frac{27q^2}{256} - \frac{401q}{6144} + \frac{\pi^2 q}{64} + \frac{4391}{294912} - \frac{9\pi^2}{1024} \right) \right. \tag{4.79}$$

$$+ \left(-\frac{3q^2}{64} + \frac{13q}{256} - \frac{187}{12288} + \frac{\pi^2}{256} \right) \alpha_0^2$$

$$\left. + \left(-\frac{5q}{384} + \frac{37}{6144} \right) \alpha_0^4 - \frac{7\alpha_0^6}{4608} \right\} + \dots \Bigg],$$

$$t_2 = \frac{\pi^2}{(\pi^2+4q^2)\beta \sin \theta_0} \Bigg[1+\beta^2 \left(\frac{1}{4} - \frac{\alpha_0^2}{2} \right)$$

$$+ \beta^4 \left\{ \left(\frac{3}{256} + \frac{q}{16(\pi^2+4q^2)} \right) + \left(\frac{q^2}{8} - \frac{3}{32} + \frac{\pi^2}{32} \right) \alpha_0^2 + \frac{3\alpha_0^4}{32} \right\}$$

$$+ \beta^6 \left\{ \left(-\frac{29}{9216} + \frac{q}{64(\pi^2+4q^2)} \right) \right. \tag{4.80}$$

$$+ \left(\frac{q^3}{16} - \frac{q^2}{128} + \frac{\pi^2 q}{64} + \frac{1}{512} - \frac{\pi^2}{512} - \frac{q}{32(\pi^2+4q^2)} \right) \alpha_0^2$$

$$\left. + \left(-\frac{q^2}{32} + \frac{5}{384} - \frac{\pi^2}{128} \right) \alpha_0^4 - \frac{5\alpha_0^6}{576} \right\} + \dots \Bigg]$$

in which $q = \log (\beta\gamma/4)$ and $\log \gamma = 0.577215 \dots$ (Euler's constant). The results (4.79), (4.80) are in complete agreement with Millar's [38] values. Moreover, the result (4.80) for t_2 was checked by an independent calculation using the method of Jones and Noble [24].

§ 5. The circular wing in steady incompressible flow

5.1. Consider an aerofoil of circular planform moving with constant velocity U in an incompressible and non-viscous medium. Rectangular coordinates x, y, z with coordinate axes fixed to the wing are used. The positive direction of the x-axis is taken opposite to the direction of motion of the wing; the y-axis is taken in the spanwise direction. The projection of the aerofoil on the plane $z = 0$ is a circle having radius a with its centre at the origin of the coordinates.

Following the formulation of the boundary value problem as presented by van Spiegel [44], the velocity vector $\underline{q}$ of the medium with respect to the x, y, z-system is derived from a perturbation velocity potential Φ, viz.

$$\underline{q} = \underline{U} + \operatorname{grad} \Phi. \tag{5.1}$$

In the same way, the acceleration vector of the medium may be derived from an acceleration potential Ψ. In linearized aerofoil theory, for the case of steady flow, the potentials Φ and Ψ are connected by the relations,

$$\Psi = U\partial\Phi/\partial x, \quad \Phi(x, y, z) = (1/U) \int_{-\infty}^{x} \Psi(\xi, y, z) d\xi. \tag{5.2}$$

The acceleration potential Ψ is related to the pressure p, viz.

$$\Psi = (p_0 - p)/\rho_0 \tag{5.3}$$

where p_0 and ρ_0 denote the pressure in the undisturbed medium and the density of the medium respectively.

The boundary value problem can now be formulated in terms of Φ and Ψ. The potentials Φ and Ψ are required to satisfy the following conditions:

(i) Φ, Ψ are solutions of Laplace's equation, viz.

$$\Delta\Phi \equiv \frac{\partial^2\Phi}{\partial x^2} + \frac{\partial^2\Phi}{\partial y^2} + \frac{\partial^2\Phi}{\partial z^2} = 0, \quad \Delta\Psi = 0; \tag{5.4}$$

(ii) Φ, Ψ are continuous outside the circular disk $z = 0$, $x^2+y^2 \leqq a^2$, except that Φ possesses a discontinuity across the linearized wake i.e. the surface determined by $x > \sqrt{a^2-y^2}$, $|y| \leqq a$, $z = 0$;

(iii) $\Psi(x, y, z) = 0$ for (x, y, z) at infinity;

(iv) $\partial\Phi/\partial z = w(x, y)$ for $z = \pm 0$, $x^2+y^2 < a^2$;

(v) grad Φ, Ψ are quadratically integrable over any domain of three-dimensional space, including the edges of the circular disk and of the linearized wake; according to the Kutta condition the pressure p and therefore (cf. (5.3)) the acceleration potential Ψ must remain finite at the trailing edge of the circular disk.

The downwash distribution $w(x, y)$ will be a given function. At the present formulation the so-called lifting problem is considered. From the condition (iv) it is obvious that Φ is an odd function of z. Owing to (5.2) and (ii) Ψ too will be an odd function of z and $\Psi = 0$, when $z = 0$, $x^2+y^2 > a^2$. The Kutta condition will determine the discontinuity of Φ across the linearized wake.

We introduce cylindrical coordinates ρ, φ, z with

$$(5.5) \qquad x = \rho \cos \varphi,\ y = \rho \sin \varphi,\ -\pi \leqq \varphi < \pi.$$

The velocity potential $\Phi(\rho, \varphi, z)$ is split into a regular part Φ^{reg} and a singular part Φ^{sing},

$$(5.6) \qquad \Phi = \Phi^{\text{reg}} + \Phi^{\text{sing}}.$$

Concerning $\Phi^{\text{reg}}(\rho, \varphi, z)$, this function is required to satisfy the following conditions:
(i) Φ^{reg} is a solution of Laplace's equation, viz. $\Delta\Phi^{\text{reg}} = 0$;
(ii) $\Phi^{\text{reg}} = 0$, when $z = 0$, $\rho > a$;
(iii) $\Phi^{\text{reg}} = 0$ for (ρ, φ, z) at infinity;
(iv) $\partial\Phi^{\text{reg}}/\partial z = w(\rho, \varphi)$, when $z = 0$, $0 \leqq \rho < a$;
(v) Φ^{reg} is everywhere finite.
Hence $\Phi^{\text{reg}}(\rho, \varphi, z)$ will be an odd function of z.

Now we expand $w(\rho, \varphi)$ and $\Phi^{\text{reg}}(\rho, \varphi, z)$ in Fourier series with respect to φ. It is assumed that the downwash distribution $w(\rho, \varphi)$ is even in φ, hence the Fourier series for $w(\rho, \varphi)$ and $\Phi^{\text{reg}}(\rho, \varphi, z)$ may be represented by

$$(5.7) \quad w(\rho, \varphi) = \sum_{n=0}^{\infty} w_n(\rho) \cos n\varphi,\ \Phi^{\text{reg}}(\rho, \varphi, z) = \sum_{n=0}^{\infty} \Phi_n^{\text{reg}}(\rho, z) \cos n\varphi.$$

The case of a downwash distribution $w(\rho, \varphi)$ odd in φ will be treated in section 5.5. Further we assume that the functions $w_n(\rho)/\rho^n$ are even functions of ρ, regular for $|\rho| \leqq a(1+\Delta)$ with Δ arbitrary small, positive and that the series (5.7) are in fact finite sums i.e. the functions $w_n(\rho)$ will vanish for n sufficiently large. For the downwash distributions which are of interest in practice, these assumptions are certainly fulfilled.

Following Bazer and Hochstadt [3] we state the integral representation,

$$(5.8) \quad \Phi_n^{\text{reg}}(\rho, z) = \rho^n \left(\frac{1}{\rho}\frac{\partial}{\partial\rho}\right)^n \int_{-1}^{1} \frac{f_n(t)}{\sqrt{\rho^2+(z+iat)^2}}\, dt,\ n = 0, 1, 2, \ldots,$$

valid for $z \geqq 0$. For $z \leqq 0$ we define, $\Phi_n^{\text{reg}}(\rho, z) = -\Phi_n^{\text{reg}}(\rho, -z)$. Compared with Bazer and Hochstadt's original integral representation the wave number k has been set equal to zero. The unknown function $f_n(t)$ is required to be an odd function of t, to be regular in t for $|t| \leqq 1+\Delta$, $\Delta > 0$ and to satisfy the conditions,

$$(5.9) \qquad d^k f_n(1)/dt^k = 0,\ \text{for } k = 0, 1, \ldots, n-1.$$

It can easily be shown that Φ_n^{reg} as given by (5.8) satisfies all the conditions (i) to (v) except the condition (iv), which may be written as

$$\partial\Phi_n^{\text{reg}}/\partial z = w_n(\rho), \text{ when } z = 0,\ 0 \leqq \rho < a. \tag{5.10}$$

According to § 2, section 2.2 the condition (5.10) leads to the Fredholm integral equation (2.29) (m being replaced by n) for the function $f_n(t)/(1-t^2)^n$ with $u_1^{(n)}(\rho) = w_n(\rho)/\rho^n$ to be substituted into (2.30). However, because k has been set equal to zero, the parameter α and the kernel of the integral equation will vanish, hence, the function $f_n(t)$ can be solved immediately,

$$\begin{aligned} f_n(t) = {} & \frac{(-1)^n i a^{n+2}}{2^n \pi\sqrt{\pi}\,\Gamma(n+\frac{1}{2})}(1-t^2)^n \oint_{-1}^{1} \frac{1}{s-t}\,\frac{ds}{(1-s^2)^n} \\ & \cdot \left[\int_{|s|}^{1} (u^2-s^2)^{n-\frac{1}{2}}\,\frac{w_n(au)}{u^{n-1}}\,du\right]. \end{aligned} \tag{5.11}$$

The result (5.11) being rather complicated, we present another expression for $f_n(t)$. Using Bazer and Hochstadt's [3] original method, the condition (5.10) leads to an integral equation for the function $f_n(t)$. When α is set equal to zero in this integral equation, it can easily be derived that

$$\left(\frac{1}{t}\frac{d}{dt}\right)^n \left\{\frac{f_n(t)}{t}\right\} = \frac{a^{n+2}}{\pi i t^{2n+1}} \int_0^t \frac{u^{n+1} w_n(au)}{\sqrt{t^2-u^2}}\,du. \tag{5.12}$$

The equivalence of (5.11) and (5.12) may be verified in an independent manner. Finally, according to Bazer and Hochstadt [3] the velocity potential on the disk is given by

$$\begin{aligned} \Phi_n^{\text{reg}}(\rho, +0) & = \frac{2\rho^n}{ia}\left(\frac{1}{\rho}\frac{d}{d\rho}\right)^n \int_{\rho/a}^{1} \frac{f_n(t)}{\sqrt{t^2-(\rho/a)^2}}\,dt \\ & = \frac{2\rho^n}{ia^{2n+1}} \int_{\rho/a}^{1} \frac{t}{\sqrt{t^2-(\rho/a)^2}} \left(\frac{1}{t}\frac{d}{dt}\right)^n \left\{\frac{f_n(t)}{t}\right\} dt, \end{aligned} \tag{5.13}$$

valid for $0 \leqq \rho \leqq a$, where the latter result has been derived by integration by parts using the conditions (5.9).

Similar to (5.6) the acceleration potential Ψ will be split according to

$$\Psi = \Psi^{\text{reg}} + \Psi^{\text{sing}} \tag{5.14}$$

where Ψ^{reg} is defined by (compare form. (5.2))

$$\Psi^{\text{reg}} = U\partial\Phi^{\text{reg}}/\partial x. \tag{5.15}$$

Contrary to its name Ψ^{reg} becomes infinite at the edge $\rho = a$, $z = 0$ of the disk. According to (2.6) the following expansion holds for $\Psi^{\text{reg}}(\rho, \varphi, z)$ when $\rho = a+\delta \cos \gamma$, $z = \delta \sin \gamma$, $\delta > 0$, $-\pi \leqq \gamma \leqq \pi$,

$$\Psi^{\text{reg}} = U \sum_{n=0}^{\infty} \left[\left\{ \frac{\partial \Phi_n^{\text{reg}}}{\partial \delta} \cos \gamma - \frac{1}{\delta} \frac{\partial \Phi_n^{\text{reg}}}{\partial \gamma} \sin \gamma \right\} \cos n\varphi \cos \varphi \right.$$

$$(5.16) \qquad \left. + \frac{n}{\rho} \Phi_n^{\text{reg}} \sin n\varphi \sin \varphi \right]$$

$$= Ui\sqrt{2} \left(\frac{\delta}{a} \right)^{-\frac{1}{2}} \sin \tfrac{1}{2}\gamma \cos \varphi \sum_{n=0}^{\infty} \frac{f_n^{(n)}(1)}{a^{n+2}} \cos n\varphi + O(1),$$

valid for small values of δ. $f_n^{(n)}(1)$, denoting the derivative $d^n f_n(1)/dt^n$ can be represented by

$$(5.17) \qquad f_n^{(n)}(1) = \frac{a^{n+2}}{\pi i} \int_0^1 \frac{u^{n+1} w_n(au)}{\sqrt{1-u^2}} \, du,$$

owing to (5.9), (5.12).

The function $\Psi^{\text{sing}}(\rho, \varphi, z)$ is required to satisfy the following conditions:

(i) Ψ^{sing} is a solution of Laplace's equation, viz. $\Delta \Psi^{\text{sing}} = 0$;

(ii) $\Psi^{\text{sing}} = 0$, when $z = 0$, $\rho > a$;

(iii) $\partial \Psi^{\text{sing}} / \partial z = 0$, when $z = 0$, $0 \leqq \rho < a$;

(iv) Ψ^{sing} becomes infinite at the edge $\rho = a$, $z = 0$ of the disk; Ψ^{sing} is quadratically integrable over any domain of three-dimensional space, including the edge of the disk.

