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Some intuitionistic remarks about transformations of sequences

by

J. G. Dijkman

1. The transformations of sequences $\{s_n\}$ by infinite matrices $(c_{m,n})$ are well known (cf: G. H. Hardy: Divergent Series, Oxford University Press 1949, Ch. III).

From a given sequence $\{s_n\}$ and a given matrix $(c_{m,n})$ a new sequence $\{t_m\}$ is defined by:

$$t_m = \sum_{n=0}^{\infty} c_{m,n} s_n. \tag{1}$$

Of course such a definition makes sense only if for every m the series involved are convergent (in some sense).

It is obvious that the type of convergence of the sequence $\{t_m\}$ depends in general both on the given sequence $\{s_n\}$ and on the matrix $(c_{m,n})$.

2.1. In this paper transformations by matrices $(c_{m,n})$ are considered, which transform every convergent sequence $\{s_n\}$ into a convergent sequence $\{t_m\}$.

Schur (Journal f.d.r. und angew. Mathematik, **151** (1921) 79—111) proved a theorem about this subject (Kojima, Tokohu M.J. 12: 291—326 for semi-matrices). We shall analyse the formulation and proof of this theorem as given by Hardy (page 43).

THEOREM: In order that a matrix $(c_{m,n})$ transforms every convergent sequence $\{s_n\}$ into a convergent sequence $\{t_m\}$ the following conditions are necessary and sufficient:

(2.1.1) $\sum_{n=0}^{\infty} |c_{m,n}|$ is convergent for every m and a number H exists independent of m with $\sum_{n=0}^{\infty} |c_{m,n}| < H$ for all m,

(2.1.2) for every integer n the sequence $\{c_{m,n}\}$ is convergent and $\lim_{m\to\infty} c_{m,n} = \delta_n$,

(2.1.3) the sequence $\{c_m\}$ is convergent and $\lim_{m\to\infty} c_m = \delta$ with $c_m = \sum_{n=0}^{\infty} c_{m,n}$.

If these conditions are fulfilled $\sum_{n=0}^{\infty} \delta_n$ is absolutely convergent and the sequence $\{t_m\}$ is convergent with limit value:

$$t = \delta s + \sum_{n=0}^{\infty} \delta_n (s_n - s) = s(\delta - \sum_{n=0}^{\infty} \delta_n) + \sum_{n=0}^{\infty} \delta_n s_n \tag{2}$$

with $s = \lim_{n\to 0} s_n$.

2.2. From the intuistionistic point of view we are able to prove

$$^{+}\lim_{m\to\infty} t_m = \delta \cdot s + \sum_{n=0}^{\infty} \delta_n (s_n - s),$$

if we read positive convergence instead of convergence, but, surprisingly, difficulties are met in proving the second right-hand member of (2). (For the intuitionistic definitions of convergence I refer to A. Heyting: Intuitionism, North-Holland Publ. Comp. Amsterdam 1956. For the intuitionistic theory of series I refer to M. J. Belinfante's papers mentioned in Heyting's book. It may be useful to observe the difference between the definition of negative convergence used by Belinfante and the one given in Heyting's book).

2.3.1. The proof given by Hardy can almost be rewritten and we shall see that the absolute convergence of the series $\sum_{n=0}^{\infty} \delta_n$ is not substantial. This is important for it is not possible (nowadays) to prove the absolute convergence of $\sum \delta_n$.

2.3.2. The proof runs as follows:

We prove that the elements $t_m = \sum_{n=0}^{\infty} c_{m,n} s_n$ are defined. From $^{+}\lim_{n\to\infty} s_n = s$ it follows that the sequence $\{s_n\}$ is bounded. The series $\sum_{n=0}^{\infty} |c_{m,n}|$ is positively convergent and this implies together with the boundedness of $\{s_n\}$ that $\sum_{n=0}^{\infty} c_{m,n} s_n$ is positively convergent for every value of m. First we restrict ourselves to the case: $s = 0$.

Choose an arbitrary value for $\varepsilon > 0$ and take the integer $N = N(\varepsilon)$ in such a way that: $|s_n| < \varepsilon : 4H$ for all $n > N$. Take $k > N$, then we get:

$$t_m - \sum_{n=0}^{k} \delta_n s_n = \sum_{n=0}^{N} (c_{m,n} - \delta_n) s_n + \sum_{N+1}^{k} (c_{m,n} - \delta_n) s_n + \sum_{k+1}^{\infty} c_{m,n} s_n.$$

$\sum_{n=0}^{\infty} c_{m,n} s_n$ being positively convergent we are able to determine $N_1 = N_1(\varepsilon, m)$ with:

$$\Big| \sum_{n=k+1}^{\infty} c_{m,n} s_n \Big| < \tfrac{1}{4}\varepsilon \text{ for every } k > N_1.$$

From $^{+}\lim_{m\to\infty} c_{m,n} = \delta_n$ it follows that there is a N_2 such that

$\Big| \sum_{n=0}^{N} (c_{m,n} - \delta_n) s_n \Big| < \frac{1}{4}\varepsilon$ for every $m > N_2$ (use: the sequence $\{s_n\}$ is bounded).

Obviously:

$$\Big|\sum_{N_1+1}^{k}(c_{m,n}-\delta_n)s_n\Big| \not> \sum_{N_1+1}^{k}(|c_{m,n}|+|\delta_n|)\cdot|s_n| < \frac{\varepsilon}{4H}\sum_{N_1+1}^{k}(|c_{m,n}|+|\delta_n|),$$

and from

$$\sum_{n=0}^{k}|\delta_n| = \sum_{n=0}^{k}{}^{+}\lim_{m\to\infty}|c_{m,n}| = {}^{+}\lim_{m\to\infty}\sum_{n=0}^{k}|c_{m,n}| \not> H$$

it follows:

$$\Big|\sum_{n=0}^{k}(c_{m,n}-\delta_n)s_n\Big| < \frac{\varepsilon}{4H}\cdot 2H = \tfrac{1}{2}\varepsilon.$$

Now we have reached a point where we can find an integer M such that

$$\Big|t_m-\sum_{n=0}^{k}\delta_n s_n\Big| < \varepsilon \text{ for every } k > M \text{ and } m > M;$$

because k and m are mutually independent we obtain:

$$|t_{m_1}-t_{m_2}| < 2\varepsilon \text{ for every } m_i > M;$$

from this and from Cauchy's general principle of convergence it follows that the sequence $\{t_m\}$ is positively convergent.

