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On the Convergence of a Lacunary Trigonometrical Series

by

Fu Cheng Hsiang

1. A lacunary trigonometrical series is a series for which the terms different from zero are very sparse. This kind of trigonometrical series may be put in the form:

$$\sum_{\nu=1}^{\infty} (a_{\nu} \cos n_{\nu} x + b_{\nu} \sin n_{\nu} x).$$

We here assume, for simplicity, that the constant term of the series also vanishes. A series $\sum c_i$ is said to possess a gap (u, v) if $c_i = 0$ for $u < i < v$. It is known that [3, p. 251, § 10.31] *if a series $\sum C_i$ possesses infinitely many gaps (m_k, m'_k) such that $m'_k/m_k > \mu > 1$ for some μ and is summable $(C, 1)$ to sum S , then S_{m_k} and also $S_{m'_k}$ converges to S .*

2. Suppose that the above trigonometrical series is the Fourier series of an integrable function $f(x)$. This series possesses infinitely many gaps $(n_{\nu}, n_{\nu+1})$ such that $n_{\nu+1}/n_{\nu} > \lambda > 1$ for some λ . Write, for a given point x_0 ,

$$\varphi(t) = \varphi_{x_0}(t) = \frac{1}{2} \{f(x_0 + t) + f(x_0 - t) - 2f(x_0)\}.$$

Since

$$\varphi^*(t) = \frac{1}{t} \int_0^t |\varphi| du = o(1) \quad (t \rightarrow +0)$$

holds for almost all x , the Fourier series of an integrable function is therefore almost everywhere summable $(C, 1)$ by Fejér's theorem. From this fact, we draw immediately the following well-known

KOLMOGOROFF'S THEOREM [1, 2]. *If the Fourier series of an integrable function $f(x)$ possesses infinitely many gaps $(n_{\nu}, n_{\nu+1})$ such that $n_{\nu+1}/n_{\nu} > \mu > 1$, then the partial sum $S_{n_{\nu}}$ converges almost everywhere to $f(x)$.*

We can also conclude that, at a given point x_0 at which $\varphi^*(t) = o(1)$ ($t \rightarrow +0$), if $n_{\nu+1}/n_{\nu} > \lambda > 1$, then $S_{n_{\nu}}(x_0) \rightarrow f(x_0)$ as $\nu \rightarrow \infty$.

3. We write

$$\varphi_1(t) = \frac{1}{t} \int_0^t \varphi \, du,$$

$$\varphi_2(t) = \frac{1}{t} \int_0^t \varphi_1 \, du.$$

In this note, we replace the condition $\varphi^*(t) = O(1)$ by the weaker condition $\varphi_1(t) = O(1) (t \rightarrow +0)$. We develop Kolmogoroff's theorem into the following manner.

THEOREM. *If the lacunary Fourier series of an integrable function $f(x)$ possesses infinitely many gaps $(n_\nu, n_{\nu+1})$ such that $n_{\nu+1}/n_\nu > \lambda > 1$, and if, at a given point x_0 , (i) $\varphi_1(t) = O(1) (t \rightarrow +0)$ and*

(ii)
$$\int_0^t |d\varphi_2| = O(1)$$

when $0 < t \leq \eta$ for some η , then $S_{n_\nu}(x_0) \rightarrow f(x_0)$ as $\nu \rightarrow \infty$.

4. Now, we are in a position to prove the theorem. Take, for instance, $n_\nu = n$, $n_{\nu+1} = m$ and denote respectively by $D_n(t)$ and $K_n(t)$ Dirichlet's and Fejér's kernels. Then