$\Psi^{\text{sing}}(\rho, \varphi, z)$ is expanded in a Fourier series,

$$(5.18) \qquad \Psi^{\text{sing}}(\rho, \varphi, z) = \sum_{n=0}^{\infty} \Psi_n^{\text{sing}}(\rho, z) \cos n\varphi.$$

Owing to the condition (iv) we modify Bazer and Hochstadt's integral representation by stating,

$$(5.19) \qquad \Psi_n^{\text{sing}}(\rho, z) = \frac{\partial}{\partial z} \rho^n \left(\frac{1}{\rho} \frac{\partial}{\partial \rho} \right)^n \int_{-1}^{1} \frac{g_n(t)}{\sqrt{\rho^2 + (z+iat)^2}} \, dt, \qquad n = 0, 1, 2, \ldots,$$

valid for $z \geqq 0$, whereas for $z \leqq 0$ we define $\Psi_n^{\text{sing}}(\rho, z) = -\Psi_n^{\text{sing}}(\rho, -z)$. The unknown function $g_n(t)$ is required to be an even function of t, to be regular in t for $|t| \leqq 1+\Delta$, $\Delta > 0$ and to satisfy the conditions,

$$(5.20) \qquad d^k g_n(1)/dt^k = 0 \text{ for } k = 0, 1, \ldots, n-1.$$

It can easily be shown that the representation (5.19) satisfies the conditions (i), (ii), (iv). The condition (iii) which may be written as $\partial \Psi_n^{\text{sing}}/\partial z = 0$ for $z = 0$, $0 \leqq \rho < a$, can be reduced to

$$\lim_{z\to+0} \left(\frac{\partial^2}{\partial\rho^2} + \frac{1}{\rho}\frac{\partial}{\partial\rho} - \frac{n^2}{\rho^2}\right) \rho^n \left(\frac{1}{\rho}\frac{\partial}{\partial\rho}\right)^n \int_{-1}^{1} \frac{g_n(t)}{\sqrt{\rho^2+(z+iat)^2}}\, dt = 0,$$

hence,

$$\lim_{z\to+0} \rho^n \left(\frac{1}{\rho}\frac{\partial}{\partial\rho}\right)^n \int_{-1}^{1} \frac{g_n(t)}{\sqrt{\rho^2+(z+iat)^2}}\, dt = \bar{C}_n \rho^n, \tag{5.21}$$

valid for $0 \leqq \rho < a$, where $\bar{C}_n$ is an arbitrary constant. According to § 2, section 2.3 the condition (5.21) leads to the Fredholm integral equation (2.46) (m being replaced by n) for the function $g_n(t)/(1-t^2)^n$ with $u_2^{(n)}(\rho) = \bar{C}_n$ to be substituted into (2.47). However, the parameter α being equal to zero, the function $g_n(t)$ can be solved immediately from the integral equation, viz.

$$g_n(t) = C_n(1-t^2)^n \tag{5.22}$$

where C_n is an arbitrary constant (related to $\bar{C}_n$). Substituting this value of $g_n(t)$ into (5.19) the resulting integral can be evaluated explicitly. Introducing oblate spheroidal coordinates μ, η defined by

$$\rho = a\sqrt{(1-\mu^2)(1+\eta^2)},\quad z = a\mu\eta,\quad -1 \leqq \mu \leqq 1,\quad \eta \geqq 0, \tag{5.23}$$

we derived the following formula for $\Psi_n^{\text{sing}}(\rho, z)$,

$$\Psi_n^{\text{sing}}(\rho, z) = A_n U^2 \frac{\mu}{\mu^2+\eta^2} \left(\frac{1-\mu^2}{1+\eta^2}\right)^{\frac{1}{2}n}, \tag{5.24}$$

where the dimensionless constant A_n is given by

$$A_n = \frac{(-2)^{n+1} n!}{a^{n+2} U^2} C_n. \tag{5.25}$$

A proof of (5.24) will not be given here. It may be remarked that the functions $\Psi_n^{\text{sing}}(\rho, z)$ agree with Kinner's [25] potential functions of the second kind.

In the plane $z = 0$ and in the neighbourhood of the edge $\rho = a$, $z = 0$ of the disk, the following special results hold,

$$\Psi_n^{\text{sing}}(\rho, +0) = A_n U^2 \frac{(\rho/a)^n}{\sqrt{1-(\rho/a)^2}}, \text{ for } 0 \leqq \rho < a; \tag{5.26}$$

$$(5.27)\quad \partial\Psi_n^{\text{sing}}(\rho, 0)/\partial z = A_n \frac{U^2}{a} \frac{1}{(\rho/a)^n\{(\rho/a)^2-1\}^{\frac{3}{2}}}, \text{ for } \rho > a;$$

$$(5.28)\quad \Psi_n^{\text{sing}}(a+\delta\cos\gamma, \delta\sin\gamma) = \frac{A_n}{\sqrt{2}} U^2 \left(\frac{\delta}{a}\right)^{-\frac{1}{2}} \sin\tfrac{1}{2}\gamma + O(1),$$

valid for $-\pi \leqq \gamma \leqq \pi$, δ sufficiently small, positive. The latter result can be derived from formula (2.7).

The part Φ^{sing} of the velocity potential Φ, introduced in (5.6), will be derived from Ψ^{sing} according to (compare form. (5.2))

$$(5.29)\qquad \Phi^{\text{sing}}(x, y, z) = (1/U)\int_{-\infty}^{x} \Psi^{\text{sing}}(\xi, y, z)d\xi.$$

We are especially interested in the normal derivative $\partial\Phi^{\text{sing}}/\partial z$ on the disk $z = 0$, $x^2+y^2 < a^2$. Differentiating (5.29) with respect to z and taking the limit for $z \to +0$, assuming $x^2+y^2 < a^2$, the resulting integral will be divergent. However, it can be shown that the common integral sign may be replaced by the sign $^*\!\int$ denoting that the finite part (in the sense of Hadamard) of the divergent integral is meant. Hence, we obtain the result,

$$(5.30)\quad \partial\Phi^{\text{sing}}(x, y, +0)/\partial z = (1/U)\,{}^*\!\!\int_{-\infty}^{-\sqrt{a^2-y^2}} \partial\Psi^{\text{sing}}(\xi, y, 0)/\partial z\, d\xi$$

where $x^2+y^2 < a^2$. Substituting for $\partial\Psi^{\text{sing}}/\partial z$ its Fourier series with the nth term given by (5.27), the integration will be performed term by term. Therefore we consider the integral,

$$(5.31)\qquad R_n(y) = {}^*\!\!\int_{-\infty}^{-\sqrt{a^2-y^2}} \frac{e^{in\varphi}}{(\rho/a)^n\{(\rho/a)^2-1\}^{\frac{3}{2}}}\, d\xi$$

where $\rho = \sqrt{\xi^2+y^2}$, $\varphi = \pm\pi - \arcsin(y/\rho)$ according to $y \gtrless 0$, $-a < y < a$. Introducing ρ into (5.31) as the new variable, the resulting integral can be integrated by parts. Omitting the infinite contribution of the lower limit $\rho = a$, we obtain

$$(5.32)\quad \begin{aligned} R_n(y) &= (-1)^n a^{n+3} \int_a^{\infty} \frac{d}{d\rho}\left[\frac{\{\rho\exp(i\arcsin(y/\rho))\}^{-n}}{\sqrt{\rho^2-y^2}}\right] \frac{d\rho}{\sqrt{\rho^2-a^2}} \\ &= -i^n a^{n+3} \frac{d}{dy} \int_a^{\infty} \frac{(y-i\sqrt{\rho^2-y^2})^{-n-1}}{\sqrt{\rho^2-y^2}} \frac{\rho d\rho}{\sqrt{\rho^2-a^2}}. \end{aligned}$$

In (5.32) we make the substitution $\sqrt{\rho^2-y^2} = \sqrt{a^2-y^2}\cosh u$, leading to

$$R_n(y) = -i^n a^{n+3}(d/dy)\int_0^\infty (y-i\sqrt{a^2-y^2}\cosh u)^{-n-1}du$$

$$(5.33) \qquad = -i^n a^2(d/dy)Q_n(y/a-i0)$$

$$= -i^n a[Q_n'(y/a)+(\pi i/2)P_n'(y/a)],$$

owing to [12], form. 3.7(12), 3.4(9). Using the real part of (5.31), (5.33) we obtain the result,

(5.34)

$$\partial\Phi^{\text{sing}}(x, y, +0)/\partial z$$

$$= -U\left[\sum_{n=0}^{\infty}(-1)^n A_{2n}Q_{2n}'(y/a)-(\pi/2)\sum_{n=0}^{\infty}(-1)^n A_{2n+1}P_{2n+1}'(y/a)\right],$$

valid for $x^2+y^2 < a^2$.

The velocity potential Φ, as given by (5.6), will solve the boundary value problem, provided that the following conditions are satisfied:

(i) According to the Kutta condition the acceleration potential Ψ must remain finite at the trailing edge of the wing yielding the equation,

(5.35)

$$\sum_{n=0}^{\infty}A_n\cos n\varphi+\frac{2}{\pi U}\cos\varphi\int_0^1\sum_{n=0}^{\infty}\{u^{n+1}w_n(au)\cos n\varphi\}\frac{du}{\sqrt{1-u^2}} = 0,$$

for $-\pi/2 \leqq \varphi \leqq \pi/2$, which we derived from (5.16), (5.17), (5.18), (5.28). In the sequel the second term in (5.35) will be shortly written as $\sum_{n=0}^{\infty} B_n\cos n\varphi$ where B_n, like A_n, is dimensionless.

(ii) $\partial\Phi^{\text{sing}}(x, y, +0)/\partial z = 0$, when $x^2+y^2 < a^2$. Integration of (5.34) leads to the equation,

$$(5.36) \quad \sum_{n=0}^{\infty}(-1)^n A_{2n}Q_{2n}(y/a)-(\pi/2)\sum_{n=0}^{\infty}(-1)^n A_{2n+1}P_{2n+1}(y/a) = 0,$$

for $-a < y < a$.

Using the orthogonality properties of the trigonometric functions and of the Legendre polynomials (cf. [12], form. 3.12(18), (19), (21)) the equations (5.35), (5.36) may be reduced to the following infinite systems of linear equations,

(5.37)

$$\frac{\pi}{2}\,\varepsilon_n(-1)^n A_{2n}+\sum_{m=0}^{\infty}(-1)^m A_{2m+1}\frac{4m+2}{(2m+2n+1)(2m-2n+1)}=\beta_{2n},$$

$$\frac{\pi}{2}\,(-1)^n A_{2n+1}-\sum_{m=0}^{\infty}(-1)^m A_{2m}\frac{4n+3}{(2n+2m+2)(2n-2m+1)}=0,$$

$$n=0,1,2,\ldots,$$

where $\varepsilon_0 = 2$, $\varepsilon_n = 1$ when $n \geqq 1$ and

$$(5.38)\qquad \beta_{2n} = -\frac{\pi}{2}\,\varepsilon_n(-1)^n B_{2n} - \sum_{m=0}^{\infty}(-1)^m B_{2m+1}\frac{4m+2}{(2m+2n+1)(2m-2n+1)}.$$

By elimination of A_{2n} or A_{2n+1} we obtain infinite systems of linear algebraic equations for the coefficients A with even and odd subscripts, viz.

$$(5.39)\qquad \sum_{m=0}^{\infty}\lambda^{e}_{nm}(-1)^m A_{2m}=\beta_{2n}/\pi,\quad \sum_{m=0}^{\infty}\lambda^{o}_{nm}(-1)^m A_{2m+1}=\gamma_{2n+1}/\pi$$

where $n = 0, 1, 2, \ldots$ and

$$(5.40)\qquad \gamma_{2n+1}=\frac{2}{\pi}\sum_{m=0}^{\infty}\frac{\beta_{2m}}{\varepsilon_m}\,\frac{4n+3}{(2n+2m+2)(2n-2m+1)}.$$

The matrix elements λ^{e}_{nm}, λ^{o}_{nm} associated with the infinite systems (5.39) are found by summation of certain series of a similar type as occurring in van Spiegel's thesis [44], section II.12, viz.

(5.41)

$$\begin{cases}\lambda^{e}_{nn}=\varepsilon_n+\dfrac{2}{\pi^2}\left[\dfrac{2n+1}{4n+1}\{\psi(n+1)-\psi(n+\tfrac12)\}-\tfrac14\varepsilon_n\psi'(n+\tfrac12)\right],\\[2ex] \lambda^{e}_{nm}=\dfrac{2}{\pi^2}\left[\dfrac{2m+1}{(2n+2m+1)(2n-2m-1)}\{\psi(n+\tfrac12)-\psi(m+1)\}\right.\\[2ex] \qquad\left.-\dfrac{2m}{(2n+2m)(2n-2m)}\{\psi(n+\tfrac12)-\psi(m+\tfrac12)\}\right],\quad n\neq m,\end{cases}$$

(5.42)

$$\begin{cases} \lambda^o_{nn} = 1+\dfrac{2}{\pi^2}\left[\dfrac{n(4n+3)}{(2n+2)(2n+1)^2}-\dfrac{2n+1}{4n+3}\{\psi(n+\tfrac{3}{2})-\psi(n+1)\}\right. \\ \qquad\qquad \left.-\tfrac{1}{4}\psi'(n+\tfrac{3}{2})\right], \\ \lambda^o_{nm} = \dfrac{2}{\pi^2}\left[-\dfrac{2m+1}{(2n+2m+2)(2n-2m)}\psi(n+\tfrac{3}{2})\right. \\ \qquad +\dfrac{2m+1}{(2n+2m+3)(2n-2m+1)}\psi(n+1) \\ \qquad -\dfrac{2n+\frac{3}{2}}{(2n+2m+2)(2n-2m+1)}\psi(m+\tfrac{1}{2}) \\ \qquad +\dfrac{2n+\frac{3}{2}}{(2n+2m+3)(2n-2m)}\psi(m+\tfrac{3}{2}) \\ \qquad \left.-\dfrac{4n+3}{(2n+1)(2n+2)(2m+1)}\right], \quad n\neq m, \end{cases}$$

where $\psi(z) = \Gamma'(z)/\Gamma(z)$.

5.2. In this section we investigate the infinite linear systems (5.39). First, we present two theorems concerning infinite systems of linear equations. These theorems are proved by means of methods from functional analysis, cf. Ljusternik and Sobolew [30].

THEOREM 1. Let the infinite system S of linear equations, $x_i+\sum_{k=1}^{\infty} a_{ik}x_k = r_i$, $(i = 1, 2, \ldots)$ satisfy the following conditions:
(i) $\sum_{i=1}^{\infty} |r_i|$ is convergent; (ii) $\sup_{k=1,2,\ldots} \sum_{i=1}^{\infty} |a_{ik}| = M < 1$. Then the system S will have a unique solution $x_1, x_2, \ldots$ and $\sum_{i=1}^{\infty} |x_i|$ will be convergent.

PROOF: The Banach space l_1 consists of all sequences $x = \{x_1, x_2, \ldots\}$ with $\sum_{i=1}^{\infty} |x_i|$ convergent. The norm of an element x of l_1 is defined by $||x|| = \sum_{i=1}^{\infty} |x_i|$. We introduce the operator A, which transforms an element x of l_1 into a sequence y viz. $y = Ax$ according to $y_i = \sum_{k=1}^{\infty} a_{ik}x_k$. The sequence y will be an element of l_1, for

$$(5.43)\quad \sum_{i=1}^{\infty} |y_i| = \sum_{i=1}^{\infty} |\sum_{k=1}^{\infty} a_{ik}x_k| \leqq \sum_{k=1}^{\infty} |x_k| \sum_{i=1}^{\infty} |a_{ik}| \leqq M \sum_{k=1}^{\infty} |x_k|$$

and the latter series is convergent. The inequality (5.43) can also be represented by $||y||=||Ax|| \leqq M||x||$, hence, the operator A

is bounded with $||A|| \leqq M$. The infinite system S can now be written as $(I+A)x = r$, where I denotes the identity operator viz. $Ix = x$ and r is an element of l_1. Owing to $||A|| \leqq M < 1$, the operator $(I+A)$ will have an inverse operator $(I+A)^{-1}$ (cf. [30], § 20) and the system S will have a unique solution in l_1, $x = (I+A)^{-1}r$. Hence the series $\sum_{i=1}^{\infty} |x_i|$ will be convergent.

THEOREM 2. Let the infinite system S of linear equations, $x_i+\sum_{k=1}^{\infty} a_{ik}x_k = r_i$, $(i = 1, 2, \ldots)$ satisfy the same conditions as stated in theorem 1. Then the solution of the truncated system $S^{(n)}$ of linear equations, $x_i+\sum_{k=1}^{n} a_{ik}x_k = r_i$, $(i = 1, 2, \ldots, n)$ converges to the solution of S when $n \to \infty$.

PROOF: We introduce the operator $A^{(n)}$, which transforms an element x of l_1 into a sequence y viz. $y = A^{(n)}x$ according to $y_i = \sum_{k=1}^{n} a_{ik}x_k$ $(i = 1, 2, \ldots, n)$, $y_i = 0$ $(i = n+1, n+2, \ldots)$. It is obvious that the operator $A^{(n)}$ is bounded with $||A^{(n)}|| \leqq M$. The truncated system $S^{(n)}$ can be written as $\{I+A^{(n)}\}x = r$. According to theorem 1 this system has a unique solution in l_1.