The general case is reduced to the special case considered by the conversion: $s'_n = s_n - s$ and $t'_m = \sum_{n=0}^{\infty} c_{m,n}s'_n$.
Then we obtain:

$$(3) \qquad {}^{+}\lim_{m\to\infty} t_m = {}^{+}\lim_{k\to\infty}\sum_{n=0}^{k}\delta_n(s_n-s)+s\delta.$$

3.1. It may seem natural to suppose that

$${}^{+}\lim_{m\to\infty} t_m = {}^{+}\lim_{p\to\infty}\sum_{n=0}^{p}\delta_n s_n - s\,{}^{+}\lim_{p\to\infty}\sum_{0}^{p}\delta_n + s\cdot\delta$$

may be deduced from (3).

It is remarkable that it is possible to construct an example of a transformation satisfying conditions (2.1.1), (2.1.2) and (2.1.3) for which however the series $\sum\delta_n$ is only negatively convergent.

It will be obvious that the absolute convergence of the constructed example is out of discussion.

3.2. To construct the promised example consider the decimal expansion of π.

Let τ indicate the sequence consisting of the integers $0\ 1\ 2\cdots 8\ 9$.

We define:

If no sequence τ occurs in the first m digits after the decimal point in π, define: $c_{m,n} = 0$ for all n.

If in the first m digits after the decimal point in π the sequence τ occurs and if k is the index of the 9 in the first sequence τ that occurs in the decimal expansion of π, then we take: $c_{m,n} = 1/k$ for $n < k$, $c_{m,n} = -1$ for $n = k$ and $c_{m,n} = 0$ for $n > k$.

We begin by verifying that the conditions (2.1.1), (2.1.2) and (2.1.3) are fulfilled.

For every value of m it is possible to verify the occurrence of τ in the first m digits of π.

If τ does not occur in the first m digits, we have: $\sum_n |c_{m,n}| = 0$; when τ occurs in the first m digits we may write

$$\sum_{n=0}^{\infty} |c_{m\ n}| = \sum_{n=0}^{k} |c_{m,n}| = 2.$$

This means that the series $\sum |c_{m,n}|$ is positively convergent for every value of m, and we can choose $H = 3$ (cf. (2.1.1)).

From the absolute positive convergence it follows that the series $\sum_{n=0}^{\infty} c_{m,n}$ is positively convergent for every m.

Now we consider $\delta_n = {}^{+}\lim_{m\to\infty} c_{m,n}$.

We have to prove that δ_n is defined by $\delta_n = {}^{+}\lim_{m\to\infty} c_{m,n}$ for every fixed value of n. Therefore, we consider δ_n (n fixed). The existence of δ_n becomes clear by calculating it as accurately as desired. Take a value $\varepsilon > 0$, an integer $m_0 > 1/\varepsilon$ with $m_0 > n$, and calculate the first m_0 digits of the decimal expansion of π. This calculation informs us about the occurrence of τ among the m_0 digits of π. Does τ occur, then we know all $c_{m,n}$ and δ_n has one of the values $1/k$, -1 or 0 and we can decide which of them.

If τ does not occur, then from $m_0 > n$ and $m_0 > 1/\varepsilon$ it follows $0 \not> \delta_n < \varepsilon$. From $c_m = \sum c_{m,n} = 0$ for every value of m it follows: $\delta = {}^{+}\lim c_m = 0$. We now consider the series $\sum \delta_n$.

From the definition of the matrix we see:

$\delta_n = 0$ if τ does not occur in the decimal expansion of π;

$\delta_n = 1/k$ for $n < k$ if τ occurs;

$\delta_n = -1$ for $n = k$;

$\delta_n = 0$ for $n > k$;

and it is obvious that the series $\sum \delta_n$ is negatively convergent with limitvalue 0.

This example shows that the series $\sum |\delta_n|$ need not be positively convergent when the matrix $(c_{m,n})$ fulfills the conditions required by the theorem. The series $\sum|\delta_n|$ is negatively convergent with limitvalues 0 and 2.

We have thus derived an example for which it is not allowed to write

$$(4) \qquad \sum_{n=0}^{\infty} \delta_n(s_n-s) = \sum_{n=0}^{\infty} \delta_n s_n - s \sum_{n=0}^{\infty} \delta_n.$$

However, it follows that the two series occurring in the righthand member of (4) have the same type of convergence. This means: they are simultaneously positively or negatively convergent or non-oscillating. In the same way it follows that we shall not succeed in proving the positive convergence of $\sum\delta_n s_n$ in the case $s \neq 0$.

3.3. In section 9 attention will be paid to the necessity of the conditions mentioned in the theorem.

4.1. A transformation satisfying the conditions (2.1.1), (2.1.2) and (2.1.3) may change the character of convergence of a negatively convergent sequence. This will be elaborated now.

To this aim we shall construct an example of a negatively convergent sequence, which is transformed into a twofold negatively convergent sequence.

It is possible to extend this example to an example of a transformation showing that a negatively convergent sequence may be transformed into a p-fold negatively convergent sequence. Here p is an arbitrarily chosen natural number.

4.2.1. To construct the example we take the matrix $(c_{m,n})$ as defined in 3.2. and we define a sequence $\{s_n\}$ in the following way:

If τ does not occur in the first n digits of π then we define $s_n = 1$;

if τ occurs in the first n digits of π and if k is the index of the 9 in the first sequence τ that occurs then we define: $s_k = (-1)^k$ and $s_n = 1$ for $n > k$.

For every value of m the element $t_m = \sum c_{m,n}s_n$ of the sequence $\{t_m\}$ is defined. However, this sequence is twofold negatively convergent, for if τ does not occur in the decimal expansion of π, then $t_m = 0$ for every m and the limitvalue of the sequence is 0, but if τ occurs and k has the indicated meaning, then $t_m = 0$ for $m \geqq k$ if k is even and $t_m = 2$ for $m \geqq k$ if k is odd.