$$D_n(t) = \frac{1}{2} + \sum_{\nu=1}^n \cos \nu t,$$

$$(n+1)K_n(t) = \sum_{\nu=0}^n D_\nu(t) = \frac{\sin^2(n+1)t/2}{2 \sin^2 t/2}.$$

Then, from the identity

$$mK_{m-1}(t) - nK_{n-1}(t) = \sum_{\nu=n}^{m-1} D_\nu(t)$$

$$= (m-n)D_n(t) + \sum_{\nu=1}^{m-n-1} (m-n-\nu) \cos(n+\nu)t$$

and in virtue of the special property of the lacunary Fourier series, we obtain

$$S_n(x_0) - f(x_0) = \frac{1}{\pi} \int_0^\pi \varphi(t) D_n(t) dt$$

$$= \frac{1}{\pi(m-n)} \int_0^\pi \varphi(t) (mK_{m-1}(t) - nK_{n-1}(t)) dt.$$

Write

$$\Phi(t) = \int_0^t \varphi \, du.$$

Integration by parts gives

$$\begin{aligned}
 S_n(x_0) - f(x_0) &= \frac{1}{\pi(m-n)} [\Phi(t)(mK_{m-1}(t) - nK_{n-1}(t))]_0^\pi \\
 &\quad - \frac{1}{\pi(m-n)} \int_0^\pi \Phi(t) \frac{d}{dt} (mK_{m-1}(t) - nK_{n-1}(t)) dt \\
 &= 0(1) - \frac{1}{\pi(m-n)} \int_0^\pi \Phi(t) (mK'_{m-1}(t) - nK'_{n-1}(t)) dt \\
 &= 0(1) - \frac{1}{2\pi(m-n)} \left(\frac{1}{2} \int_0^\pi \Phi(t) \frac{m \sin mt - n \sin nt}{\sin^2 t/2} dt \right. \\
 &\quad \left. - \int_0^\pi \Phi(t) \frac{\sin^2 mt/2 - \sin^2 nt/2}{\sin^2 t/2} \cos t/2 dt \right) \\
 &= 0(1) - \frac{1}{2\pi(m-n)} (I_{1/2} - I_2),
 \end{aligned}$$

say. We are going to estimate the orders of the integrals I_1 and I_2 respectively. We write

$$\begin{aligned}
 I_1 &= m \int_0^\pi \Phi \frac{\sin mt}{\sin^2 t/2} dt - n \int_0^\pi \Phi \frac{\sin nt}{\sin^2 t/2} dt \\
 &= mI_3 - nI_4,
 \end{aligned}$$

say. Since $(2 \sin t/2)^{-2} - t^{-2}$ is bounded, we obtain, by Riemann-Lebesgue's theorem,

$$\begin{aligned}
 I_3 &= 4 \int_0^\pi \Phi \frac{\sin mt}{t^2} dt + 0(1) \\
 &= 4 \int_0^\pi \varphi_1 \frac{\sin mt}{t} dt + 0(1) \\
 &= 0(1)
 \end{aligned}$$

as $m \rightarrow \infty$ by De la Vallée Poussin's test [3, p. 33, § 2.8] for the convergence of Fourier series at a given point by the condition (ii) and $\varphi_2(t) = 0(1)$ as $t \rightarrow +0$. Similarly,

$$\begin{aligned}
 I_4 &= 4 \int_0^\pi \varphi_1 \frac{\sin nt}{t} dt + 0(1) \\
 &= 0(1)
 \end{aligned}$$

as $n \rightarrow \infty$ by the same test. It follows that

$$\begin{aligned} \frac{1}{m-n} \cdot I_1 &= o\left(\frac{m}{m-n}\right) + o\left(\frac{n}{m-n}\right) \\ &= o\left(\frac{1}{1-\lambda^{-1}}\right) + o\left(\frac{1}{\lambda^{-1}-1}\right) \\ &= o(1) \end{aligned}$$

as $n \rightarrow \infty$. In estimating the order of I_2 , let us write

$$\begin{aligned} I_2 &= \int_0^\pi \varphi_1 \frac{\sin^2 mt/2 - \sin^2 nt/2}{\sin^2 t/2} \cdot \frac{t}{\tan t/2} dt \\ &= \left(\int_0^{\pi/(m-n)} + \int_{\pi/(m-n)}^\delta + \int_\delta^\pi \right) \\ &= I_5 + I_6 + I_7, \end{aligned}$$

say. We have

$$\begin{aligned} |I_5| &\leq 4 \operatorname{Max}_{0 \leq t \leq \delta} |\varphi_1(t)| \int_0^{\pi/2(m-n)} \frac{|\sin^2 mt - \sin^2 nt|}{\sin^2 t} \left| \frac{t}{\tan t} \right| dt \\ &= 4 \operatorname{Max}_{0 \leq t \leq \delta} |\varphi_1(t)| I'_5, \end{aligned}$$

