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On non-normal numbers *

by

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It was first discovered by D. D. Wall [13] in 1949 that a real number x is normal to the base $g \geqq 2$ if and only if the sequence $g^n x (n = 1, 2, \ldots)$ is uniformly distributed mod 1. But it had been known since the occurrence of Borel's [2] celebrated theorem in 1909 that almost all real numbers are normal to any base, and consequently the problem arose to investigate various types of sets of non-normal numbers. In particular, several authors have determined the Hausdorff (or fractional) dimension (cf. Hausdorff [6]) of such sets. The following results are typical:

1. Given digit frequencies

For any real number $x \in (0, 1]$, we consider the g-adic expansion

$$(*) \qquad x = \sum_{i=1}^{\infty} \frac{e_i}{g^i}$$

where $g \geqq 2$ is a fixed integer and the digits e_i, $0 \leqq e_i < g$, are so chosen that infinitely many of them are different from zero. Let

$$A_j(x, n) = \sum_{i=1,\, e_i=j}^{n} 1 \quad (j = 0, \ldots, g-1)$$

and define, for given non-negative numbers $\zeta_0, \zeta_1, \ldots, \zeta_{g-1}$ with $\sum_{j=0}^{g-1} \zeta_j = 1$, $G = G(\zeta_0, \ldots, \zeta_{g-1})$ to be the set of all such x satisfying

$$\lim_{n \to \infty} \frac{A_j(x, n)}{n} = \zeta_j \qquad (j = 0, \ldots, g-1).$$

Then, as was shown in 1949 by H. G. Eggleston [4],

$$\dim G = -\frac{1}{\log g} \sum_{j=0}^{g-1} \zeta_j \log \zeta_j \qquad (\text{"}0 \log 0\text{"} = 1).$$

* Nijenrode lecture.

For reference, we denote the function in this equation by $d(\zeta_0, \ldots, \zeta_{g-1}) = d(\zeta)$.

In the special case $g = 2$ this theorem has been proved independently by V. Knichal [7] in 1933 and by A. S. Besicovitch [1] in 1934, in slightly different forms.

2. Missing digits.

The Cantor ternary set C may be interpreted as the set of all $x \in (0, 1]$ in whose expansion (*) to the base $g = 3$ all digits e_i are different from 1, united with a certain countable set. It was shown by F. Hausdorff [6] in 1918 that

$$\dim C = \frac{\log 2}{\log 3}.$$

The following generalization was proved by the speaker [9] in 1953: Let $g \geqq 2$ be fixed and let $F = f_1 f_2 \ldots f_i$ be any finite block of (not necessarily distinct) g-adic digits. Furthermore, let K_F be the set of all $x \in (0, 1]$ in whose g-adic expansion no block of i consecutive digits equals F. Then, if $P(F)$ denotes the set of all integers p for which the block of the first p digits and the block of the last p digits of F are equal, and if $\gamma(F)$ is the greatest positive root of the i-th degree equation

$$\sum_{p \in P(F)} (z^p - g z^{p-1}) + 1 = 0,$$

then

$$\dim K_F = \frac{\log \gamma(F)}{\log g}.$$

In a later paper [11] the case was studied where a finite set $\mathfrak{F} = \{F_1, \ldots, F_n\}$ of such blocks are excluded.

3. Given digit averages

For each $x \in (0, 1]$ with the g-adic expansion (*), we define $S(x, n) = \sum_{i=1}^{n} e_i$ and consider, for a given $\zeta \in [0, g-1]$, the set $M(\zeta)$ of all x satisfying

$$\lim_{n \to \infty} \frac{1}{n} S(x, n) = \zeta.$$

It was shown in 1951 by H. G. Eggleston [5] that

$$\dim M(\zeta) = \frac{\log(1+r+\ldots r^{g-1})-\zeta \log r}{\log g}$$

where r is the greatest positive root of the equation.

$$\sum_{j=0}^{g-1} (j-\zeta)x^j = 0.$$

This theorem was generalized by the speaker [10] to the case where the set $\{0, 1, \ldots, g-1\}$ is subdivided into mutually disjoint subsets $\mathfrak{G}_1, \mathfrak{G}_2, \ldots, \mathfrak{G}_m$, and weighted averages

$$S_\mu(x, n) = \sum_{i=1, e_i \in \mathfrak{G}_\mu}^{n} \lambda_{e_i} \quad (\mu = 1, \ldots, m)$$

with given non-negative weights $\lambda_0, \lambda_1, \ldots, \lambda_{g-1}$ are considered. Then the dimension of the set $M(\zeta_1, \ldots, \zeta_m)$ of all $x \in (0, 1]$ was determined for which the limits

$$\lim_{n\to\infty} \frac{1}{n} S_\mu(x, n) \quad (\mu = 1, \ldots, m)$$

exist and have given values $\zeta_1, \ldots, \zeta_m$.

4. Oscillating digit frequencies

In order to study real numbers x for which some or all of the frequencies $A_j(x, n)/n$ oscillate, the speaker [12] used the following method: For any index n let $p_n(x)$ be the point in the simplex $H_g = \{0 \leqq \zeta_j \leqq 1 \ (j = 0, \ldots, g-1); \sum_{j=0}^{g-1} \zeta_j = 1\}$ which has coordinates $(A_0(x, n)/n, \ldots, A_{g-1}(x, n)/n)$. Furthermore, let $V_g(x)$ be the set of limit points of the sequence $p_1(x), p_2(x), \ldots$. Obviously, $V_g(x)$ may consist of a single point, and this happens, in particular, whenever x is normal. But it was shown in 1957 (cf. [12]) that, given any continuum (i.e. a closed, connected set) $C \subseteqq H_g$, there exists a non-empty set $G(C)$ of numbers $x \in (0, 1]$ for which $V_g(x) = C$. Furthermore,

$$\dim G(C) = \min_{\zeta \in C} d(\zeta)$$

where $d(\zeta)$ is the function defined above. Conversely, for any number x, the set $V_g(x)$ is a continuum contained in the simplex H_g.

5. Unsolved problems

In connection with results mentioned above, the following questions appear to be of interest:

A) Given two integers $g \geqq 2$, $h \geqq 2$ such that $g^n \neq h^m$ for all positive integers m, n, and two continua $C_1 \subseteqq H_g$, $C_2 \subseteqq H_h$, do there exist numbers $x \in (0, 1]$ for which

$$V_g(x) = C_1 \text{ and } V_h(x) = C_2?^*$$

B) If so, what is the Hausdorff dimension of the set of all such x?

C) Which of the sets $G(C)$ contain, and which do not contain, any algebraic number?

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* It was shown by W. Schmidt [8] and also by J. W. S. Cassels [3] that x need not be normal to the base h if it is normal to the base g.

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