4.2.2. The possibility that there are more sequences τ in the decimal expansion of π enables us to construct a matrix which transforms a negatively convergent sequence into a p-fold negatively convergent sequence.

4.3. The examples given have been constructed in such a way that it is not allowed to say that the series $\sum\delta_n$ involved are positive ly convergent. The proof in 2.3.2. could be maintained because only the boundedness of $\sum|\delta_n|$ was used.

We saw an example transforming a negatively convergent sequence into a twofold negatively convergent sequence. In this example a series $\sum\delta_n$ appeared of which we do not know the positive convergence.

Now we investigate transformations of bounded negatively convergent sequences supposing $\sum|\delta_n|$ is positively convergent.

By this suppletion of the conditions we prove the invariance of the type of negative convergence.

Obviously the righthand-member of (2) is true in this case.

5.1. THEOREM: In order that the matrix $(c_{m,n})$ transforms every bounded negatively convergent sequence into a negatively convergent sequence, the following conditions are sufficient:

(5.1.1.) $\sum_{n=0}^{\infty}|c_{m,n}|$ is positively convergent for every m and a number H exists independent of m with $\sum_{n=0}^{\infty}|c_{m,n}| < H$.

(5.1.2.) for every integer n the sequence $\{c_{m,n}\}$ is positively convergent and $^{+}\lim_{m\to\infty} c_{m,n} = \delta_n$;

(5.1.3.) the sequence $\{c_m\}$ is positively convergent and $^{+}\lim_{m\to\infty}c_m = \delta$ with $c_m = \sum_{n=0}^{\infty} c_{m,n}$;

(5.1.4.) $\sum|\delta_n|$ is positively convergent.

PROOF: Suppose the sequence $\{s_n\}$ is bounded and negatively convergent with $^{-}\lim_{n\to\infty} s_n = s$.

Then we have: $(\exists B)(\forall n)(|s_n| < B)$.

From (5.1.4.) it follows: $\sum\delta_n s_n$ is positively convergent and (5.1.1) leads to the positive convergence of $\sum_{n=0}^{\infty} c_{m,n}$ for every m. It is easily seen that $\sum_{n=0}^{\infty}|c_{m,n}s_n|$ is positively convergent. Hence it is allowed to define $t_m = \sum_{n=0}^{\infty} c_{m,n}s_n$. We intend to prove: the sequence $\{t_m\}$ is negatively convergent to t with

$$t = \delta s+ \sum_{n=0}^{\infty} \delta_n(s_n-s) = s(\delta-\sum_{n=0}^{\infty} \delta_n)+\sum_{n=0}^{\infty} \delta_n s_n.$$

First make the restriction $s = 0$.

The sequence $\{s_n\}$ is negatively convergent to 0, hence

$$(5)\qquad (\forall\varepsilon)\neg\neg(\exists N)(\forall n)\left((|s_{N+n}| < \frac{\varepsilon}{4H}\right)$$

(for H: see (5.1.1)).

From: $(\forall n)(|s_{N+n}| < \varepsilon : 4H)$ it follows

$$\Big|\sum_{\mu=N+1}^{k} (c_{m,\mu}-\delta_\mu)s_\mu\Big| < \frac{\varepsilon}{4H}\cdot \sum_{\mu=N+1}^{k} (|c_{m,\mu}|+|\delta_\mu|) < \tfrac{1}{2}\varepsilon,$$

for:

$$\sum_{\mu=N+1}^{k} \delta_\mu = \sum_{N+1}^{k} {}^{+}\lim_{m\to\infty} c_{m,\mu} = {}^{+}\lim_{m\to\infty} \sum_{N+1}^{k} |c_{m,\mu}| < H.$$

From (5) it follows:

$$(6) \qquad (\forall\varepsilon) \neg\neg\, (\exists N)(\forall p)(\sum_{n=N+1}^{N+p} (c_{m,n}-\delta_n)s_n| < \tfrac{1}{2}\varepsilon).$$

From: $(\forall n)(|s_{N+n}| < \varepsilon\colon 4H)$ we obtain:

$$(7) \quad |\sum_{n=N+p}^{\infty} | c_{m,n}s_n| \ngtr \sum |c_{m,n}| \cdot |s_n| < H \cdot \frac{\varepsilon}{4H} = \tfrac{1}{4}\varepsilon \text{ for every } p.$$

On account of (5.1.2) it is possible to construct an integer $M(N, \varepsilon)$ for every N in such a way that

$$(8) \qquad |\sum_{n=0}^{N} (c_{m,n}-\delta_n)s_n| < \tfrac{1}{4}\varepsilon$$

holds, for every $m > M$.

Combining the results (6), (7) and (8)

$$(9) \qquad (\forall\varepsilon) \neg\neg\, (\exists N)(\exists M)(\forall m)(\forall p)(|t_{M+m}- \sum_{n=0}^{N+p} \delta_n s_n| < \varepsilon)$$

is obtained.

The positive convergence of $\sum_{n=0}^{\infty} \delta_n s_n$ is known. Let σ be the limitvalue of this series then from (9) it follows:

$$(\forall\varepsilon) \neg\neg\, (\exists M)(\forall m)(|t_{M+m}-\sigma| < 2\varepsilon),$$

and this proves the negative convergence of the sequence $\{t_m\}$ to σ. The general case is treated by applying the conversion $s_n' = s_n - s$.

5.2. The conditions (5.1.2) and (5.1.3) may be weakened by supposing negative convergence of the sequences $\{c_{m,n}\}$ (n fixed) and $\{c_m\}$ instead of positive convergence. Then we obtain the extension:

THEOREM: In order that the matrix $(c_{m,n})$ transforms every bounded negatively convergent sequence into a negatively convergent sequence, the following conditions are sufficient:

(5.2.1) = (5.1.1),

(5.2.2) for every integer n there exists a δ_n such that the sequence $\{c_{m,n}\}$ is negatively convergent to δ_n; ${}^{-}\lim_{m\to\infty} c_{m,n} = \delta_n$,

(5.2.3) the sequence $\{c_m\}$ is negatively convergent to δ with

$c_m = \sum_{n=0}^{\infty} c_{m,n}$. (Thus: $^+\lim_{p\to\infty} \sum_{n=0}^{p} c_{m,n} = c_m$ and $^-\lim_{m\to\infty} c_m = \delta$),

(5.2.4) $\sum|\delta_n|$ is positively convergent.