It can easily be shown that the sequence of operators $A^{(n)}$ converges weakly to the operator A i.e. for every element x of l_1 we have $||\{A-A^{(n)}\}x|| \to 0$ when $n \to \infty$. We denote the inverse operators $(I+A)^{-1}$, $\{I+A^{(n)}\}^{-1}$ by B, $B^{(n)}$, respectively. Let x be an arbitrary element of l_1, then we have

$$\{I+A^{(n)}\}\{B-B^{(n)}\}x = -\{A-A^{(n)}\}Bx$$

and

$$||\{I+A^{(n)}\}\{B-B^{(n)}\}x|| \geqq (1-M)||\{B-B^{(n)}\}x||,$$

hence,

$$||\{B-B^{(n)}\}x|| \leqq ||\{A-A^{(n)}\}Bx||/(1-M). \tag{5.44}$$

When $n \to \infty$ the right-hand side of (5.44) converges to zero. Consequently the sequence of operators $B^{(n)}$ converges weakly to the operator B and for the solutions of the systems S, $S^{(n)}$, which are given by Br, $B^{(n)}r$, respectively, we have $B^{(n)}r \to Br$ when $n \to \infty$.

The theorems 1 and 2 will be applied to the systems (5.39), written in the form,

$$\left.\begin{aligned} (-1)^n A_{2n} + \sum_{m=0}^{\infty} (\lambda_{nm}^{e}/\varepsilon_n - \delta_{nm})(-1)^m A_{2m} \\ = \beta_{2n}/(\varepsilon_n \pi), \\ (-1)^n A_{2n+1} + \sum_{m=0}^{\infty} (\lambda_{nm}^{o} - \delta_{nm})(-1)^m A_{2m+1} \\ = \gamma_{2n+1}/\pi, \end{aligned}\right\} n = 0, 1, 2, \ldots, \tag{5.45}$$

where δ_{nm} denotes Kronecker's symbol and $\varepsilon_0 = 2$, $\varepsilon_n = 1$ when $n \geqq 1$. Starting from (5.41) and using the inequality,

$$(5.46)\quad \log \frac{q}{p} + \frac{q-p}{2pq} < \psi(q)-\psi(p) < \log \frac{q}{p} + \frac{q-p}{2pq} + \frac{q^2-p^2}{12p^2q^2},$$

valid for $q > p > 0$, which may be derived from Binet's second expression for $\log \Gamma(z)$ (cf. [49], §§12.32, 12.33), we proved the following results,

$$(5.47)\quad \begin{gathered} \lambda^e_{nn} < \lambda^e_{n+1,n+1}\ (n \geqq 1),\quad \lambda^e_{nm} < \lambda^e_{n+1,m+1}\ (n < m), \\ \lambda^e_{nm} > \lambda^e_{n+1,m+1}\ (n > m). \end{gathered}$$

It follows easily from (5.41) and the asymptotic behaviour of the function $\psi(z)$ (cf. [12], form. 1.18(7)) that

$$(5.48)\quad \lim_{n\to\infty} \lambda^e_{nn} = 1,\ \lim_{n\to\infty} \lambda^e_{n,n+k} = 0$$

where k is a fixed integer, $\neq 0$. Further it follows by calculation that $0 < \lambda^e_{00}/2 < \lambda^e_{11}$. Hence, (5.47) may be supplemented to

$$(5.49)\quad \begin{cases} 0 < \lambda^e_{00}/2 < \lambda^e_{nn} < \lambda^e_{n+1,n+1} < 1\ (n \geqq 1), \\ \lambda^e_{nm} < \lambda^e_{n+1,m+1} < 0\ (n < m),\ \lambda^e_{nm} > \lambda^e_{n+1,m+1} > 0\ (n > m). \end{cases}$$

According to these inequalities the following estimate can be made,

$$(5.50)\quad \sum_{n=0}^{\infty} |\lambda^e_{nm}/\varepsilon_n - \delta_{nm}| \leqq \sum_{n=1}^{\infty} \lambda^e_{n0} - \sum_{m=1}^{\infty} \lambda^e_{0m} + 1 - \lambda^e_{00}/2 < 0.5051\ldots,$$

valid for $m = 0, 1, 2, \ldots$. The upper limit 0.5051 was obtained from a closed form result for $\sum_{n=1}^{\infty} \lambda^e_{n0}$, whereas $\{-\sum_{m=1}^{\infty} \lambda^e_{0m}\}$ was estimated upwards.

Similarly we proved for the elements λ^o_{nm},

$$(5.51)\quad 0 < \lambda^o_{00} < \lambda^o_{nn} < \lambda^o_{n+1,n+1} < 1\ (n \geqq 1),\ \lambda^o_{nm} < 0\ (n \neq m).$$

The following series appeared to be summable in closed form,

$$(5.52)\quad \begin{aligned} \sum_{n=0}^{\infty} |\lambda^o_{nm} - \delta_{nm}| &= 1 - \lambda^o_{mm} - \sum_{\substack{n=0 \\ n\neq m}}^{\infty} \lambda^o_{nm} \\ &= \frac{1}{\pi^2}\left[\frac{\psi(m+\frac{3}{2})-\psi(1)}{2m+1} + \tfrac{1}{2}\psi'(m+\tfrac{3}{2})\right]. \end{aligned}$$

As the latter result decreases with m increasing, we have

$$(5.53)\quad \sup_{m=0,1,\ldots} \sum_{n=0}^{\infty} |\lambda^o_{nm} - \delta_{nm}| = \frac{1}{\pi^2}\left[\psi(\tfrac{3}{2}) - \psi(1) + \tfrac{1}{2}\psi'(\tfrac{3}{2})\right] = 0.1095\ldots$$

Finally, we consider the right-hand sides of the systems (5.39) viz. β_{2n}, γ_{2n+1}. According to their definition (cf. (5.35)) the coefficients B_n will vanish for n sufficiently large and $\sum_{n=0}^{\infty} (-1)^n B_{2n} = 0$. Hence, it is obvious from (5.38) that $\beta_{2n} = O(1/n^2)$ for $n \to \infty$ and $\sum_{n=0}^{\infty} |\beta_{2n}|$ will be convergent. The representation (5.40) for γ_{2n+1} may be transformed by substituting β_{2n} from (5.38), yielding

$$\gamma_{2n+1} = -\sum_{m=0}^{\infty} (-1)^m B_{2m} \frac{4n+3}{(2n+2m+2)(2n-2m+1)} - \pi \sum_{m=0}^{\infty} \lambda^o_{nm} (-1)^m B_{2m+1} + \frac{\pi}{2} (-1)^n B_{2n+1}. \tag{5.54}$$

Both series occurring in (5.54) are in fact finite sums. Using the property $\sum_{n=0}^{\infty} (-1)^n B_{2n} = 0$ and the asymptotic behaviour of λ^o_{nm} for $n \to \infty$, m fixed, the first and second sum are of order $O(1/n^3)$ and $O[(\log n)/n^3]$ for $n \to \infty$, respectively. Hence, $\sum_{n=0}^{\infty} |\gamma_{2n+1}|$ will be convergent.

The systems (5.39) satisfy the conditions stated in theorems 1 and 2. Therefore these systems will have a unique solution and the series $\sum_{n=0}^{\infty} |A_{2n}|$, $\sum_{n=0}^{\infty} |A_{2n+1}|$ will be convergent. Further, these systems may be solved by truncation to a finite system, the solution of this truncated system being convergent to the solution of the infinite system.

In a similar manner it can be shown, that even the series $\sum_{n=0}^{\infty} n|A_{2n+1}|$ and $\sum_{n=1}^{\infty} n \log n \, |A_{2n+1}|$ are convergent. Concerning the coefficients A_{2n}, the first equation (5.37) with β_{2n} substituted from (5.38), may be written as follows,

$$A_{2n} = \frac{(-1)^n}{\pi n^2} \sum_{m=0}^{\infty} (-1)^m (2m+1)(A_{2m+1}+B_{2m+1}) - B_{2n} - \frac{(-1)^n}{\pi n^2} \sum_{m=0}^{\infty} (-1)^m (A_{2m+1}+B_{2m+1}) \frac{(2m+1)^3}{(2m+2n+1)(2m-2n+1)}, \tag{5.55}$$

valid for $n \geqq 1$. Introducing

$$K = (1/\pi) \sum_{m=0}^{\infty} (-1)^m (2m+1)(A_{2m+1}+B_{2m+1}), \tag{5.56}$$

it can be shown that A_{2n} may be represented by

$$A_{2n} = (-1)^n K/n^2 + \bar{A}_{2n}, \tag{5.57}$$

valid for $n \geqq 1$, where $\sum_{n=1}^{\infty} n|\bar{A}_{2n}|$ will be convergent.

In a similar manner we proved that there exist constants L_1, L_2 such that

$$(5.58)\quad A_{2n+1} = (-1)^n L_1(\log n)/n^3 + (-1)^n L_2/n^3 + \bar{A}_{2n+1},$$

valid for $n \geqq 1$, where $\sum_{n=1}^{\infty} n^2|\bar{A}_{2n+1}|$ will be convergent.

5.3. Using the results of the preceding section it can be shown, that the reduction of the boundary value problem to the infinite linear systems (5.39), which has been performed in section 5.1 in a purely formal way, is completely correct. First, according to (5.18), (5.24) the acceleration potential Ψ^{sing} is given by

$$(5.59)\qquad \Psi^{\text{sing}} = U^2 \frac{\mu}{\mu^2+\eta^2} \sum_{n=0}^{\infty} A_n \left(\frac{1-\mu^2}{1+\eta^2}\right)^{\frac{1}{2}n} \cos n\varphi.$$

Owing to the convergence of $\sum_{n=0}^{\infty} |A_n|$ the series in (5.59) will be uniformly convergent everywhere. Due to the factor $\mu/(\mu^2+\eta^2)$, Ψ^{sing} will be defined and continuous everywhere except at the edge $\rho = a$, $z = 0$ or $\mu = \eta = 0$ of the disk. Further, in each point not on the edge, the right-hand side of (5.59) may be differentiated term by term and it can easily be shown, that Ψ^{sing} satisfies all the conditions stated in section 5.1. The behaviour of Ψ^{sing} in a point (ρ, φ, z) with $\rho = a+\delta \cos \gamma$, $z = \delta \sin \gamma$, $\delta > 0$, $-\pi \leqq \gamma \leqq \pi$ near the edge of the disk, can be derived from (5.59), using the convergence of $\sum_{n=0}^{\infty} n|A_{2n+1}|$ and $\sum_{n=1}^{\infty} n|\bar{A}_{2n}|$ (cf. (5.57)), viz.

$$(5.60)\quad \Psi^{\text{sing}} = \frac{U^2}{\sqrt{2}} \left(\frac{\delta}{a}\right)^{-\frac{1}{2}} \sin \tfrac{1}{2}\gamma \sum_{n=0}^{\infty} A_n \cos n\varphi + O\left[\left(\frac{\delta}{a}\right)^{\frac{1}{2}} \log \left(\frac{\delta}{a}\right)\right].$$

Hence, the edge singularity of Ψ^{sing} is indeed given by the series formed by the singularities of Ψ_n^{sing} (compare (5.28)) and consequently the equation (5.35) will be correct. Similarly, the correctness of the limit procedure, leading to the finite part (5.30) of a divergent integral, and of the resulting equation (5.36), can be verified. Finally, the reduction of the equations (5.35), (5.36) to the infinite systems (5.37), (5.39) may be set on a rigorous foundation.

The velocity vector grad Φ becomes infinite at the leading edge $\rho = a$, $\pi/2 \leqq |\varphi| \leqq \pi$, $z = 0$ of the wing and along the edges $x \geqq 0$, $y = \pm a$, $z = 0$ of the linearized wake. The following expansions will hold in the neighbourhood of these edges:

$$(5.61)\quad \begin{cases} \partial\Phi/\partial x = (U/\sqrt{2})(\delta/a)^{-\frac{1}{2}} \sin \tfrac{1}{2}\gamma\, R(\varphi)+O(1), \\ \partial\Phi/\partial y = (U/\sqrt{2})(\delta/a)^{-\frac{1}{2}} \sin \tfrac{1}{2}\gamma\, R(\varphi) \operatorname{tg}\varphi+O(1), \\ \partial\Phi/\partial z = (-U/\sqrt{2})(\delta/a)^{-\frac{1}{2}} \cos \tfrac{1}{2}\gamma\, R(\varphi) \sec\varphi+O(1), \end{cases}$$

valid in a point (ρ, φ, z) with $\rho = a+\delta\cos\gamma$, $\pi/2 \leqq |\varphi| \leqq \pi$, $z = \delta \sin\gamma$, $\delta > 0$, $-\pi \leqq \gamma \leqq \pi$;

$$(5.62)\quad \begin{cases} \partial\Phi/\partial y = \pm\pi UK\sqrt{2}(\delta^*/a)^{-\frac{1}{2}} \sin \tfrac{1}{2}\gamma^*+O(1), \\ \partial\Phi/\partial z = -\pi UK\sqrt{2}(\delta^*/a)^{-\frac{1}{2}} \cos \tfrac{1}{2}\gamma^*+O(1), \end{cases}$$

valid in a point (x, y, z) with $x \geqq 0$, $y = \pm(a+\delta^*\cos\gamma^*)$, $z = \delta^*\sin\gamma^*$, $\delta^* > 0$, $-\pi \leqq \gamma^* \leqq \pi$. $R(\varphi)$ stands for the left-hand side of (5.35), viz.

$$(5.63)\qquad R(\varphi) = \sum_{n=0}^{\infty} A_n \cos n\varphi + \sum_{n=0}^{\infty} B_n \cos n\varphi.$$

K is given by (5.56).

The expansion for $\partial\Phi/\partial x = \Psi/U$ follows from (5.16), (5.17), (5.60). At the trailing edge of the wing $\partial\Phi/\partial x$ will be finite according to the Kutta condition (cf. (5.35)). The proof of the expansions for $\partial\Phi/\partial y$, $\partial\Phi/\partial z$, which uses the convergence properties of the sequences $\{A_{2n}\}$ and $\{A_{2n+1}\}$ as stated in (5.57), (5.58), will not be given here.