say. Now,

$$I'_5 \leq \int_0^{\pi/2(m-n)} \frac{|\sin(m+n)t \sin(m-n)t|}{\sin^2 t} dt.$$

Considering that

$$\left| \frac{\sin \alpha t}{t} \right| \leq \alpha$$

for $0 \leq t \leq \pi/2$ and $\alpha \geq 0$, we get

$$\begin{aligned} I'_5 &\leq (m^2 - n^2) \int_0^{\pi/2(m-n)} dt \\ &= \frac{\pi}{2} (m + n). \end{aligned}$$

Therefore,

$$|I_5| \leq 2\pi \operatorname{Max}_{0 \leq t \leq \delta} |\varphi_1(t)| (m + n).$$

Moreover, we have

$$\begin{aligned} |I_6| &\leq 4 \operatorname{Max}_{0 \leq t \leq \delta} |\varphi_1(t)| \int_{\pi/2(m-n)}^{\delta/2} \csc^2 t \, dt \\ &< 8 \operatorname{Max}_{0 \leq t \leq \delta} |\varphi_1(t)| \cdot \frac{\pi}{2} \cdot \frac{2(m-n)}{\pi} \\ &= 8 \operatorname{Max}_{0 \leq t \leq \delta} |\varphi_1(t)| (m-n). \end{aligned}$$

Last, by the second mean value theorem, we obtain

$$|I_7| \leq \frac{A}{\delta^3},$$

where A is an absolute constant. From the above analysis, it follows that

$$\begin{aligned} |S_n(x_0) - f(x_0)| &< 0(1) + \frac{1}{2\pi(m-n)} (m |I_3| + n |I_4| \\ &\quad + 2\pi \operatorname{Max}_{0 \leq t \leq \delta} |\varphi_1(t)| (m+n) \\ &\quad + 8 \operatorname{Max}_{0 \leq t \leq \delta} |\varphi_1(t)| (m-n) \\ &\quad + A/\delta^3) \\ &= 0(1) + \frac{1}{2\pi} \left(\frac{m}{m-n} |I_3| + \frac{n}{m-n} |I_4| \right. \\ &\quad + 2\pi \operatorname{Max}_{0 \leq t \leq \delta} |\varphi_1(t)| \frac{m+n}{m-n} \\ &\quad \left. + 8 \operatorname{Max}_{0 \leq t \leq \delta} |\varphi_1(t)| + \frac{A}{(m-n)\delta^3} \right) \\ &< 0(1) + \frac{1}{2\pi} \left(\frac{1}{1-\lambda^{-1}} |I_3| + \frac{1}{\lambda-1} |I_4| \right. \\ &\quad + 2\pi \operatorname{Max}_{0 \leq t \leq \delta} |\varphi_1(t)| \frac{1+\lambda^{-1}}{1-\lambda^{-1}} \\ &\quad \left. + 8 \operatorname{Max}_{0 \leq t \leq \delta} |\varphi_1(t)| + \frac{A}{(m-n)\delta^3} \right). \end{aligned}$$

For a given $\varepsilon > 0$, we can choose a δ so small that

$$\operatorname{Max}_{0 \leq t \leq \delta} |\varphi_1(t)| < \varepsilon$$

by the condition (i). After fixing δ , we take a sufficiently large

integer n which makes $|I_3|$, $|I_4|$ and $(m-n)^{-1}\delta^{-3}$ all less than ε . Thus, we obtain finally

$$|S_n(x_0) - f(x_0)| < 0(1) + \frac{1}{2\pi} \left(\frac{1}{1-\lambda^{-1}} + \frac{1}{\lambda-1} \right. \\ \left. + 2\pi \frac{1+\lambda^{-1}}{1-\lambda^{-1}} + 8 + A \right) \varepsilon.$$

Since ε is an arbitrary small quantity, letting it tend to zero, we get

$$S_n(x_0) - f(x_0) \rightarrow 0,$$

i.e.,

$$\lim_{n \rightarrow \infty} S_n(x_0) = f(x_0).$$

This proves the theorem.

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