In particular: $^-\lim_{m\to\infty} t_m = t = \delta s + \sum_{n=0}^{\infty} \delta_n(s_n - s)$ with $s = {}^-\lim_{n\to\infty} s_n$.

Proof: $\{s_n\}$ is a bounded sequence, hence:

(10) $$\exists B \forall n(|s_n| < B).$$

For every m we have:

$$(\forall\mu)(\forall\nu)(|\sum_{n=\mu}^{\mu+\nu} c_{m,n}s_n| < B.\sum_{n=\mu}^{\mu+\nu} |c_{m,n}|)$$

and from (5.2.1) it follows:

$$(\forall\varepsilon)(\exists\mu)(\forall\nu)\left(\sum_{n=\mu}^{\mu+\nu} |c_{m,n}| < \frac{\varepsilon}{B}\right),$$

hence

$$(\forall\varepsilon)(\exists\mu)(\forall\nu)(|\sum_{n=\mu}^{\mu+\nu} c_{m,n}s_n| < \varepsilon);$$

proving that $t_m = \sum_{n=0}^{\infty} c_{m,n}s_n$ is defined for every m.

Consider the case: $s = 0$. We have to prove:

$$^-\lim_{m\to\infty} t_m = \sum_{n=0}^{\infty} \delta_n s_n.$$

The series $\sum_{n=0}^{\infty} \delta_n s_n$ is positively convergent on account of (5.2.4) and (10).

From the sequence $\{s_n\}$ we know:

(11) $$(\forall\varepsilon) \neg\neg (\exists N)(\forall n)\left(|s_{N+n}| < \frac{\varepsilon}{4H}\right) \quad (H\text{: see 5.1.1}).$$

From: $(\forall n)(|s_{N+n}| < \varepsilon/4H)$ it follows for $k > N$:

$$\left|\sum_{n=N+1}^{k} (c_{m,n} - \delta_n)s_n\right| < \frac{\varepsilon}{4H} \sum_{n=N+1}^{k} (|c_{m,n}| + |\delta_n|) < 2H.\frac{\varepsilon}{4H} = \tfrac{1}{2}\varepsilon, \text{ for:}$$

$$\sum_{N+1}^{k} |\delta_n| = \sum_n {}^-\lim_{m\to\infty} |c_{m,n}| = {}^-\lim_{m\to\infty} \sum_{n=N+1}^{k} |c_{m,n}| < H$$

and now we may say:

From $(\forall n)(|s_{N+n}| < \varepsilon/4H)$ it follows:

(12) $$\left|\sum_{n=N+1}^{k} (c_{m,n} - \delta_n)s_n\right| < \tfrac{1}{2}\varepsilon \text{ for } k > N.$$

$^-\lim_{m\to\infty} c_{m,n} = \delta_n$ leads to:

$$(\forall\varepsilon)(\forall n)\neg\neg(\exists M_1)(\forall m > M_1)(|c_{m,n}-\delta_n| < \varepsilon)$$

and we may say:

(13) $(\forall\varepsilon)(\forall N)\neg\neg(\exists M)(\forall m > M_1)(|\sum_{n=0}^{N}(c_{m,n}-\delta_n)s_n| < \frac{1}{4}\varepsilon)$

because $|s_n| < B$.

From $\forall n(|s_{N+n}| < \varepsilon/4H)$ it follows too:

(14) $(\forall\varepsilon)(\exists N_1)(\forall k > N_1)(|\sum_{n=k+1}^{\infty} c_{m,n}s_n| < \frac{1}{4}\varepsilon$

for

$$|\sum_n c_{m,n}s_n| < \frac{\varepsilon}{4H}\sum |c_{m,n}| < \tfrac{1}{4}\varepsilon.$$

From (11), (12), (13), (14) and the evident relation:

$$|t_m - \sum_{0}^{k}\delta_n s_n| \leqq |\sum_{n=0}^{N}(c_{m,n}-\delta_n)s_n| + |\sum_{n=N+1}^{k}(c_{m,n}-\delta_n)s_n| + |\sum_{n=k+1}^{\infty} c_{m,n}s_n|$$

we obtain:

(15) $(\forall\varepsilon)\neg\neg(\exists N)\neg\neg(\exists M)(\forall k)(\forall m)(|t_{M+m} - \sum_{n=0}^{N+k}\delta_n s_n| < \varepsilon)$

The series $\sum \delta_n s_n$ is positively convergent and its limitvalue is denoted by t. Combining this with (15) we have:

$$(\forall\varepsilon)\neg\neg(\exists N)\neg\neg(\exists M)(\forall k)(\forall m)(|t_{M+m}-t| < 2\varepsilon)$$

and this is equivalent to

$$(\forall\varepsilon)\neg\neg(\exists M)(\forall m)(|t_{M+m}-t| < 2\varepsilon)$$

proving the statement in the case $s = 0$.

The general case is treated by considering $s'_n = s_n - s$.

Then we know ${}^{-}\lim s'_n = 0$ and we obtain:

$$t_m = {}^{+}\lim_{p\to\infty}\sum_{n=0}^{p} c_{m,n}s_n = {}^{+}\lim\sum_{n=0}^{p} c_{m,n}(s'_n+s)$$
$$= {}^{+}\lim_{p\to\infty}(\sum_{n=0}^{p} c_{m,n}s'_n + s\sum_{n=0}^{p} c_{m,n}) = t'_m + s\cdot c_m,$$

hence:

$${}^{-}\lim_{m\to\infty} t_m = {}^{-}\lim t'_m + s\cdot{}^{-}\lim c_m = {}^{-}\lim t'_m + s\cdot\delta.$$

5.3. By weakening one or some of the conditions of the original theorem we have derived some results. Up to now we have not weakened the conditions (2.1.1) = (5.1.1).

Accepting the condition: $(\exists H)(\forall m)(\sum_{n=0}^{\infty}|c_{m,n}| < H)$, then from the intuitionistic point of view we have the possibilities that $\sum_{n=0}^{\infty}|c_{m,n}|$ is negatively convergent or non-oscillating for fixed m.