5.4. The pressure difference Π between the lower and upper side of the aerofoil follows from (5.3), viz.

$$(5.64)\quad \Pi(\rho, \varphi) = p(\rho, \varphi, -0)-p(\rho, \varphi, +0) = 2\rho_0\Psi(\rho, \varphi, +0).$$

A Fourier series for $\Psi(\rho, \varphi, +0)$ can be derived from (5.13), (5.15), (5.26),

$$(5.64\text{a})\quad \begin{aligned} \Psi(\rho, \varphi, +0) = U\sum_{n=0}^{\infty} \Bigg[& \frac{d\Phi_n^{\text{reg}}(\rho, +0)}{d\rho} \cos n\varphi \cos\varphi \\ & + \frac{n}{\rho}\Phi_n^{\text{reg}}(\rho, +0) \sin n\varphi \sin\varphi \Bigg] \\ & + U^2 \sum_{n=0}^{\infty} A_n \frac{(\rho/a)^n}{\sqrt{1-(\rho/a)^2}} \cos n\varphi, \end{aligned}$$

valid for $0 \leqq \rho < a$. Integration of the pressure difference over the wing surface yields the following expressions for the lift L and the moment about the y-axis, M_y,

$$(5.65)\qquad L = \int_0^a \int_{-\pi}^{\pi} \Pi(\rho, \varphi)\rho d\varphi d\rho = 4\pi\rho_0 U^2 a^2 A_0,$$

$$(5.66)\quad \begin{aligned} M_y &= \int_0^a \int_{-\pi}^{\pi} \Pi(\rho, \varphi)\rho \cos \varphi \; \rho d\varphi d\rho \\ &= 4\pi\rho_0 U^2 a^3 \left[2/(\pi U) \int_0^1 u\sqrt{1-u^2}\, w_0(au)du + \tfrac{1}{3}A_1\right]. \end{aligned}$$

According to Ward [48] the induced drag D, being the x-component of the aerodynamic force acting on the wing, can be represented by

$$(5.67)\quad D = \rho_0 \iint_S \{\tfrac{1}{2}(\text{grad } \Phi)^2 \cos (n, x) - (\partial\Phi/\partial x)(\partial\Phi/\partial n)\} dS,$$

where S is an arbitrary surface enclosing the disk $z = 0$, $0 \leqq \rho \leqq a$. n denotes the outer normal to S. (n, x) represents the angle between n and the positive direction of the x-axis. It can easily be shown that the integral (5.67) is invariant for changes of S. The vector grad Φ being infinite at the leading edge of the wing and along the edges of the linearized wake, we choose S to be a surface consisting of both sides $z = \pm 0$ of the disk and of an arbitrary part of the linearized wake plus small tube-like surfaces of radius δ which enclose the above-mentioned edges. The contributions to the integral from the linearized wake and from the tubes around its edges become zero when $\delta \to 0$ owing to (5.62). On the tube around the leading edge the following expansion holds,

$$(5.68)\quad \partial\Phi/\partial n = (-U/\sqrt{2})(\delta/a)^{-\frac{1}{2}} \sin \tfrac{1}{2}\gamma \; R(\varphi) \sec \varphi + O(1),$$

according to (5.61). Taking the limit for $\delta \to 0$, we obtain the following formula for D,

$$(5.69)\quad \begin{aligned} D = (-2\rho_0/U) &\int_0^a \int_{-\pi}^{\pi} \Psi(\rho, \varphi, +0)\, w(\rho, \varphi) \rho d\varphi d\rho \\ &+ \rho_0 U^2 a^2 \int_{\pi/2}^{\pi} \int_{-\pi}^{\pi} [\tfrac{1}{2}\{R(\varphi)\}^2 \sec \varphi \cos \gamma \\ &\qquad + \{R(\varphi)\}^2 \sec \varphi \sin^2 \tfrac{1}{2}\gamma] d\gamma d\varphi. \end{aligned}$$

The first integral in (5.69) may be determined by replacing $\Psi(\rho, \varphi, +0)$, $w(\rho, \varphi)$ by their Fourier series (5.64a), (5.7). The remaining integration with respect to ρ deals a.o. with integrals of the following type,

$$(5.70)\quad \begin{aligned} &\int_0^a \Bigg[w_{n+1}(\rho) \left\{ \rho \frac{d\Phi_n^{\text{reg}}(\rho, +0)}{d\rho} - n\Phi_n^{\text{reg}}(\rho, +0) \right\} \\ &\qquad + w_n(\rho) \left\{ \rho \frac{d\Phi_{n+1}^{\text{reg}}(\rho, +0)}{d\rho} + (n+1)\Phi_{n+1}^{\text{reg}}(\rho, +0) \right\} \Bigg] d\rho \\ &= -2\pi a^2 \frac{f_n^{(n)}(1)}{a^{n+2}} \frac{f_{n+1}^{(n+1)}(1)}{a^{n+3}}, \end{aligned}$$

the latter result being derived by substituting power series expansions for the functions $w_n(\rho)$, $w_{n+1}(\rho)$, while corresponding expansions following from (5.12), (5.13), will replace $\Phi_n^{\mathrm{reg}}(\rho, +0)$, $\Phi_{n+1}^{\mathrm{reg}}(\rho, +0)$ and their derivatives. The first term of the second integral in (5.69) will vanish. The second term of this integral has been determined by substituting for the two factors $R(\varphi)$,

$$(5.71)\quad \begin{cases} R(\varphi) = 2\sum_{m=0}^{\infty} A_{2m}\cos 2m\varphi + 2\sum_{m=0}^{\infty} B_{2m}\cos 2m\varphi, \\ R(\varphi) = 2\sum_{n=0}^{\infty} A_{2n+1}\cos(2n+1)\varphi + 2\sum_{n=0}^{\infty} B_{2n+1}\cos(2n+1)\varphi, \end{cases}$$

respectively, and using the integral,

(5.72)

$$\int_{\pi/2}^{\pi} \cos 2m\varphi \cos(2n+1)\varphi \sec\varphi \, d\varphi = \begin{Bmatrix} (\pi/2)(-1)^{n+m}, \text{ when } n \geqq m, \\ 0, \text{ when } n < m. \end{Bmatrix}$$

The relations (5.71) which are valid for $\pi/2 \leqq |\varphi| \leqq \pi$ follow from (5.35); formula (5.72) can be proved by induction.

Ultimately we derived the following simple formula for D,

$$(5.73)\qquad D = 2\pi^2\rho_0 U^2 a^2 \sum_{n=0}^{\infty} (-1)^n A_{2n+1} \sum_{m=0}^{n} (-1)^m A_{2m}.$$

5.5. For a downwash distribution $w(\rho, \varphi)$ odd in φ, the boundary value problem can be solved in the same manner as described in section 5.1. In the various Fourier series $\cos n\varphi$ must be replaced by $\sin n\varphi$. The final equations (5.35), (5.36) will change into

(5.74)

$$\sum_{n=1}^{\infty} A_n \sin n\varphi + \frac{2}{\pi U}\cos\varphi \int_0^1 \sum_{n=1}^{\infty} \{u^{n+1} w_n(au)\sin n\varphi\} \frac{du}{\sqrt{1-u^2}} = 0,$$

for $-\pi/2 \leqq \varphi \leqq \pi/2$, and

$$(5.75)\quad (\pi/2)\sum_{n=1}^{\infty} (-1)^n A_{2n} P_{2n}(y/a) + \sum_{n=0}^{\infty} (-1)^n A_{2n+1} Q_{2n+1}(y/a) = C,$$

for $-a < y < a$. C is a constant which disappears at the further reduction.

Denoting the second term in the left-hand side of (5.74) by $\sum_{n=1}^{\infty} D_n \sin n\varphi$, the equations (5.74), (5.75) can be reduced to the following infinite systems of linear equations,

$$(5.76) \quad \sum_{m=1}^{\infty} \mu^e_{nm}(-1)^m A_{2m} = \theta_{2n}/\pi, \sum_{m=0}^{\infty} \mu^o_{nm}(-1)^m A_{2m+1} = \delta_{2n+1}/\pi,$$

where $n = 1, 2, 3, \ldots$, $n = 0, 1, 2, \ldots$, respectively and

$$(5.77) \quad \begin{cases} \mu^e_{nn} = 1 + \dfrac{2}{\pi^2}\left[\dfrac{2n}{4n+1}\{\psi(n+1)-\psi(n+\tfrac{1}{2})\}-\tfrac{1}{4}\psi'(n+\tfrac{1}{2})\right], \\ \mu^e_{nm} = \dfrac{2}{\pi^2}\left[\dfrac{2m}{(2n+2m+1)(2n-2m+1)}\{\psi(n+1)-\psi(m+\tfrac{1}{2})\}\right. \\ \left. - \dfrac{2m}{(2n+2m)(2n-2m)}\{\psi(n+\tfrac{1}{2})-\psi(m+\tfrac{1}{2})\}\right], \ n \neq m, \end{cases}$$

$$(5.78) \quad \begin{cases} \mu^o_{nn} = 1 - \dfrac{2}{\pi^2}\left[\dfrac{4n+1}{2(2n+1)^2} - \dfrac{2n+2}{4n+3}\{\psi(n+2)-\psi(n+\tfrac{1}{2})\}\right. \\ \qquad \left. + \tfrac{1}{4}\psi'(n+\tfrac{1}{2})\right], \\ \mu^o_{nm} = \dfrac{2}{\pi^2}\left[\dfrac{2n+\frac{1}{2}}{(2n+2m+2)(2n-2m-1)}\psi(n+\tfrac{3}{2})\right. \\ \qquad - \dfrac{2n+\frac{3}{2}}{(2n+2m+3)(2n-2m)}\psi(n+\tfrac{1}{2}) \\ \qquad + \dfrac{2m+1}{(2n+2m+2)(2n-2m)}\psi(m+\tfrac{1}{2}) \\ \qquad \left. - \dfrac{2m+2}{(2n+2m+3)(2n-2m-1)}\psi(m+2)\right], \ n \neq m, \end{cases}$$

$$(5.79) \quad \begin{aligned} \delta_{2n+1} &= -\frac{\pi}{2}(-1)^n D_{2n+1} \\ &\quad - \sum_{m=1}^{\infty}(-1)^m D_{2m}\frac{4m}{(2n+2m+1)(2n-2m+1)}, \end{aligned}$$

$$(5.80) \quad \theta_{2n} = -\frac{2}{\pi}\sum_{m=0}^{\infty}\delta_{2m+1}\frac{4n+1}{(2n+2m+2)(2n-2m-1)}.$$

Similar properties as stated in section 5.2 can be derived for the infinite linear systems (5.76).

Finally we present the following expressions for the moment with respect to the x-axis, M_x, and for the induced drag D,

$$(5.81) \quad M_x = \int_0^a \int_{-\pi}^{\pi} \Pi(\rho, \varphi)\, \rho \sin\varphi\, \rho\, d\varphi\, d\rho = \frac{4\pi}{3}\rho_0 U^2 a^3 A_1,$$

$$D = -2\pi^2 \rho_0 U^2 a^2 \sum_{m=1}^{\infty} (-1)^m A_{2m} \sum_{n=0}^{m-1} (-1)^n A_{2n+1}. \tag{5.82}$$

5.6. For some simple downwash distributions numerical results were derived for lift, moment and induced drag. The infinite linear systems (5.39), (5.76) were truncated to finite systems of order 5, 10, 15, . . ., 35. The solutions of these systems showed a sufficiently rapid convergence to yield accurate values for lift, moment and induced drag [2]).

(i) Downwash distribution, $w(x, y) = -\alpha U$.
According to (5.35) the coefficients B_n are given by $B_1 = -2\alpha/\pi$, while all other coefficients vanish. The following results were calculated,

$$L = 2.81176\, \rho_0 \alpha U^2 a^2,\ M_y = -1.46453\, \rho_0 \alpha U^2 a^3,$$
$$D = 1.25886\, \rho_0 \alpha^2 U^2 a^2.$$

(ii) Downwash distribution, $w(x, y) = Ux/a$.
According to (5.35) the coefficients B_n are given by $B_0 = B_2 = 2/(3\pi)$, while all other coefficients vanish. The following results were calculated,

$$L = -1.464527\, \rho_0 U^2 a^2,\ M_y = -0.689269\, \rho_0 U^2 a^3,$$
$$D = 0.373766\, \rho_0 U^2 a^2.$$

(iii) Downwash distribution, $w(x, y) = Uy/a$.
According to (5.74) the coefficients D_n are given by $D_2 = 2/(3\pi)$, while all other coefficients vanish. The following results were calculated,

$$M_x = -0.384786\, \rho_0 U^2 a^3,\ D = 0.188535 \rho_0 U^2 a^2.$$

(iv) Downwash distribution, $w(x, y) = Uxy/a^2$.
According to (5.74) the coefficients D_n are given by $D_1 = D_3 = 4/(15\pi)$, while all other coefficients vanish. The following results were calculated,

$$M_x = -0.1806993\, \rho_0 U^2 a^3,\ D = 0.0482754\, \rho_0 U^2 a^2.$$

Our results are more accurate than the corresponding values presented by van Spiegel [44]. The agreement between the numerical values occurring in M_y and L for the downwash distributions (i) and (ii), respectively is a consequence of the reciprocity relation, cf. van de Vooren [46].

[2]) The numerical calculations in sections 5.6 and 6.4 were performed on the digital computer ZEBRA. The programming was done by Mr. A. van Deemter.

§ 6. The elliptic wing in unsteady compressible flow

6.1. We consider a wing of elliptic planform which moves with constant velocity U in a compressible, non-viscous fluid and performing at the same time harmonic oscillations of small amplitude in the transverse direction. Similarly as in § 5 we use rectangular coordinates x_1, y_1, z_1 with coordinate axes fixed to the aerofoil. The positive direction of the x_1-axis is again taken opposite to the direction of motion of the wing; the y_1-axis is taken in the spanwise direction. Further, we introduce the Mach number M, given by $M = U/c_0$, where c_0 is the speed of sound in the undisturbed medium. The motion of the aerofoil will be subsonic, hence, $M < 1$. The projection of the aerofoil on the plane $z_1 = 0$ is an ellipse with semi-axes a and $a/\sqrt{1-M^2}$ in the directions of the x_1-axis and of the y_1-axis, respectively. The time will be denoted by t_1.

The velocity vector $\underline{q}$ of the medium is again derived from a perturbation velocity potential Φ_1 viz.

$$\underline{q} = \underline{U} + \operatorname{grad} \Phi_1. \tag{6.1}$$

According to van de Vooren [46] the function $\Phi_1(x_1, y_1, z_1, t_1)$ satisfies the linearized equation,

$$(1-M^2)\frac{\partial^2 \Phi_1}{\partial x_1^2} + \frac{\partial^2 \Phi_1}{\partial y_1^2} + \frac{\partial^2 \Phi_1}{\partial z_1^2} - 2\frac{M}{c_0}\frac{\partial^2 \Phi_1}{\partial x_1 \partial t_1} - \frac{1}{c_0^2}\frac{\partial^2 \Phi_1}{\partial t_1^2} = 0. \tag{6.2}$$

Let the equation of the wing surface be given by

$$z_1 = g_1(x_1, y_1, t_1) = g_{1,0}(x_1, y_1)e^{-i\nu t_1} \tag{6.3}$$

where ν denotes the frequency of the oscillating wing. Then the linearized boundary condition for Φ_1 reads,

$$\partial \Phi_1/\partial z_1 = (-i\nu g_{1,0} + U \partial g_{1,0}/\partial x_1)e^{-i\nu t_1} = w_1(x_1, y_1)e^{-i\nu t_1} \tag{6.4}$$

for $z_1 = \pm 0$, $x_1^2 + (1-M^2)y_1^2 < a^2$. From the condition (6.4) it is obvious that Φ_1 is an odd function of z_1, whereas Φ_1 depends on t_1 through a factor $e^{-i\nu t_1}$. Hence, we substitute

$$\Phi_1(x_1, y_1, z_1, t_1) = \Phi_{1,0}(x_1, y_1, z_1)e^{-i\nu t_1}, \tag{6.5}$$

leading to the boundary condition,

$$\partial \Phi_{1,0}/\partial z_1 = w_1(x_1, y_1) \text{ for } z_1 = \pm 0,\ x_1^2 + (1-M^2)y_1^2 < a^2. \tag{6.6}$$

Similarly we introduce an acceleration potential,

$$\Psi_1(x_1, y_1, z_1, t_1) = \Psi_{1,0}(x_1, y_1, z_1)e^{-i\nu t_1} \tag{6.7}$$

which satisfies the same equation (6.2). In linearized theory $\Phi_{1,0}$ and $\Psi_{1,0}$ are connected by

$$\Psi_{1,0} = -i\nu\Phi_{1,0}+U\partial\Phi_{1,0}/\partial x_1. \tag{6.8}$$

Application of a Lorentz transformation,

$$\begin{aligned} x = x_1,\ y = \sqrt{1-M^2}\,y_1,\ z = \sqrt{1-M^2}\,z_1,& \\ t = Mx_1+c_0(1-M^2)t_1,& \end{aligned} \tag{6.9}$$

changes the functions Φ_1, Ψ_1 into Φ, Ψ, viz.

$$\begin{aligned} &\Phi_1(x_1, y_1, z_1, t_1) \\ &= \Phi_1\left(x, \frac{y}{\sqrt{1-M^2}}, \frac{z}{\sqrt{1-M^2}}, \frac{t-Mx}{c_0(1-M^2)}\right) = \Phi(x, y, z, t), \end{aligned} \tag{6.10}$$

while a similar result holds for Ψ_1 and Ψ. Then the function $\Phi(x, y, z, t)$ satisfies the equation,

$$\frac{\partial^2\Phi}{\partial x^2}+\frac{\partial^2\Phi}{\partial y^2}+\frac{\partial^2\Phi}{\partial z^2}-\frac{\partial^2\Phi}{\partial t^2} = 0, \tag{6.11}$$

the same equation holding for $\Psi(x, y, z, t)$.