In this direction some theorems are developed in the sections 6, 7 and 8.

6. THEOREM: From

6.1. $\sum_n|c_{m,n}|$ is negatively convergent to γ_m for every m,

6.2. $\sum_n|c_{m,n}| < H$ with H independent of m,

6.3. $\{s_n\}$ is a positively convergent sequence,

it follows:

the sequence $\sum_n c_{m,n}s_n$ is negatively convergent for every m.

PROOF: Put $\gamma_m^\nu = \sum_{n=0}^{\nu}|c_{m,n}|\cdot s_n$ and $\bar{\gamma}_m^\nu = \sum_{n=0}^{\nu}|c_{m,n}|$.

Consider the sequences

$$a_{m,\nu} = \gamma_m^\nu - \bar{\gamma}_m^\nu \cdot s \text{ for fixed } m.$$

We prove that every sequence $\{a_{m,\nu}\}$ with m fixed is positively convergent. To this aim choose an $\varepsilon > 0$ and calculate a N in such a way that $|s_n - s| < \varepsilon/4H$ for $n > N$. Then we have:

$$\begin{aligned}|a_{m,N+p+q} - a_{m,N+q}| &= \left|\sum_{N+q+1}^{n+p+q} s_n|c_{m,n}| - s\cdot\sum_{N+q+1}^{N+p+q}|c_{m,n}|\right| \\ &= \left|\sum_{N+q+1}^{N+p+q}(s_n - s)\cdot|c_{m,n}|\right| < \frac{\varepsilon}{H}\sum_{N+q+1}^{N+p+q}|c_{m,n}|\end{aligned}$$

and for every m the positive convergence of the sequence $\{a_{m,n}\}$ follows from Cauchy's general principle of convergence. (Observe that the convergence is uniform in m).

In order to prove the convergence of the series $\sum_n|c_{m,n}|s_n$ the limitvalue of the series ought to be calculated. We state that $\sum_n|c_{m,n}|s_n$ is negatively convergent to $a_m + s\gamma_m$ with $a_m = {}^+\lim_{n\to\infty} a_{m,n}$.

Therefore we prove:

$$(\forall\varepsilon)(\forall m)\neg\neg(\exists N)(\forall r)(|s\gamma_m + a_m - \sum_{n=0}^{N+r}|c_{m,n}|\cdot s_n| < \varepsilon).$$

We know ${}^+\lim_{p\to\infty} a_{m,p} = a_m$, hence to every $\varepsilon_1 > 0$ an integer N_1 can be calculated with the property $|a_m - a_{m,p}| < \varepsilon_1$ for $p > N_1$.

Take $N+r > N_1$, then

$$\left|s\gamma_m+a_n-\sum_{n=0}^{N+r}|c_{m,n}|\cdot s_n\right|=\left|a_m-a_{m,N+r}+a_{m,N+r}+s\gamma_m-\sum_{n=0}^{N+r}|c_{m,n}|\cdot s_n\right|$$
$$<\varepsilon_1+\left|a_{m,N+r}+s\gamma_m-\sum_{n=0}^{N+r}|c_{m,n}|\cdot s_n\right|=$$
$$\varepsilon_1+\left|\gamma_m^{N+r}-\bar{\gamma}_m^{N+r}\cdot s+s\gamma_m-\sum_{n=0}^{N+r}|c_{m,n}n|s_n\right|$$
$$\not>\varepsilon_1+|s|\cdot|\bar{\gamma}_m^{N+r}-\gamma_m|+\left|\gamma_m^{N+r}-\sum_{n=0}^{N+r}|c_{m,n}|\cdot s_n\right|$$
$$=\varepsilon_1+|s|\cdot\left|\bar{\gamma}_m^{N+r}-\gamma_m\right|,$$

hence:

$$(16)\qquad |s\gamma_m+a_m-\sum_{n=0}^{N+r}|c_{m,n}|\cdot s_n| < \varepsilon_1+|s|\cdot|\bar{\gamma}_m^{N+r}-\gamma_m|.$$

From 6.1 we know:

$$(\forall\varepsilon)(\forall m)\neg\neg(\exists N)(\forall r)(|\bar{\gamma}_m^{N+r}-\gamma_m| < \varepsilon)$$

and combining this with (16) we have:

$$(\forall\varepsilon)(\forall m)\neg\neg(\exists N)(\forall r)(|s\gamma_m+a_m-\sum_{n=0}^{N+r}|c_{m,n}|\cdot s_n| < \varepsilon_1+|s|\varepsilon)$$

and this relation is equivalent to:
the series $\sum_n|c_{m,n}|s_n$ is negatively convergent to $s\gamma_m+a_m$.

7.1. Definition: The series $\sum|c_{m,n}|$ are called uniformly non-oscillating with respect to m if

$$(17)\qquad (\forall\varepsilon)\neg\neg(\exists N)(\forall m)(\forall r)(\sum_{n=N+1}^{N+r}|c_{m,n}| < \varepsilon).$$

7.2. Theorem: In order that the matrix $(c_{m,n})$ transforms a bounded sequence $\{s_n\}$ into a non-oscillating sequence $\{t_m\}$ the following conditions are sufficient:

(7.2.1) $\sum_{n=0}^{\infty}|c_{m,n}|$ are uniformly non-oscillating with respect to m,
(7.2.2) the series $\sum_{n=0}^{\infty}c_{m,n}s_n$ is positively convergent.

Proof. Choose the integers M and r in an arbitrary way then from

$$(\exists B)(\forall n)(|s_n| < B)$$

it follows:

$$(18)\qquad |t_{M+r}-t_M|=|\sum_{M+1}^{M+r}c_{m,n}s_n| \not> B\sum_{M+1}^{M+r}|c_{m,n}|.$$

Applying (17) to (18) we obtain

$$(\forall \varepsilon) \neg\neg (\exists M)(\forall r)(|t_M - t_{M+r}| < \varepsilon)$$

proving that the sequence $\{t_m\}$ is non-oscillating.

8.1. In the following we shall use the theorem:
If the sequence $\{s_n\}$ is non-oscillating then the sequence is negatively bounded, hence: $\neg\neg(\exists B)(\forall n)(|s_n| < B)$.