Introducing dimensionless quantities,

$$\omega = \nu a/U,\ \varkappa = \omega M/(1-M^2), \tag{6.12}$$

the functions Φ, Ψ will depend on t through a factor $\exp(-i\varkappa t/a)$. Hence, we write

$$\begin{aligned} \Phi(x, y, z, t) &= \Phi_0(x, y, z)\exp(-i\varkappa t/a), \\ \Psi(x, y, z, t) &= \Psi_0(x, y, z)\exp(-i\varkappa t/a). \end{aligned} \tag{6.13}$$

The functions $\Phi_0(x, y, z)$, $\Psi_0(x, y, z)$ will be solutions of Helmholtz' equation,

$$\frac{\partial^2\Phi_0}{\partial x^2}+\frac{\partial^2\Phi_0}{\partial y^2}+\frac{\partial^2\Phi_0}{\partial z^2}+\frac{\varkappa^2}{a^2}\Phi_0 = 0. \tag{6.14}$$

The function $\Phi_0(x, y, z)$ satisfies the boundary condition,

$$\partial\Phi_0/\partial z = w(x, y) \text{ for } z = \pm 0,\ x^2+y^2 < a^2, \tag{6.15}$$

where

$$w(x, y) = w_1(x, y/\sqrt{1-M^2})\exp(i\varkappa Mx/a)/\sqrt{1-M^2}. \tag{6.16}$$

It can easily be shown that Φ_0 and Ψ_0 are connected by

(6.17) $$\Psi_0 = -ik(U/a)\Phi_0 + U\partial\Phi_0/\partial x,$$

where

(6.18) $$k = \omega/(1-M^2).$$

Integrating (6.17) we obtain the inverse formula,

(6.19) $$\Phi_0(x, y, z) = (1/U)\exp(ikx/a)\int_{-\infty}^{x} \Psi_0(\xi, y, z)\exp(-ik\xi/a)d\xi.$$

Finally, for later reference we state the following relation between $\Phi_{1,0}$ and Φ_0,

(6.20) $$\begin{aligned}\Phi_{1,0}(x_1, y_1, z_1) &= \Phi_{1,0}\left(x, \frac{y}{\sqrt{1-M^2}}, \frac{z}{\sqrt{1-M^2}}\right)\\ &= \Phi_0(x, y, z)\exp(-i\varkappa Mx/a),\end{aligned}$$

while a similar result holds for $\Psi_{1,0}$ and Ψ_0.

The boundary value problem formulated in terms of Φ_0 and Ψ_0 can be solved in a similar way as described in § 5. The solution will be presented in the form of a series expansion in powers of ω. Two terms of this expansion have been calculated. It is obvious that such a solution is only valid for small values of ω.

We introduce cylindrical coordinates ρ, φ, z as stated in (5.5). The function $\Phi_0(\rho, \varphi, z)$ is split into a regular part Φ_0^{reg} and a singular part Φ_0^{sing},

(6.21) $$\Phi_0 = \Phi_0^{\text{reg}} + \Phi_0^{\text{sing}}.$$

The function $\Phi_0^{\text{reg}}(\rho, \varphi, z)$ is required to satisfy the following conditions:

(i) Φ_0^{reg} is a solution of Helmholtz' equation, viz. $\Delta\Phi_0^{\text{reg}} + (\varkappa/a)^2\Phi_0^{\text{reg}} = 0$;
(ii) $\Phi_0^{\text{reg}} = 0$, when $z = 0$, $\rho > a$;
(iii) Φ_0^{reg} satisfies Sommerfeld's radiation condition at infinity;
(iv) $\partial\Phi_0^{\text{reg}}/\partial z = w(\rho, \varphi)$, when $z = 0$, $0 \leqq \rho < a$;
(v) Φ_0^{reg} is everywhere finite.

The functions $w(\rho, \varphi)$ and $\Phi_0^{\text{reg}}(\rho, \varphi, z)$ are again expanded in Fourier series with respect to φ. In this section we confine ourselves to the case of a downwash distribution $w(\rho, \varphi)$ even in φ, hence, the Fourier series may be represented by

(6.22) $$w(\rho, \varphi) = \sum_{n=0}^{\infty} w_n(\rho)\cos n\varphi, \quad \Phi_0^{\text{reg}}(\rho, \varphi, z) = \sum_{n=0}^{\infty} \Phi_n^{\text{reg}}(\rho, z)\cos n\varphi.$$

The case of a downwash distribution $w(\rho, \varphi)$ odd in φ will be treated in section 6.3. It is assumed that the functions $w_n(\rho)$ vanish for n sufficiently large and satisfy the same conditions as stated in § 5.

According to Bazer and Hochstadt [3] we present the following integral representation for $\Phi_n^{\text{reg}}(\rho, z)$,

$$(6.23)\quad \Phi_n^{\text{reg}}(\rho, z) = \rho^n \left(\frac{1}{\rho}\frac{\partial}{\partial\rho}\right)^n \int_{-1}^{1} \frac{\exp\{i(\varkappa/a)\sqrt{\rho^2+(z+iat)^2}\}}{\sqrt{\rho^2+(z+iat)^2}} f_n(t)dt,$$

$n = 0, 1, 2, \ldots$, valid for $z \geqq 0$. For $z \leqq 0$ we define $\Phi_n^{\text{reg}}(\rho, z) = -\Phi_n^{\text{reg}}(\rho, -z)$. The unknown function $f_n(t)$ is required to satisfy the same conditions as in (5.8).

The representation (6.23) satisfies all the conditions (i) to (v) except the condition (iv), which may be written as

$$(6.24)\qquad \partial\Phi_n^{\text{reg}}/\partial z = w_n(\rho), \text{ when } z = 0,\ 0 \leqq \rho < a.$$

According to § 2, section 2.2 the condition (6.24) leads to the Fredholm integral equation (2.29) (m being replaced by n) for the function $f_n(t)/(1-t^2)^n$ with $u_1^{(n)}(\rho) = w_n(\rho)/\rho^n$ to be substituted into (2.30). The parameter α must be replaced by $\varkappa$. The kernel of the integral equation being of order $\varkappa^2$, it is clear that the solution $f_n(t)$ of the integral equation is given by (5.11) multiplied by a factor $\{1+O(\varkappa^2)\}$. The formula (5.12), being equivalent to (5.11) remains also valid after multiplication by this factor. Similarly, according to Bazer and Hochstadt [3] we have,

$$(6.25)\quad \begin{aligned} \Phi_n^{\text{reg}}(\rho, +0) &= \frac{2\rho^n}{ia}\left(\frac{1}{\rho}\frac{d}{d\rho}\right)^n \int_{\rho/a}^{1} \frac{\cosh(\varkappa\sqrt{t^2-(\rho/a)^2})}{\sqrt{t^2-(\rho/a)^2}} f_n(t)dt \\ &= \frac{2\rho^n}{ia^{2n+1}} \int_{\rho/a}^{1} \frac{t}{\sqrt{t^2-(\rho/a)^2}} \left(\frac{1}{t}\frac{d}{dt}\right)^n \left\{\frac{f_n(t)}{t}\right\} dt\, \{1+O(\varkappa^2)\}, \end{aligned}$$

valid for $0 \leqq \rho < a$, cf. (5.13).

Similar to (6.21) the function $\Psi_0(\rho, \varphi, z)$ will be split according to

$$(6.26)\qquad \Psi_0 = \Psi_0^{\text{reg}} + \Psi_0^{\text{sing}}$$

where Ψ_0^{reg} is defined by (compare form. (6.17))

$$(6.27)\qquad \Psi_0^{\text{reg}} = -ik(U/a)\Phi_0^{\text{reg}} + U\partial\Phi_0^{\text{reg}}/\partial x.$$

Near the edge $\rho = a$, $z = 0$ the expansion (5.16) holds for Ψ_0^{reg}.

The function $\Psi_0^{\text{sing}}(\rho, \varphi, z)$ is required to satisfy the same conditions as the function $\Psi^{\text{sing}}(\rho, \varphi, z)$ considered in § 5, except that Ψ_0^{sing} is a solution of Helmholtz' equation, $\Delta\Psi_0^{\text{sing}}+(\varkappa/a)^2\Psi_0^{\text{sing}}=0$. The function $\Psi_0^{\text{sing}}(\rho, \varphi, z)$ is expanded in a Fourier series,

$$\Psi_0^{\text{sing}}(\rho, \varphi, z) = \sum_{n=0}^{\infty} \Psi_n^{\text{sing}}(\rho, z) \cos n\varphi. \tag{6.28}$$

Similar to (5.19) we state the integral representation,

(6.29)
$$\Psi_n^{\text{sing}}(\rho, z) = \frac{\partial}{\partial z}\rho^n\left(\frac{1}{\rho}\frac{\partial}{\partial \rho}\right)^n \int_{-1}^{1} \frac{\exp\{i(\varkappa/a)\sqrt{\rho^2+(z+iat)^2}\}}{\sqrt{\rho^2+(z+iat)^2}}\, g_n(t)dt,$$

$n = 0, 1, 2, \ldots$, valid for $z \geqq 0$, whereas for $z \leqq 0$ we define $\Psi_n^{\text{sing}}(\rho, z) = -\Psi_n^{\text{sing}}(\rho, -z)$. The unknown function $g_n(t)$ is required to satisfy the same conditions as in (5.19). The representation (6.29) satisfies all the prescribed conditions except the condition, $\partial\Psi_n^{\text{sing}}/\partial z = 0$ for $z = 0$, $0 \leqq \rho < a$. Similar to (5.21), the latter condition can be reduced to

(6.30)
$$\lim_{z\to+0} \rho^n\left(\frac{1}{\rho}\frac{\partial}{\partial \rho}\right)^n \int_{-1}^{1} \frac{\exp\{i(\varkappa/a)\sqrt{\rho^2+(z+iat)^2}\}}{\sqrt{\rho^2+(z+iat)^2}}\, g_n(t)dt = \bar{C}_n J_n(\varkappa\rho/a),$$

valid for $0 \leqq \rho < a$, where $\bar{C}_n$ is an arbitrary constant. According to § 2, section 2.3 the condition (6.30) leads to the Fredholm integral equation (2.46) (m being replaced by n) for the function $g_n(t)/(1-t^2)^n$ with $u_2^{(n)}(\rho) = \bar{C}_n J_n(\varkappa\rho/a)/\rho^n$ to be substituted into (2.47). The parameter α must be replaced by $\varkappa$. It can easily be shown, that the solution $g_n(t)$ of the integral equation is given by (5.22) multiplied by a factor $\{1+O(\varkappa^2)\}$.

Substituting this value of $g_n(t)$ into (6.29), we state the following results for Ψ_n^{sing} and $\partial\Psi_n^{\text{sing}}/\partial z$ in the plane $z = 0$, viz.

$$\Psi_n^{\text{sing}}(\rho, +0) = A_n U^2 \frac{(\rho/a)^n}{\sqrt{1-(\rho/a)^2}}\{1+O(\varkappa^2)\}, \quad \text{for } 0 \leqq \rho < a, \tag{6.31}$$

(6.32)
$$\frac{\partial\Psi_n^{\text{sing}}(\rho, 0)}{\partial z} = A_n \frac{U^2}{a} \frac{\exp\left(i\varkappa\sqrt{(\rho/a)^2-1}\right)}{(\rho/a)^n\{(\rho/a)^2-1\}^{\frac{3}{2}}}\{1-i\varkappa\sqrt{(\rho/a)^2-1}+O(\varkappa^2)\}, \quad \text{for } \rho > a,$$

where A_n is given by (5.25). The formula (6.31) follows from

$$\lim_{z\to+0}\frac{\partial}{\partial z}\rho^n\left(\frac{1}{\rho}\frac{\partial}{\partial\rho}\right)^n\int_{-1}^{1}\frac{\exp\{i(\varkappa/a)\sqrt{\rho^2+(z+iat)^2}\}}{\sqrt{\rho^2+(z+iat)^2}}C_n(1-t^2)^n dt$$

$$(6.33)\qquad = 2C_n\rho^n\left(\frac{1}{\rho}\frac{d}{d\rho}\right)^{n+1}\int_{\rho/a}^{1}\frac{\cosh\left(\varkappa\sqrt{t^2-(\rho/a)^2}\right)}{\sqrt{t^2-(\rho/a)^2}}t(1-t^2)^n dt$$

$$= A_n U^2\frac{(\rho/a)^n\cosh\left(\varkappa\sqrt{1-(\rho/a)^2}\right)}{\sqrt{1-(\rho/a)^2}},\quad \text{for } 0\leqq\rho<a.$$

The formula (6.32), valid for $\rho > a$ can be derived by considering the integral,

$$I_n = \lim_{z\to+0}\frac{\partial^2}{\partial z^2}\rho^n\left(\frac{1}{\rho}\frac{\partial}{\partial\rho}\right)^n\int_{-1}^{1}\frac{\exp\{i(\varkappa/a)\sqrt{\rho^2+(z+iat)^2}\}}{\sqrt{\rho^2+(z+iat)^2}}C_n(1-t^2)^n dt$$

$$(6.34)\qquad = \frac{C_n}{a}\rho^n\left[\left(\frac{1}{\rho}\frac{d}{d\rho}\right)^{n+1}\int_{-1}^{1}\frac{\exp\left(i\varkappa\sqrt{(\rho/a)^2-t^2}\right)}{\sqrt{(\rho/a)^2-t^2}}(1-t^2)^n dt\right.$$

$$\left.-a^2\left(\frac{1}{\rho}\frac{d}{d\rho}\right)^{n+2}\int_{-1}^{1}\frac{\exp\left(i\varkappa\sqrt{(\rho/a)^2-t^2}\right)}{\sqrt{(\rho/a)^2-t^2}}t^2(1-t^2)^n dt\right].$$

Using [12], form. 7.11(3), (9) we obtain the derivative,

$$(6.35)\qquad \left(\frac{1}{\rho}\frac{d}{d\rho}\right)^j\left\{\frac{\exp\left(i\varkappa\sqrt{(\rho/a)^2-t^2}\right)}{\sqrt{(\rho/a)^2-t^2}}\right\}$$

$$= \frac{(-2)^j(\frac{1}{2})_j}{a^{2j}}\frac{\exp\left(i\varkappa\sqrt{(\rho/a)^2-t^2}\right)}{\{(\rho/a)^2-t^2\}^{j+\frac{1}{2}}}\left[1-i\varkappa\sqrt{(\rho/a)^2-t^2}+O(\varkappa^2)\right],$$

valid for $j = 1, 2, 3, \ldots$. Substituting this derivative with $j = n+1$, $j = n+2$ into (6.34) and expanding $\exp\left(i\varkappa\sqrt{(\rho/a)^2-t^2}\right)$ according to

$$(6.36)\qquad \exp\left(i\varkappa\sqrt{(\rho/a)^2-t^2}\right) = \exp\left(i\varkappa\sqrt{(\rho/a)^2-1}\right)$$

$$\cdot\left[1+i\varkappa\{\sqrt{(\rho/a)^2-t^2}-\sqrt{(\rho/a)^2-1}\}+O(\varkappa^2)\right],$$

the integral I_n changes into

$$I_n = A_n\frac{U^2}{a}\frac{(\frac{1}{2})_{n+1}}{n!}(\rho/a)^n\exp\left(i\varkappa\sqrt{(\rho/a)^2-1}\right)$$

$$(6.37)\qquad \cdot\{1-i\varkappa\sqrt{(\rho/a)^2-1}+O(\varkappa^2)\}$$

$$\cdot\left[\int_{-1}^{1}\frac{(1-t^2)^n}{\{(\rho/a)^2-t^2\}^{n+\frac{3}{2}}}dt+(2n+3)\int_{-1}^{1}\frac{t^2(1-t^2)^n}{\{(\rho/a)^2-t^2\}^{n+\frac{5}{2}}}dt\right].$$

The integrals occurring in (6.37) can be evaluated by expanding the denominator of the integrand in a binomial series, leading ultimately to the result (6.32). In the preceding derivation we have avoided to use the common power series expansion for $\exp\left(i\varkappa\sqrt{(\rho/a)^2-t^2}\right)$, because the remainder term of this expansion will become large when ρ is large. For the expansion (6.36) this objection does not hold, because $\{\sqrt{(\rho/a)^2-t^2}-\sqrt{(\rho/a)^2-1}\}$ is bounded for $\rho > a$, $0 \leqq t \leqq 1$. For the rest no investigation has been made concerning the dependence on ρ of the remainder term in (6.32) and in the resulting expansion for $\partial\Psi_0^{\text{sing}}/\partial z$.

According to formula (2.7), the expansion (5.28) multiplied by a factor $\{1+O(\varkappa^2)\}$ holds for Ψ_n^{sing} in the neighbourhood of the edge $\rho = a$, $z = 0$.

The part Φ_0^{sing} of the function Φ_0, introduced in (6.21), will be derived from Ψ_0^{sing} according to (compare form. (6.19))

$$(6.38)\quad \Phi_0^{\text{sing}}(x, y, z) = (1/U)\exp(ikx/a)\int_{-\infty}^{x}\Psi_0^{\text{sing}}(\xi, y, z)\exp(-ik\xi/a)d\xi.$$

The normal derivative $\partial\Phi_0^{\text{sing}}/\partial z$ on the disk $z = 0$, $x^2+y^2 < a^2$ is again represented by the finite part of a divergent integral, viz.

$$(6.39)\quad \begin{aligned}&\partial\Phi_0^{\text{sing}}(x, y, +0)/\partial z\\ &\quad = (1/U)\exp(ikx/a)\,{}^*\!\!\int_{-\infty}^{-\sqrt{a^2-y^2}}\partial\Psi_0^{\text{sing}}(\xi, y, 0)/\partial z\exp(-ik\xi/a)d\xi.\end{aligned}$$

Substituting for $\partial\Psi_0^{\text{sing}}/\partial z$ its Fourier series with the nth term given by (6.32), the integration will be performed term by term. Hence, we consider the integral,

$$(6.40)\quad \begin{aligned}R_n(y) = {}^*\!\!\int_{-\infty}^{-\sqrt{a^2-y^2}} &\frac{\exp\left(i\varkappa\sqrt{(\rho/a)^2-1}\right)\{1-i\varkappa\sqrt{(\rho/a)^2-1}\}}{(\rho/a)^n\{(\rho/a)^2-1\}^{\frac{3}{2}}}\\ &\cdot e^{in\varphi}\exp(-ik\xi/a)d\xi,\end{aligned}$$

where $\rho = \sqrt{\xi^2+y^2}$, $\varphi = \pm\pi-\arcsin(y/\rho)$ according to $y \gtrless 0$, $-a < y < a$. Introducing ρ into (6.40) as the new variable, the resulting integral can be integrated by parts, yielding

$$(6.41)\quad \begin{aligned}R_n(y) = i^n a^{n+3}\Bigg[&-\frac{\exp\left(i\varkappa\sqrt{(\rho/a)^2-1}\right)}{\sqrt{\rho^2-a^2}}\,\frac{\exp\{i(k/a)\sqrt{\rho^2-y^2}\}}{(y-i\sqrt{\rho^2-y^2})^n\sqrt{\rho^2-y^2}}\Bigg|_a^\infty\\ &+\int_a^\infty\frac{\exp\left(i\varkappa\sqrt{(\rho/a)^2-1}\right)}{\sqrt{\rho^2-a^2}}\,\frac{d}{d\rho}\left\{\frac{\exp\{i(k/a)\sqrt{\rho^2-y^2}\}}{(y-i\sqrt{\rho^2-y^2})^n\sqrt{\rho^2-y^2}}\right\}d\rho\Bigg].\end{aligned}$$

The first term in the right-hand side of (6.41) must be treated with some care. The contribution of the upper limit $\rho = \infty$ will vanish. The contribution of the lower limit $\rho = a$ will be determined from the expansion,

$$\text{(6.42)}\quad \begin{aligned} &\exp\,(i\varkappa\sqrt{(\rho/a)^2-1})/\sqrt{\rho^2-a^2} \\ &\qquad = 1/\sqrt{\rho^2-a^2}+i\varkappa/a+O(\sqrt{\rho^2-a^2}). \end{aligned}$$

When $\rho \to a$, the first term of this expansion becomes infinite and is omitted, hence, there remains only the term $i\varkappa/a$. The integral in the right-hand side of (6.41) can be reduced in the same manner as a similar integral examined by van Spiegel, cf. [44], section III.3. We obtain the result,

$$\text{(6.43)}\quad \begin{aligned} R_n(y) &= i^{n+1}a^{n+2}\varkappa\,\frac{\exp\,\{i(k/a)\sqrt{a^2-y^2}\}}{(y-i\sqrt{a^2-y^2})^n\sqrt{a^2-y^2}} \\ &\quad - \frac{i^n a^{n+3}}{n+ky/a}\,\frac{d}{dy}\,\{(k/a)I_n(\varkappa,\, k)+nI_{n+1}(\varkappa,\, k)\} \end{aligned}$$

where

$$\text{(6.44)}\quad \begin{aligned} &I_n(\varkappa,\, k) \\ &= \int_a^\infty \frac{\exp\,(i\varkappa\sqrt{(\rho/a)^2-1})}{\sqrt{\rho^2-a^2}}\,\frac{\exp\,\{i(k/a)\sqrt{\rho^2-y^2}\}}{(y-i\sqrt{\rho^2-y^2})^n}\,\frac{\rho d\rho}{\sqrt{\rho^2-y^2}}. \end{aligned}$$

Substituting $\sqrt{\rho^2-y^2} = \sqrt{a^2-y^2}\cosh u$, (6.44) changes into

$$\text{(6.45)}\quad I_n(\varkappa,\, k) = \int_0^\infty \frac{\exp\,\{(i/a)\sqrt{a^2-y^2}(\varkappa\sinh u+k\cosh u)\}}{(y-i\sqrt{a^2-y^2}\cosh u)^n}\,du.$$

The integral (6.45) is certainly defined for $n = 0, 1, 2, \ldots$, $\varkappa \geqq 0$, $k \geqq 0$ except for the combination $n = 0$, $\varkappa = k = 0$. It can be shown, using [12], form. 7.12(17) that the following expansion holds for $I_0(\varkappa, k)$ when ω is small i.e. when $\varkappa$ and k are small (cf. (6.12), (6.18)),

$$\text{(6.46)}\quad \begin{aligned} I_0(\varkappa,\, k) &= -\log\,\{\tfrac{1}{2}(\varkappa+k)\sqrt{1-(y/a)^2}\}+\frac{\pi i}{2}-\gamma \\ &\quad -i\varkappa\sqrt{1-(y/a)^2}+O(\omega^2\log\,\omega), \end{aligned}$$

where γ denotes Euler's constant. Similar to van Spiegel, the recurrence relation,

$$(6.47)\quad \begin{aligned}&(\partial/\partial k)[\exp(-ky/a)\{(k/a)I_n(\varkappa, k)+nI_{n+1}(\varkappa, k)\}]\\ &\qquad = (-1/a)\exp(-ky/a)\{(k/a)I_{n-1}(\varkappa, k)+(n-1)I_n(\varkappa, k)\},\end{aligned}$$

which is valid for $n \geqq 1$, $\varkappa \geqq 0$, $k \geqq 0$ except for $n = 1$, $\varkappa = k = 0$, may be integrated with respect to k and applied n times, yielding the representation,

$$(6.48)\quad \begin{aligned}&(k/a)I_n(\varkappa, k)+nI_{n+1}(\varkappa, k)\\ &\quad = \frac{(-1)^n}{a^n}\int_0^k \frac{(k-k_1)^{n-1}}{(n-1)!}\exp\{(k-k_1)y/a\}\frac{k_1}{a} I_0(\varkappa, k_1)dk_1\\ &\qquad + \exp(ky/a)\sum_{j=0}^{n-1}\frac{(-1)^j(k/a)^j}{j!}(n-j)I_{n-j+1}(\varkappa, 0),\end{aligned}$$

valid for $n \geqq 1$, $\varkappa \geqq 0$, $k \geqq 0$. The right-hand side of (6.48) is expanded in powers of ω, taking into account only the terms of order ω^0 and ω. Starting from the expansion,

$$(6.49)\quad \begin{aligned}I_n(\varkappa, 0) &= I_n(0, 0)+\varkappa\, \partial I_n(0, 0)/\partial\varkappa+\dots\\ &= (1/a^n)Q_{n-1}(y/a-i0)-\frac{\varkappa}{(n-1)a}(y-i\sqrt{a^2-y^2})^{-n+1}+\dots,\end{aligned}$$

(cf. [12], form. 3.7(12)) where $n \geqq 2$, we obtain

$$(6.50)\quad \begin{aligned}&(k/a)I_n(\varkappa, k)+nI_{n+1}(\varkappa, k) = (n/a^{n+1})Q_n(y/a-i0)\\ &\quad +(nky/a^{n+2})Q_n(y/a-i0)-(n-1)(k/a^{n+1})Q_{n-1}(y/a-i0)\\ &\quad -(\varkappa/a)(y-i\sqrt{a^2-y^2})^{-n}+\dots,\end{aligned}$$

valid for $n \geqq 1$. Substituting (6.50) into (6.43) and using [12], form. 3.8(5), 3.4(9) we are led to the following expansion for $R_n(y)$,

$$(6.51)\quad \begin{aligned}R_n(y) = &-i^n a[\{Q'_n(y/a)+(\pi i/2)P'_n(y/a)\}\\ &+nk\{Q_n(y/a)+(\pi i/2)P_n(y/a)\}]+\dots,\end{aligned}$$

valid for $n \geqq 1$. A prime denotes differentiation with respect to the argument. It follows easily from (6.43), (6.46) that (6.51) also holds for $n = 0$. It can be verified that the remainder term in (6.51) is of order ω^2 for $n \geqq 1$ and of order $\omega^2 \log \omega$ for $n = 0$.

Splitting (6.51) into a part even in y and a part odd in y, the even part leads to the following expansion for $\partial\Phi_0^{\text{sing}}(x, y, +0)/\partial z$ on the disk $x^2+y^2 < a^2$,

$$\partial\Phi_0^{\text{sing}}(x, y, +0)/\partial z = -U \exp(ikx/a)$$

$$(6.52)\qquad \cdot\Bigg[\sum_{n=0}^{\infty}(-1)^n A_{2n}\{Q'_{2n}(y/a)+n\pi ik\, P_{2n}(y/a)\} - \sum_{n=0}^{\infty}(-1)^n A_{2n+1}\{(\pi/2)P'_{2n+1}(y/a) - (2n+1)ikQ_{2n+1}(y/a)\}\Bigg] + \ldots.$$

The function Φ_0, as given by (6.21), will solve the boundary value problem, provided that the following conditions are satisfied:

(i) According to the Kutta condition the pressure must remain finite at the trailing edge of the elliptic wing. The same condition holds for the acceleration potential Ψ_1 and for the function Ψ_0, the latter remaining finite at the "trailing edge" $\rho = a$, $|\varphi| \leqq \pi/2$, $z = 0$ of the circular disk. This condition leads again to the equation (5.35).

(ii) $\partial\Phi_0^{\text{sing}}(x, y, +0)/\partial z = 0$, when $x^2+y^2 < a^2$. Integration of (6.52) yields the equation (cf. [12], form. 10.10(15)),

$$(6.53)\qquad \sum_{n=0}^{\infty}(-1)^n A_{2n}\left[Q_{2n}(y/a)+\frac{n\pi ik}{4n+1}\{P_{2n+1}(y/a)-P_{2n-1}(y/a)\}\right] - \sum_{n=0}^{\infty}(-1)^n A_{2n+1}\left[(\pi/2)P_{2n+1}(y/a) - \frac{(2n+1)ik}{4n+3}\{Q_{2n+2}(y/a)-Q_{2n}(y/a)\}\right] = 0,$$

for $-a < y < a$.

The second term in (5.35) will be written shortly as $\sum_{n=0}^{\infty} B_n \cos n\varphi$. Then the coefficients B_n may be expanded in powers of ω. Assuming that the expansions consist of only one term, of order ω^j, we substitute

$$(6.54)\qquad A_n = A_n^{(0)} + ikA_n^{(1)}$$

where $A_n^{(0)}$, $A_n^{(1)}$ depend on ω only through a factor ω^j. The equations (5.35), (6.53) may be split by collecting terms containing the same power of k,

$$(6.55)\qquad \begin{cases} \displaystyle\sum_{n=0}^{\infty} A_n^{(0)} \cos n\varphi + \sum_{n=0}^{\infty} B_n \cos n\varphi = 0, \text{ for } -\pi/2 \leqq \varphi \leqq \pi/2, \\ \displaystyle\sum_{n=0}^{\infty}(-1)^n A_{2n}^{(0)} Q_{2n}(y/a) - (\pi/2)\sum_{n=0}^{\infty}(-1)^n A_{2n+1}^{(0)} P_{2n+1}(y/a) = 0, \\ \qquad\qquad \text{for } -a < y < a, \end{cases}$$

$$
(6.56)\quad \begin{cases} \sum_{n=0}^{\infty} A_n^{(1)} \cos n\varphi = 0, \text{ for } -\pi/2 \leqq \varphi \leqq \pi/2, \\ \sum_{n=0}^{\infty} (-1)^n A_{2n}^{(1)} Q_{2n}(y/a) - (\pi/2) \sum_{n=0}^{\infty} (-1)^n A_{2n+1}^{(1)} P_{2n+1}(y/a) \\ \quad = -\pi \sum_{n=0}^{\infty} (-1)^n A_{2n}^{(0)} \frac{n}{4n+1} \{P_{2n+1}(y/a) - P_{2n-1}(y/a)\} \\ \quad - \sum_{n=0}^{\infty} (-1)^n A_{2n+1}^{(0)} \frac{2n+1}{4n+3} \{Q_{2n+2}(y/a) - Q_{2n}(y/a)\}, \\ \qquad\qquad \text{for } -a < y < a. \end{cases}
$$

Similar as in § 5 the equations (6.55), (6.56) may be reduced to infinite systems of linear equations, viz.

$$
(6.57)\quad \sum_{m=0}^{\infty} \lambda_{nm}^{e} (-1)^m A_{2m}^{(i)} = \beta_{2n}^{(i)}/\pi, \quad \sum_{m=0}^{\infty} \lambda_{nm}^{o} (-1)^m A_{2m+1}^{(i)} = \gamma_{2n+1}^{(i)}/\pi,
$$

where $n = 0, 1, 2, \ldots$, $i = 0, 1$. λ_{nm}^{e}, λ_{nm}^{o}, $\beta_{2n}^{(0)}$, $\gamma_{2n+1}^{(0)}$ are given by (5.41), (5.42), (5.38), (5.40), the latter formula with β_{2m} replaced by $\beta_{2m}^{(0)}$. Similarly we have,

$$
(6.58)\quad \begin{aligned} \gamma_{2n+1}^{(1)} &= \pi \left\{ (-1)^n A_{2n}^{(0)} \frac{n}{4n+1} - (-1)^{n+1} A_{2n+2}^{(0)} \frac{n+1}{4n+5} \right\} \\ &+ \sum_{m=0}^{\infty} (-1)^m A_{2m+1}^{(0)} \\ &\cdot \frac{(4n+3)(4m+2)}{(2n+2m+2)(2n+2m+4)(2n-2m-1)(2n-2m+1)}, \end{aligned}
$$

$$
(6.59)\quad \beta_{2n}^{(1)} = -\frac{2}{\pi} \sum_{m=0}^{\infty} \gamma_{2m+1}^{(1)} \frac{4m+2}{(2m+2n+1)(2m-2n+1)}.
$$

If the expansions of the coefficients B_n contain more than one term, each of these terms may be treated in the manner described above. The final results for the coefficients A_n will follow by superposition.