The easy proof will be omitted.

8.2. THEOREM: If

(8.2.1) the sequence $\{s_n\}$ is non-oscillating and

(8.2.2)

$$(\exists H)(\forall m)(\forall p)(\sum_{n=0}^{p} |c_{m,n}| < H), \tag{19}$$

then for every fixed m the sequence $\{t_{m,p}\}$ is non-oscillating with

$$t_{m,p} = \sum_{n=0}^{p} c_{m,n} s_n$$

PROOF: We choose an arbitrary fixed value for m and we have:

$$|t_{m,N+q} - t_{m,N}| = |\sum_{n=N+1}^{N+q} c_{m,n} s_n| \ngtr \sum_{n=N+1}^{N+q} |c_{m,n}| \cdot |s_n|.$$

Using the theorem mentioned in 8.1 we obtain

$$(\forall m) \neg\neg (\exists P)(|t_{m,N+q} - t_{m,N}| < P \cdot \sum_{n=N+1}^{N+q} |c_{m,n}|) \tag{20}$$

From (19) we see, that the sequence $\{u_{m,p}\}$ with $u_{m,p} = \sum_{n=0}^{p} |c_{m,n}|$ is for every m a monotone non-decreasing bounded sequence; hence the sequence $\{u_{m,p}\}$ is non-oscillating for every fixed m (cf. A. Heyting, l.c. page 110, theorem 4).

So we have:

$$(\forall m)(\forall P)(\forall \varepsilon) \neg\neg (\exists N)(\forall q)(\left(\sum_{n=N+1}^{N+q} |c_{m,n}| < \frac{\varepsilon}{P}\right) \tag{21}$$

Combining (20) and (21) we obtain:

$$(\forall m) \neg\neg (\exists P)(\forall \varepsilon) \neg\neg (\exists N)(\forall q)(|t_{m,N+q} - t_{m,N}| < \varepsilon)$$

and omitting P : $(\forall m) \neg\neg (\forall \varepsilon) \neg\neg (\exists N)(\forall q)(|t_{m,N+q} - t_{m,N}| < \varepsilon)$ which is equivalent to:

$$(\forall m)(\forall \varepsilon) \neg\neg (\exists N)(\forall q)(|t_{m,N+q} - t_{m,N}| < \varepsilon)$$

proving the statement that the sequence $\{t_{m,n}\}$ is non-oscillating for every fixed m.

8.3. For the transformation of a non-oscillating sequence $\{s_n\}$ by a matrix $(c_{m,n})$ a theorem resembling that of Schur can be proved.

THEOREM: If $\{s_n\}$ is a non-oscillating sequence transformed by a matrix $(c_{m,n})$,which fulfills the following conditions

8.3.1. $$(\exists H)(\forall m)(\forall p)\left(\sum_{n=0}^{p}|c_{m,n}| < H\right).$$

8.3.2. $$(\forall n)(\forall \varepsilon)\neg\neg(\exists M)(\forall m)(|c_{M+m,n}-c_{M,n}| < \varepsilon),$$

8.3.3. $$(\forall \varepsilon)\neg\neg(\exists M)(\forall m)\neg\neg(\exists N)(\forall n)(|\sum_{i=N+1}^{N+n}(c_{M+m,i}-c_{M,i})|<\varepsilon),$$

then we have:

$$(\forall \varepsilon)\neg\neg(\exists M)(\forall m)\neg\neg(\exists N)(\forall \rho)(\forall r)(|t_{M+m,N+r+\rho}-t_{M,N+r}| < \varepsilon)$$

with $t_{i,j} = \sum_{n=0}^{j} c_{i,n}s_n$.

PROOF:

$$|t_{M+m,N+r+\rho}-t_{M,N+r} = \left|\sum_{n=0}^{N+r+\rho} c_{M+m,n}s_n - \sum_{n=0}^{N+r} c_{M,n}s_n\right|$$

$$\not> \left|\sum_{n=0}^{k}(c_{M+m,n}-c_{M,n})s_n\right| + \left|\sum_{n=k+1}^{N+r}(c_{M+m,n}-c_{M,n})(s_n-s_p)\right|$$

$$+|s_p|\cdot\left|\sum_{k+1}^{N+r}(c_{M+m,n}-c_{M,n})\right| + \left|\sum_{n=N+r+1}^{N+r+\rho} c_{M+m,n}s_n\right|$$

$$(k < N+r,\ p \text{ arbitrary}).$$

We denote the four terms in the righthand member of this inequality by I, II, III and IV. First consider II.

The sequence $\{s_n\}$ is non-oscillating, hence:

$$(\forall \varepsilon)\neg\neg(\exists p)(\forall n)(|s_{n+p}-s_p| < \varepsilon)$$

and this leads to

$$(\forall \varepsilon)\neg\neg(\exists p)(\forall k \geqq p)(\text{II} < \varepsilon\cdot\sum_{k+1}^{N+1}(|c_{M,n}|+|c_{M+m,n}|) < 2\varepsilon H).$$

Now we have obtained:

$$(\exists p)(\forall k \geqq p)(|t_{M+m,N+r+\rho}-t_{M,N+r}| < \text{I}+2\varepsilon H+\text{III}+\text{IV}).$$

On account of 7.2. we may assert:

$$(\forall \varepsilon)(\forall M)(\forall m)\neg\neg(\exists N)(\forall \rho)(\forall r)(\text{IV} < \varepsilon),$$

in this result we may write of course $N+\sigma$, with σ an arbitrary integer, instead of N.