The solutions of the infinite linear systems (6.57) will have similar properties as stated in section 5.2. Using these properties it can be shown, that the vector grad Φ_0 becomes infinite at the "leading edge" $\rho = a$, $\pi/2 \leqq |\varphi| \leqq \pi$, $z = 0$ of the circular disk and along the edges $x \geqq 0$, $y = \pm a$, $z = 0$ of the "linearized wake". In the neighbourhood of these edges the expansions (5.61), (5.62) hold for the derivatives of Φ_0, the expansions (5.62) being multiplied by a factor $\exp(ikx/a)$.

6.2. For the original problem of the oscillating elliptic wing moving in a compressible medium, the pressure difference Π between the lower and upper side of the aerofoil is given by (cf. (5.64))

(6.60)

$$\begin{aligned}\Pi(x_1, y_1) &= 2\rho_0 \Psi_1(x_1, y_1, +0, t_1) = 2\rho_0 \Psi_{1,0}(x_1, y_1, +0)e^{-i\nu t_1} \\ &= 2\rho_0 \Psi_0(x, y, +0) \exp(-i\varkappa Mx/a)e^{-i\nu t_1},\end{aligned}$$

according to (6.20). ρ_0 denotes the density of the undisturbed medium. A Fourier series for $\Psi_0(\rho, \varphi, +0)$ can be derived from (6.25), (6.27), (6.31). Replacing the function $\exp(-i\varkappa Mx/a)$ by the first two terms of its expansion, viz. $\{1-i\varkappa Mx/a\}$, we obtain approximate values for the lift L and the moment about the y_1-axis, M_y,

$$\begin{aligned}L &= 2\rho_0 e^{-i\nu t_1}\iint_{\Sigma_1} \Psi_{1,0}(x_1, y_1, +0)dx_1 dy_1 \\ &\approx \frac{4\pi\rho_0 U^2 a^2 e^{-i\nu t_1}}{\sqrt{1-M^2}}\left[A_0 + \frac{2}{\pi U} ik(1-M^2)\right. \\ &\quad \left.\cdot \int_0^1 u\sqrt{1-u^2}\, w_0(au)du - \tfrac{1}{3}ikM^2 A_1\right],\end{aligned} \tag{6.61}$$

$$\begin{aligned}M_y &= 2\rho_0 e^{-i\nu t_1}\iint_{\Sigma_1} \Psi_{1,0}(x_1, y_1, +0)x_1 dx_1 dy_1 \\ &\approx \frac{4\pi\rho_0 U^2 a^3 e^{-i\nu t_1}}{\sqrt{1-M^2}}\left[\frac{2}{\pi U}\int_0^1 u\sqrt{1-u^2}\, w_0(au)du + \tfrac{1}{3}A_1\right. \\ &\quad + \frac{2}{3\pi U} ik(1-2M^2)\int_0^1 u^2\sqrt{1-u^2}\, w_1(au)du \\ &\quad \left. - \tfrac{1}{3}ikM^2 A_0 - \tfrac{2}{15}ikM^2 A_2\right],\end{aligned} \tag{6.62}$$

where Σ_1 denotes the elliptic disk $x_1^2+(1-M^2)y_1^2 \leqq a^2$, $z_1 = 0$.

We now derive a general formula for the induced drag acting on a three-dimensional aerofoil in compressible flow. Similar to Ward (cf. [48], § 4.6) we consider an arbitrary aerofoil moving with a velocity $\underline{U}$ in a compressible medium in a direction opposite to the positive direction of the x-axis, where x, y, z will be rectangular coordinates with coordinate axes fixed to the aerofoil. t represents the time. The velocity vector of the medium, denoted by $\underline{u}$, is derived from a velocity potential Φ, $\underline{u} = \text{grad}\,\Phi$. Integration of the equation of continuity and the equation of motion, viz.

$$(6.63) \quad \partial\rho/\partial t + \text{div}\,(\rho\underline{u}) = 0, \quad \rho\partial\underline{u}/\partial t + \rho(\underline{u}\cdot\text{grad})\underline{u} + \text{grad}\,p = 0$$

(ρ, p are the density of the medium [3]) and the pressure, respectively) over an arbitrary volume V, bounded by a closed surface S, inside which there are no sources, yields

$$(6.64) \quad \iint_S \{p\underline{n} + \rho(\underline{u}-\underline{U})\underline{u}\cdot\underline{n}\}dS + \iiint_V \partial/\partial t\{\rho(\underline{u}-\underline{U})\}dV = 0,$$

where $\underline{n}$ denotes the outer normal to S. Let S consist of the surface of the aerofoil, Σ^*, any surface S_1, which completely encloses the aerofoil and a surface σ^*, which lies close to both sides of the wake between Σ^* and S_1. Since p is continuous across the wake and $\underline{u}\cdot\underline{n} = \underline{q}\cdot\underline{n}$ on Σ^*, where $\underline{q}$ denotes the velocity of the aerofoil due to the oscillation, we obtain the following expression for the aerodynamic force $\underline{F}$ acting on the aerofoil,

$$(6.65) \quad \begin{aligned} \underline{F} &= \iint_{\Sigma^*} p\underline{n}dS \\ &= -\iint_{\Sigma^*} \rho(\underline{u}-\underline{U})\underline{q}\cdot\underline{n}dS - \iint_{\sigma^*} \rho(\underline{u}-\underline{U})\underline{u}\cdot\underline{n}dS \\ &\quad - \iint_{S_1} \{p\underline{n} + \rho(\underline{u}-\underline{U})\underline{u}\cdot\underline{n}\}dS \\ &\quad - \iiint_{V_1} \partial/\partial t\{\rho(\underline{u}-\underline{U})\}dV, \end{aligned}$$

where V_1 is the volume contained between S_1 and $\Sigma^* + \sigma^*$. The normal $\underline{n}$ is directed outward from the volume V_1, hence for Σ^*, $\underline{n}$ is the inward normal.

The formula (6.65) will now be linearized. Introducing a linearization parameter ε, it is assumed that the distance of both sides of the aerofoil to the plane $z = 0$ and the velocity $\underline{q}$ are of order ε. The velocity potential Φ and the velocity vector $\underline{u}$ are expanded with respect to ε,

$$(6.66) \quad \begin{cases} \Phi = Ux + \Phi_{\text{I}} + \Phi_{\text{II}} + \ldots, & \underline{u} = \underline{U} + \underline{v}_{\text{I}} + \underline{v}_{\text{II}} + \ldots, \\ \underline{v}_{\text{I}} = \text{grad}\,\Phi_{\text{I}}, & \underline{v}_{\text{II}} = \text{grad}\,\Phi_{\text{II}}, \end{cases}$$

where the subscripts I, II refer to terms of order ε and ε^2 respectively. According to Ward [48] §§ 1.8, 1.9 the functions Φ_{I},

[3]) Only in this section the symbol ρ stands for the density of the medium. In the other sections ρ denotes the cylindrical coordinate as given by (5.5).

Φ_{II} satisfy linear partial differential equations, the boundary conditions being prescribed on both sides $z = \pm 0$ of the surface Σ, which is the projection of Σ^* on the plane $z = 0$. In fact Φ_{I} is the common perturbation velocity potential, which satisfies an equation similar to (6.2). Φ_{I}, Φ_{II} are discontinuous across $\Sigma+\sigma$, where σ is the projection of σ^* on the plane $z = 0$.

From (6.66) and Bernoulli's equation (cf. van de Vooren [46], form. (2.2)) the following expansions can be derived for p and ρ,

$$\begin{aligned}\frac{p-p_0}{\rho_0} = &-\left\{\frac{\partial\Phi_{\mathrm{I}}}{\partial t}+\underline{U}\cdot\underline{v}_{\mathrm{I}}\right\}-\left\{\frac{\partial\Phi_{\mathrm{II}}}{\partial t}+\underline{U}\cdot\underline{v}_{\mathrm{II}}+\tfrac{1}{2}\underline{v}_{\mathrm{I}}\cdot\underline{w}_{\mathrm{I}}\right.\\ &\left.-\frac{1}{2c_0^2}\left(\frac{\partial\Phi_{\mathrm{I}}}{\partial t}\right)^2-\frac{1}{c_0^2}\underline{U}\cdot\underline{v}_{\mathrm{I}}\frac{\partial\Phi_{\mathrm{I}}}{\partial t}\right\}+\dots,\end{aligned} \tag{6.67}$$

$$\rho = \rho_0-\frac{\rho_0}{c_0^2}\left\{\frac{\partial\Phi_{\mathrm{I}}}{\partial t}+\underline{U}\cdot\underline{v}_{\mathrm{I}}\right\}+\dots, \tag{6.68}$$

where

$$\underline{w}_{\mathrm{I}} = \underline{v}_{\mathrm{I}}-(\underline{U}/c_0^2)\underline{U}\cdot\underline{v}_{\mathrm{I}}. \tag{6.69}$$

c denotes the speed of sound defined by $c^2 = dp/d\rho$. A subscript 0 refers to values assumed in the undisturbed medium.

Substituting the expansions (6.66), (6.67), (6.68) into (6.65), we first collect the terms of order ε, yielding

$$\underline{F}_{\mathrm{I}} = \rho_0\underline{U}\times\iint_{S_1}\underline{n}\times\underline{v}_{\mathrm{I}}dS+\rho_0\iint_{S_1}\frac{\partial\Phi_{\mathrm{I}}}{\partial t}\underline{n}dS-\rho_0\iiint_{V_1}\frac{\partial\underline{v}_{\mathrm{I}}}{\partial t}dV. \tag{6.70}$$

It can easily be derived that the x-component of $\underline{F}_{\mathrm{I}}$ is given by

$$(\underline{F}_{\mathrm{I}})_x = -\rho_0\iint_{\Sigma^*}\frac{\partial\Phi_{\mathrm{I}}}{\partial t}\cos\,(\underline{n},\,\underline{x})dS \tag{6.71}$$

where $\underline{n}$ is the inward normal to Σ^* and $(\underline{n},\,\underline{x})$ denotes the angle between $\underline{n}$ and the positive direction of the x-axis. It is obvious that the result (6.71) is of order ε^2.

Secondly the terms of order ε^2 are collected yielding $\underline{F}_{\mathrm{II}}$. It can easily be shown, that the x-component of $\underline{F}_{\mathrm{II}}$ is given by

$$\begin{aligned}(\underline{F}_{\mathrm{II}})_x = &-\rho_0\iint_{\Sigma^*}\frac{\partial\Phi_{\mathrm{I}}}{\partial x}\underline{q}\cdot\underline{n}\,dS-\rho_0\iint_{\sigma^*}\frac{\partial\Phi_{\mathrm{I}}}{\partial x}\underline{u}\cdot\underline{n}dS\\ &+\rho_0\iint_{S_1}\left\{\tfrac{1}{2}\underline{v}_{\mathrm{I}}\cdot\underline{w}_{\mathrm{I}}\cos\,(\underline{n},\,\underline{x})-\frac{\partial\Phi_{\mathrm{I}}}{\partial x}\underline{w}_{\mathrm{I}}\cdot\underline{n}\right\}dS\\ &+\frac{\rho_0}{c_0^2}\iiint_{V_1}\left\{\frac{\partial^2\Phi_{\mathrm{I}}}{\partial t^2}\frac{\partial\Phi_{\mathrm{I}}}{\partial x}+U\frac{\partial}{\partial t}\left(\frac{\partial\Phi_{\mathrm{I}}}{\partial x}\right)^2\right\}dV.\end{aligned} \tag{6.72}$$

The latter result may be simplified. Let the equation of Σ^* be given by $z = g(x, y, t)$, both for the upper and lower side of the aerofoil. Then we have on Σ^*,

$$\text{(6.73)} \qquad \underline{q} \cdot \underline{n} = \mp \partial g/\partial t, \ \cos(\underline{n}, \underline{x}) = \pm \partial g/\partial x$$

where the upper and lower sign refer to the upper and lower side of Σ^*. In (6.73) $\underline{n}$ will be the inward normal to Σ^*. Substituting these results the integrals over Σ^* and σ^* in (6.71), (6.72) may be replaced by integrals over Σ and σ, the errors being of order ε^3. Similarly, V_1 may be taken as the complete volume enclosed by the surface S_1. Using the oddness in z of the function Φ_{I}, we ultimately obtain the following result for the induced drag D i.e. the sum of $(\underline{F}_{\mathrm{I}})_x$ and $(\underline{F}_{\mathrm{II}})_x$,

(6.74)

$$\begin{aligned} D = &-\frac{2\rho_0}{U}\iint_{\Sigma^+}\left\{\Psi w - \Psi\frac{\partial g}{\partial t}\right\}dxdy + 2\rho_0\iint_{\Sigma^+ + \sigma^+}\frac{\partial\Phi}{\partial x}\frac{\partial\Phi}{\partial z}dxdy \\ &+\rho_0\iint_{S_1}\left[\tfrac{1}{2}\left\{(1-M^2)\left(\frac{\partial\Phi}{\partial x}\right)^2 + \left(\frac{\partial\Phi}{\partial y}\right)^2 + \left(\frac{\partial\Phi}{\partial z}\right)^2\right\}\cos(\underline{n}, \underline{x}) \right. \\ &-\frac{\partial\Phi}{\partial x}\left\{(1-M^2)\frac{\partial\Phi}{\partial x}\cos(\underline{n}, \underline{x}) + \frac{\partial\Phi}{\partial y}\cos(\underline{n}, \underline{y}) \right. \\ &\left.\left. + \frac{\partial\Phi}{\partial z}\cos(\underline{n}, \underline{z})\right\}\right] dS + \frac{\rho_0}{c_0^2}\iiint_{V_1}\left\{\frac{\partial^2\Phi}{\partial t^2}\frac{\partial\Phi}{\partial x} + 2U\frac{\partial\Phi}{\partial x}\frac{\partial^2\Phi}{\partial x\partial t}\right\}dV, \end{aligned}$$

where we introduced the downwash w and the acceleration potential Ψ given by

$$w = \partial g/\partial t + U\partial g/\partial x, \ \Psi = \partial\Phi/\partial t + U\partial\Phi/\partial x.$$

In (6.74) the subscript I in Φ_{I} has been omitted. $\underline{n}$ is the outer normal to S_1 and $M = U/c_0$. Σ^+, σ^+ are the upper sides $z = +0$ of Σ and σ. Owing to the function Φ satisfying an equation similar to (6.2), the formula (6.74) for D will be invariant for changes of S_1 as can easily be shown.

For the oscillating elliptic wing in compressible flow, as considered in section 6.1, the induced drag is given by (6.74) with a subscript 1 to be attached to Φ, Ψ, w, g, x, y, z, t. Σ^+, σ^+ change into Σ_1^+, σ_1^+ i.e. the upper side $z_1 = +0$ of the elliptic disk Σ_1 and of the linearized wake σ_1 within S_1. The complete linearized wake is determined by $x_1 \geqq \sqrt{a^2-(1-M^2)y_1^2}$, $|y_1| \leqq a/\sqrt{1-M^2}$, $z_1 = \pm 0$.

According to (6.3), (6.4), (6.5), (6.7) the equation of the wing surface, the downwash, the velocity potential and the acceleration potential contain a complex time factor $e^{-i\nu t_1}$. Only the real part of the corresponding functions has a physical meaning. Therefore, in order to calculate D, all complex quantities in (6.74) must be replaced by their real parts. This replacement can be performed using the identity $\mathrm{Re}\, w_1 \cdot \mathrm{Re}\, w_2 = \frac{1}{2}\mathrm{Re}(w_1 w_2 + w_1 \bar{w}_2)$, valid for any two complex numbers w_1, w_2 where Re stands for real part and a bar denotes that the complex conjugate quantity is meant.

Application of the Lorentz transformation (6.9) to the integral (6.74) changes the surface S_1 into a surface S enclosing the circular disk Σ, $x^2+y^2 \leqq a^2$, $z = 0$. The singularities of the vector grad Φ_0 being the same as in the incompressible case, the surface S is chosen as stated in section 5.4. Omitting the details of the derivation, we present the following final result, similar to (5.69),

$$D = \mathrm{Re}\,\{D^{(1)}e^{-2i\nu t_1} + D^{(2)}\} \tag{6.75}$$

where

(6.76)

$$\begin{aligned} D^{(1)} = (-\rho_0/U)\iint_\Sigma &\Psi_0(x, y, +0)\, w(x, y) \exp(-2i\varkappa Mx/a)dxdy \\ &- \frac{i\omega\rho_0}{a\sqrt{1-M^2}}\iint_\Sigma \Psi_0(x, y, +0)\, g_0(x, y) \exp(-i\varkappa Mx/a)\, dxdy \\ &+ \tfrac{1}{2}\rho_0 U^2 a^2 \int_{\pi/2}^{\pi}\int_{-\pi}^{\pi} [\tfrac{1}{2}\{R(\varphi)\}^2 \sec\varphi \cos\gamma \\ &\qquad + \{R(\varphi)\}^2 \sec\varphi \sin^2 \tfrac{1}{2}\gamma] \exp(-2i\varkappa M \cos\varphi) d\gamma d\varphi, \end{aligned}$$

(6.77)

$$\begin{aligned} D^{(2)} = (-\rho_0/U)\iint_\Sigma &\bar{\Psi}_0(x, y, +0)\, w(x, y)\, dxdy \\ &- \frac{i\omega\rho_0}{a\sqrt{1-M^2}}\iint_\Sigma \bar{\Psi}_0(x, y, +0) g_0(x, y) \exp(i\varkappa Mx/a)dxdy \\ &+ \tfrac{1}{2}\rho_0 U^2 a^2 \int_{\pi/2}^{\pi}\int_{-\pi}^{\pi} [\tfrac{1}{2}|R(\varphi)|^2 \sec\varphi \cos\gamma \\ &\qquad + |R(\varphi)|^2 \sec\varphi \sin^2 \tfrac{1}{2}\gamma]\, d\gamma d\varphi, \end{aligned}$$

where $R(\varphi)$ denotes the left-hand side of (5.35). The function $g_0(x, y)$ is defined by

$$g_{1,0}(x_1, y_1) = g_{1,0}(x, y/\sqrt{1-M^2}) = g_0(x, y). \tag{6.78}$$

Expanding the exponential functions, the formulae (6.76), (6.77) may be reduced in a similar manner as stated for (5.69). Ultimately, expansions in powers of ω are obtained for $D^{(1)}$, $D^{(2)}$ consisting of two terms. However, these general results corresponding to an arbitrary mode of oscillation of the wing, are of a very complicated form. Therefore it is easier to perform the above-mentioned reduction after having substituted concrete values of $g_0(x, y)$, $w(x, y)$, $\Psi_0(x, y, +0)$, $R(\varphi)$ into (6.76), (6.77).

6.3. For a downwash distribution $w(\rho, \varphi)$ odd in φ, the boundary value problem can be solved in the same manner as described in section 6.1. In the various Fourier series $\cos n\varphi$ must be replaced by $\sin n\varphi$. It can easily be shown that the Kutta condition again leads to the equation (5.74). Denoting the second term of (5.74) by $\sum_{n=1}^{\infty} D_n \sin n\varphi$, the coefficients D_n may be expanded in powers of ω. Let these expansions consist of only one term, of order ω^j, then we make the substitution (6.54) where $A_n^{(0)}$, $A_n^{(1)}$ depend on ω only through a factor ω^j. The equations (6.55), (6.56) will now change into

$$(6.79)\quad \begin{cases} \sum_{n=1}^{\infty} A_n^{(0)} \sin n\varphi + \sum_{n=1}^{\infty} D_n \sin n\varphi = 0, \text{ for } -\pi/2 \leqq \varphi \leqq \pi/2, \\ (\pi/2)\sum_{n=1}^{\infty}(-1)^n A_{2n}^{(0)} P_{2n}(y/a) + \sum_{n=0}^{\infty}(-1)^n A_{2n+1}^{(0)} Q_{2n+1}(y/a) = C_1, \\ \qquad\qquad \text{for } -a < y < a, \end{cases}$$

$$(6.80)\quad \begin{cases} \sum_{n=1}^{\infty} A_n^{(1)} \sin n\varphi = 0, \text{ for } -\pi/2 \leqq \varphi \leqq \pi/2, \\ (\pi/2)\sum_{n=1}^{\infty}(-1)^n A_{2n}^{(1)} P_{2n}(y/a) + \sum_{n=0}^{\infty}(-1)^n A_{2n+1}^{(1)} Q_{2n+1}(y/a) \\ \quad = \sum_{n=1}^{\infty}(-1)^n A_{2n}^{(0)} \dfrac{2n}{4n+1}\{Q_{2n+1}(y/a) - Q_{2n-1}(y/a)\} \\ \quad\quad -\pi\sum_{n=0}^{\infty}(-1)^n A_{2n+1}^{(0)} \dfrac{n+\frac{1}{2}}{4n+3}\{P_{2n+2}(y/a) - P_{2n}(y/a)\} + C_2, \\ \qquad\qquad \text{for } -a < y < a. \end{cases}$$

C_1 and C_2 are constants which disappear at the further reduction. Similar as in § 5 the equations (6.79), (6.80) may be reduced to infinite systems of linear equations, viz.

$$(6.81)\quad \sum_{m=1}^{\infty} \mu_{nm}^{e}(-1)^m A_{2m}^{(i)} = \theta_{2n}^{(i)}/\pi, \quad \sum_{m=0}^{\infty} \mu_{nm}^{o}(-1)^m A_{2m+1}^{(i)} = \delta_{2n+1}^{(i)}/\pi,$$

where $n = 1, 2, 3, \ldots$, $n = 0, 1, 2, \ldots$ respectively and $i = 0, 1$.

μ^e_{nm}, μ^o_{nm}, $\delta^{(0)}_{2n+1}$, $\theta^{(0)}_{2n}$ are given by (5.77), (5.78), (5.79), (5.80), the latter formula with δ_{2m+1} replaced by $\delta^{(0)}_{2m+1}$. Similarly we have,

$$\theta^{(1)}_{2n} = \pi \left\{ -(-1)^{n-1} A^{(0)}_{2n-1} \frac{n-\frac{1}{2}}{4n-1} + (-1)^n A^{(0)}_{2n+1} \frac{n+\frac{1}{2}}{4n+3} \right\}$$

(6.82)
$$+ \sum_{m=1}^{\infty} (-1)^m A^{(0)}_{2m} \cdot \frac{(4n+1)4m}{(2n+2m)(2n+2m+2)(2n-2m-1)(2n-2m+1)},$$

(6.83)
$$\delta^{(1)}_{2n+1} = -\frac{2}{\pi} \sum_{m=1}^{\infty} \theta^{(1)}_{2m} \frac{4m}{(2n+2m+1)(2n-2m+1)}.$$

Similar to (6.61), (6.62) the following expansion holds for the moment about the x_1-axis, M_x,

$$M_x = 2\rho_0 e^{-i\nu t_1} \iint_{\Sigma_1} \Psi_{1,0}(x_1, y_1, +0) y_1 dx_1 dy_1$$

(6.84)
$$\approx \frac{4\pi\rho_0 U^2 a^3 e^{-i\nu t_1}}{3(1-M^2)} \left[A_1 + \frac{2}{\pi U} ik(1-M^2) \cdot \int_0^1 u^2 \sqrt{1-u^2}\, w_1(au) du - \tfrac{2}{5} ikM^2 A_2 \right].$$

The formulae (6.75), (6.76), (6.77) for D, $D^{(1)}$, $D^{(2)}$ remain valid provided that $R(\varphi)$ stands for the left-hand side of (5.74).

6.4. For some simple modes of oscillation numerical results were derived for lift, moment and induced drag. The infinite linear systems were solved in the same manner as stated in section 5.6.

(i) Vertical translation, $z_1 = Aae^{-i\nu t_1}$.

According to (6.4), (6.16) the "downwash distribution" $w(\rho, \varphi)$ is given by

(6.85)
$$w(\rho, \varphi) = -\frac{i\omega UA}{\sqrt{1-M^2}} \{1 + i\varkappa M(\rho/a)\cos\varphi\}.$$

Hence the coefficients B_n read,

(6.86)
$$B_0 = \frac{2\omega\varkappa MA}{3\pi\sqrt{1-M^2}}, \quad B_1 = -\frac{2i\omega A}{\pi\sqrt{1-M^2}},$$
$$B_2 = \frac{2\omega\varkappa MA}{3\pi\sqrt{1-M^2}}, \quad B_n = 0 \text{ for } n \geq 3.$$

The following results were calculated,

$$L = \frac{i\omega\rho_0 U^2 a^2}{1-M^2} A e^{-i\nu t_1} \left\{2.81176 + \frac{i\omega}{1-M^2}(-2.30052+2.92905\, M^2)\right\},$$

$$M_y = \frac{i\omega\rho_0 U^2 a^3}{1-M^2} A e^{-i\nu t_1} \left\{-1.46453 - \frac{i\omega}{1-M^2}(0.18478+0.49019\, M^2)\right\},$$

$$D = \mathrm{Re}\,\{D^{(1)} e^{-2i\nu t_1}\} + D^{(2)},$$

$$D^{(1)} = \frac{\omega^2\rho_0 U^2 a^2}{1-M^2} A^2 \left\{0.77645 + \frac{i\omega}{1-M^2}(0.202646+0.547084\, M^2)\right\},$$

$$D^{(2)} = -0.77645\, \frac{\omega^2\rho_0 U^2 a^2}{1-M^2} A^2.$$

(ii) Rotation about the y_1-axis, $z_1 = Bx_1 e^{-i\nu t_1}$.
According to (6.4), (6.16) the "downwash distribution" $w(\rho, \varphi)$ is given by

$$w(\rho, \varphi) = \frac{UB}{\sqrt{1-M^2}} \{1 + (i\varkappa M - i\omega)(\rho/a) \cos \varphi\}. \tag{6.87}$$

Hence the coefficients B_n read,

$$B_0 = \frac{2(i\varkappa M - i\omega)B}{3\pi\sqrt{1-M^2}}, \quad B_1 = \frac{2B}{\pi\sqrt{1-M^2}},$$
$$B_2 = \frac{2(i\varkappa M - i\omega)B}{3\pi\sqrt{1-M^2}}, \quad B_n = 0 \text{ for } n \geqq 3. \tag{6.88}$$

The following results were calculated,

$$L = \frac{\rho_0 U^2 a^2}{1-M^2} B e^{-i\nu t_1} \left\{-2.81176 + \frac{i\omega}{1-M^2}(3.76504-4.39358\, M^2)\right\},$$

$$M_y = \frac{\rho_0 U^2 a^3}{1-M^2} B e^{-i\nu t_1} \left\{1.46453 + \frac{i\omega}{1-M^2}(0.87405-0.19908\, M^2)\right\},$$

$$D = \mathrm{Re}\,\{D^{(1)} e^{-2i\nu t_1}\} + D^{(2)},$$

$$D^{(1)} = \frac{\rho_0 U^2 a^2}{1-M^2} B^2 \left\{0.62943 + \frac{i\omega}{1-M^2}(-2.74475+2.30929\, M^2)\right\},$$

$$D^{(2)} = 0.62943\, \frac{\rho_0 U^2 a^2}{1-M^2} B^2.$$

(iii) Rotation about the x_1-axis, $z_1 = Cy_1 e^{-i\nu t_1}$.
According to (6.4), (6.16) the "downwash distribution" $w(\rho, \varphi)$ is given by

(6.89) $$w(\rho, \varphi) = -\frac{i\omega UC}{1-M^2}\{(\rho/a)\sin\varphi + \tfrac{1}{2}i\varkappa M(\rho/a)^2 \sin 2\varphi\}.$$

Hence the coefficients D_n read,

(6.90) $$D_1 = \frac{4\omega\varkappa MC}{15\pi(1-M^2)}, \quad D_2 = -\frac{2i\omega C}{3\pi(1-M^2)},$$
$$D_3 = \frac{4\omega\varkappa MC}{15\pi(1-M^2)}, \quad D_n = 0 \text{ for } n \geqq 4.$$

The following results were calculated,

$$M_x = \frac{i\omega\rho_0 U^2 a^3}{(1-M^2)^2} Ce^{-i\nu t_1}\left\{0.384786 + \frac{i\omega}{1-M^2}(-0.352239+0.361399\,M^2)\right\},$$

$$D = \text{Re}\,\{D^{(1)}e^{-2i\nu t_1}\} + D^{(2)},$$

$$D^{(1)} = \frac{\omega^2\rho_0 U^2 a^2}{(1-M^2)^2} C^2\left\{0.098126 + \frac{i\omega}{1-M^2}(0.001698+0.025944\,M^2)\right\},$$

$$D^{(2)} = -0.098126\,\frac{\omega^2\rho_0 U^2 a^2}{(1-M^2)^2}\,C^2.$$

In the special case $M = 0$ our results for lift and moment agree with van Spiegel's [44] values. It has been verified that the lifts and moments for the translational vibration (i) and the rotational vibration (ii) satisfy the reciprocity relation, cf. van de Vooren [46].

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