Putting $k = p$ and taking care that $N > p$ then we obtain

$$(\forall\varepsilon)\neg\neg(\exists p)(\forall M)(\forall m)\neg\neg(\exists N)(\forall\rho)(\forall r)\Bigl(|t_{M+m,N+r+\rho}-t_{M,N+r}| < \Bigl|\sum_{n=0}^{p}(c_{M+m,n}-c_{M,n})s_n\Bigr| + |s_p|\cdot\Bigl|\sum_{n=p+1}^{N+r}(c_{M+m,n}-c_{M,n})\Bigr| + 2\varepsilon(H+1)\Bigr)$$

From (8.3.2) it follows that for every given p:

$$(\forall n)(\forall\varepsilon)\neg\neg(\exists M)(\forall m)\Bigl(\Bigl|\sum_{n=0}^{p}(c_{M+m,n}-c_{M,n})s_n\Bigr| < \varepsilon\Bigr);$$

combining this with the foregoing results we obtain

$$\forall\varepsilon\neg\neg(\exists p)\neg\neg(\exists M)(\forall m)\neg\neg(\exists N)(\forall\rho)(\forall r)\Bigl(|t_{M+m,N+r+\rho}-t_{M,N+r}| < \varepsilon(2H+3)+|s_p|\cdot\Bigl|\sum_{n=p+1}^{N+r}(c_{M+m,n}-c_{M,n})\Bigr|\Bigr).$$

Applying 8.3.2. we obtain:

$$(\forall\varepsilon)\neg\neg(\exists A)(\forall a)(\forall d)\neg\neg(\exists B)(\forall b)\Bigl(\Bigl|\sum_{i=B+1}^{B+b}(c_{A+d,i}-c_{A+a,i})\Bigr| < 2\varepsilon\Bigr)$$

from which it is obvious that A and B may be replaced by greater values and the same applies for M and N and this leads to:

$$(\forall\varepsilon)\neg\neg(\exists p)\neg\neg(\exists M)(\forall m)\neg\neg(\exists N)(\forall\rho)(\forall r)(|t_{M+m,N+r+\rho} - t_{M,N+r}| < \varepsilon(2H+5),$$

which is equivalent to:

$$(\forall\varepsilon)\neg\neg(\exists M)(\forall m)\neg\neg(\exists N)(\forall\rho)(\forall r)(|t_{M+m,N+r+\rho}-t_{M,N+r}| < \varepsilon).$$

9.1. We now return to the necessity of the conditions mentioned in the theorem of section 2.1.

It already turned out, that the conditions are not sufficient to maintain the last part of the statement but they are sufficient to prove the essential parts of the theorem.

Now we shall investigate the necessity of the conditions. Hardy (cf. Divergent Series page 45) proved the necessity of the conditions (2.1.2) and (2.1.3) but the proof of the necessity of (2.1.1) is based on the principle of the excluded middle.

In the following an example is constructed of a transformation by an infinite matrix $(c_{m,n})$ which transforms every positively convergent sequence $\{s_n\}$ into a positively convergent sequence $\{t_m\}$ but this matrix has the property that the series $\sum|c_{m,n}|$ is twofold negatively convergent.

9.2. The sequence $\{\rho_p\}$ will be defined as follows: if the sequence τ(cf. 3.2.) does not occur in the first p digits of π, then we take $\rho_p = 1/p$, and in the other case, when τ occurs in the first p digits of π, then we define $\rho_p = 1/k$ for $p \geqq k$ (k has the same meaning as in 3.2) Now we can define ρ by $\rho = \lim_{p\to\infty} \rho_p$.

Finally define:

$c_{m,n} = (-1)^n \rho$ if τ does not occur in the first m digits of π
$c_{m,n} = 0$ if τ occurs in the first m digits of π.

We prove: Every positively convergent sequence $\{s_n\}$ with $\lim_{n\to\infty} s_n = s$ is transformed by $t_m = \sum_{n=0}^{\infty} c_{m,n} s_n$ into a positively convergent sequence $\{t_m\}$.

At first we prove that $\sum_{n=0}^{\infty} c_{m,n} s_n$ is positively convergent for m. Therefore take the partial sums $t_{m,p} = \sum_{n=0}^{p} c_{m,n} s_n$ and consider $|t_{m,p+q} - t_{m,p}|$. From the positive convergence of $\{s_n\}$ it follows:

$$(22) \qquad (\forall \varepsilon)(\exists N)(\forall \nu > N)\forall \mu(|s_{\nu+\mu} - s_\nu| < \varepsilon)$$

Choose a fixed value for ε and calculate N in such a way that (22) and $|s_n| < \varepsilon \cdot N$ for $n > N$ are satisfied. Then it is possible to determine the occurrence or non-occurrence of τ in the first N digits of π. If τ occurs in the first N decimals then we have: $t_{m,p+q} - t_{m,p} = 0$ for all $p > N$.

If τ does not cocur in the first N decimals then we prove:

$$|t_{m,p+q} - t_{m,p}| < 3\varepsilon \quad \text{for} \quad p > N.$$

Suppose:

$$|t_{m,p+q} - t_{m,p}| > 2\tfrac{1}{2}\varepsilon.$$

It is impossible that τ occurs in π, for if τ occurs then we know k (cf. 3.2) and we have:

$$c_{m,n} = (-1)^n : k \quad \text{for} \quad n < k \text{ and } c_{m,n} = 0 \quad \text{for} \quad n \geqq k.$$

Because we have: $N < k$ we can deduce the following inequalities:

a) for $p+q \leqq k$: $$|t_{m,p+q} - t_{m,p}| = \left|\sum_{n=p+1}^{p+q} (-1)^n \cdot \frac{1}{k} \cdot s_n\right| = \frac{1}{k}\left|\sum_{p+1}^{p+q} (-1)^n \cdot s_n\right|$$

$$\ngtr \frac{1}{k}(|s_{p+1} - s_{p+2}| + |s_{p+3} - s_{p+4}| + \cdots + |s_{p+q-1} - s_{p+q}|)$$

$$< \frac{1}{k} q \cdot \varepsilon \quad \text{if } q \text{ is even and}$$

$$\ngtr \frac{1}{k}(|s_{p+1} - s_{p+2}| + \cdots \quad \cdots + |s_{p+q-2} - s_{p+q-1}| + |s_{p+q}|)$$

$$< \frac{1}{k}(q \cdot \varepsilon + |s_{p+q}|) < 2\varepsilon \quad \text{if} \quad q \text{ is odd.}$$

b) for $p+q>k$: $|t_{m,p+q}-t_{m,p}| = |\sum_{p+1}^{p+q} c_{m,n}s_n| = |\sum_{p+1}^{k} c_{m,n}s_n|$ and in the same we obtain:

$$|t_{m,p+q}-t_{m,p}| < 2\varepsilon.$$

We see that from the hypothesis $|t_{m,p+q}-t_{m,p}| > 2\frac{1}{2}\varepsilon$ it follows that it is impossible that τ occurs in π, but then we have $\rho = 0$ and this means $c_{m,p} = 0$ for every m and p. However, the conclusion $t_{m,p} = 0$ for every m and p contradicts the hypothesis $|t_{m,p+q}-t_{m,p}| > 2\frac{1}{2}\varepsilon$.

The hypothesis $|t_{m,p+q}-t_{m,p}| > 2\frac{1}{2}\varepsilon$ for $p > N$ lead to a contradiction, hence $|t_{m,p+q}-t_{m,p}| < 3\varepsilon$.

Now we have proved that $^{+}\lim_{p\to\infty} t_{m,p}$ exists for every value of m. The sequence $\{t_m\}$ is a sequence of equal elements because all rows of the matrix $(c_{m,n})$ are equal, hence the sequence $\{t_m\}$ is positively convergent and by this we have proved:

Every positively convergent sequence $\{s_n\}$ is transformed into a positively convergent sequence $\{t_m\}$ by the given matrix $(c_{m,n})$.

However, the series $\sum_n |c_{m,n}|$ are twofold negatively convergent for every value of m and the limitvalues are 0 and 1, hence it is not allowed to state the positive convergnce of this series.

From this example it is obvious that the required convergence of the series mentioned in (2.1.1) is not an essential requirement in the theorem.

A theorem with intuitionistic weaker conditions and stating the same is the following:

10.1 Theorem: In order that a matrix $(c_{m,n})$ transforms every positively convergent sequence $\{s_n\}$ into a positively convergent sequence $\{t_m\}$ the following conditions are sufficient:

(10.1′) $$\forall m \forall \varepsilon\, \exists N \forall p \left(\left| \sum_{N+1}^{N+p} c_{m,n} \right| < \varepsilon \right)$$

(10.1″) $$\exists H \forall m \forall p \left(\sum_{n=0}^{p} |c_{m,n}| < H \right)$$

(10.2) $$\forall n \exists \delta_n \forall \varepsilon \exists M \forall m (|c_{M+m,n}-\delta_n| < \varepsilon)$$

(10.3) $$\exists \delta \forall \varepsilon \exists M \forall m (|c_{M+m}-\delta| < \varepsilon) \text{ with } c_m = \sum_{n=0}^{\infty} c_{m,n}$$

Remark: Condition (2.1.1) has been split up into (10.1′) and (10.1″). The condition (10.1′) tells us that the series $\sum_n c_{m,n}$ are positively convergent for every m.

If $\sigma_{m,\nu} = \sum_{n=0}^{\nu} |c_{m,n}|$ then (10.1″) is the translation of: a number H exists with $\sigma_{m,\nu} < H$ for all m and ν. This does not pronounce any convergence of the sequence $\{\sigma_{m,\nu}\}$ (m fixed).

PROOF: We consider the special case $^{+}\lim_{n\to\infty} s_n = 0$. Then we have:

$$\forall \varepsilon \exists N \forall n \left(|s_{N+n}| < \frac{\varepsilon}{H} \right). \tag{23}$$

We define:

$$t_{m,p} = \sum_{n=0}^{p} c_{m,n} s_n$$

When we fix m for a given ε we can choose N in such a way that (10.1′) and (23) are satisfied.

From (10.1′) and (23) it follows:

$$|t_{m,N+n} - t_{m,N}| = \left| \sum_{i=N+1}^{N+n} c_{m,i} s_i \right| \ngtr \frac{\varepsilon}{H} \sum_{i=N+1}^{N+n} |c_{m,i}| < \varepsilon$$

hence the sequence $\{t_{m,p}\}$ is positively convergent for every m and the limitvalue of these sequences $\{t_{m,p}\}$ are indicated by t_m. In the same way as in 2.3.2 we prove the existence of $^{+}\lim t_m$. In the general case introduce $s'_n = s_n - s$ and $t'_m = \sum_n c_{m,n} s'_n$. We prove the convergence of $\{t_{m,p}\}$ for $p \to \infty$.

Evidently:

$$|t_{m,p+q} - t_{m,p}| \ngtr \left| \sum_{p+1}^{p+q} c_{m,n} s_n - s \sum_{p+1}^{p+q} c_{m,n} + s \sum_{p+1}^{p+q} c_{m,n} \right|$$

$$\ngtr \left| \sum_{p+1}^{p+q} c_{m,n}(s_n - s) \right| + |s| \left| \sum_{p+1}^{p+q} c_{m,n} \right|$$

Choose $\varepsilon > 0$ and calculate N_1 in such a way that $|s_n - s| < \varepsilon/H$ for all $n > N_1$.

By (10.1′) we can choose a number N_2 in such a way that $|s| \cdot |\sum_{N_2+1}^{N_2+n} c_{m,p}| < \varepsilon$ for all $n > N_2$.

When we take $p > \max(N_1, N_2)$ we have: $|t_{m,p+q} - t_{m,p}| < 2\varepsilon$ hence the sequence $\{t_{m,p}\}$ is positively convergent for every fixed m and now we may write:

$$t_m = {}^{+}\lim_{p\to\infty} \sum_{n=0}^{p} c_{m,n} s_n = {}^{+}\lim_{p\to\infty} \sum_{n=0}^{p} c_{m,n}(s'_n + s)$$

$$= {}^{+}\lim_{p\to\infty} \sum_{n=0}^{p} c_{m,n} s'_n + s \lim_{p\to\infty} \sum_{n=0}^{p} c_{m,n} = t'_m + s \cdot c_m$$

hence

$${}^{+}\lim_{m\to\infty} t_m = {}^{+}\lim_{m\to\infty} (t'_m + s\, c_m) = {}^{+}\lim t'_m + s \cdot \delta$$

This proof uses only the uniform boundedness of the series $\sum_n |c_{m,n}|$ without supposing anything as to convergence.

10.2. As to the necessity of the conditions I refer to section 9.1. Of course (10.1'), (10.2) and (10.3) are necessary. I have not succeeded in proving the necessity of (10.1') nor in constructing an example with weaker conditions.

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