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Two theorems for the n-dimensionality of metric spaces*

by

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The purpose of this note is to establish two theorems that respectively give necessary and sufficient conditions for metric spaces to be n-dimensional.

1. We have proved earlier the following theorems [4] [1]).

(I) *A metric space R has* $\dim \leqq n$ [2]) *if and only if we can introduce a topology-preserving metric ρ into R such that the spherical nbds (= neighborhoods) $S_{1/i}(p)$, $i = 1, 2, \dots$ of any point p of R have boundaries of* $\dim \leqq n-1$ *and such that $\{S_{1/i}(p) | p \in R\}$ is closure preserving* [3]) *for every i.*

(II) *A metric space R has* $\dim \leqq n$ *if and only if we can introduce a topology-preserving metric ρ into R such that*

$$\dim B[S_{1/i}(F)] \leqq n-1, \quad i = 1, 2, \dots$$

for every closed set F of R. [4])

Our first problem is to refine these theorems as follows.

THEOREM 1. *A metric space R has* $\dim \leqq n$ *if and only if we can introduce a topology-preserving metric ρ into R such that the spherical nbds $S_\varepsilon(p)$, $\varepsilon > 0$ of any point p of R have boundaries of* $\dim \leqq n-1$ *and such that $\{S_\varepsilon(p) | p \in R\}$ is closure preserving for any $\varepsilon > 0$.*

* The content of this paper is a development in detail of our communication which was published at the Symposium on general topology and its relations to modern analysis and algebra, Prague, September 1961.

[1]) It follows from [8] that $\dim R \leqq n$ for a separable metric space R if and only if we can introduce a metric into R such that the boundary $B[S_\varepsilon(p)]$ of $S_\varepsilon(p) = \{q | \rho(p, q) < \varepsilon\}$ has $\dim \leqq n—1$ for almost all ε. See, for example, [9].

[2]) Dim R denotes the covering dimension of R, but it coincides with the strong inductive dimension Ind R by [2] and [3] if R is metrizable.

[3]) A collection $\mathfrak{A}$ of subsets of R is called closure preserving if $\cup \{\bar{A} | A \in \mathfrak{A}'\} = \overline{\cup \{A | A \in \mathfrak{A}'\}}$ for any subset $\mathfrak{A}'$ of $\mathfrak{A}$.

[4]) $S_{1/i}(F) = \{p | \rho, p\ q) < 1/i$ for some $q \in F\}$. We expressed in [4] this theorem in a slightly different form, i. e. we proved it for every subset F of R, but there is no essential difference.

Proof. The if part of this theorem is implied by the if part of our previous Theorem (I). [5])

To show the only if part we let dim $\leqq n$; then, as is easily seen, we can choose a sequence $\{\mathfrak{U}_i | i = 0, 1, 2, \ldots\}$ of open coverings such that [6])

1) $\{R\} = \mathfrak{U}_0 > \mathfrak{U}_1^{**} > \mathfrak{U}_1 > \mathfrak{U}_2^{**} > \mathfrak{U}_2 > \mathfrak{U}_3^{**} > \ldots$,

2) $\{S(p, \mathfrak{U}_m) | m = 0, 1, 2, \ldots\}$ is an nbd basis of each point p of R,

3) $S^2(p, \mathfrak{U}_{m+1}^{*})$ intersects at most $n+1$ members of $\mathfrak{U}_m$. Now we define $S_{m_1 m_2 \ldots m_k}(U)$ for integers $m_1, m_2, \ldots, m_k$ with $1 \leqq m_1 < m_2 < \ldots < m_k$ and for $U \in \mathfrak{U}_m$, by

$$S_{m_1}(U) = U, \quad m_1 \geqq 0;$$
$$S_{m_1 \ldots m_k}(U) = S^2(S_{m_1 \ldots m_{k-1}}(U), \ \mathfrak{U}_{m_k}), \quad 1 \leqq m_1 < m_2 < \ldots < m_k, \ k \geqq 2.$$

Then we define open coverings of R by

$$\mathfrak{S}_{m_1} = \mathfrak{U}_{m_1}, \quad m_1 \geqq 0$$
$$\mathfrak{S}_{m_1 \ldots m_k} = \{S_{m_1 \ldots m_k}(U) | U \in \mathfrak{U}_{m_1}\}, \quad 1 \leqq m_1 < m_2 < \ldots < m_k, \ k \geqq 2,$$

to define a non-negative valued function $\rho(x, y)$ on $R \times R$ by

$$\rho(x, y) = \inf\{1/2^{m_1} + \ldots + 1/2^{m_k} | y \in S(x, \mathfrak{S}_{m_1 \ldots m_k})\}.$$

We have shown [6], [7] that this function $\rho(x, y)$ is a topology-preserving metric of R.[7]) We can now prove that ρ is the desired metric.

For any countable sequence $m_1, m_2, \ldots$ of integers with $1 \leqq m_1 < m_2 < \ldots$ we define open sets $S_{m_1 m_2 \ldots}(U)$, $U \in \mathfrak{U}_{m_1}$ by

$$S_{m_1 m_2 \ldots}(U) = \bigcup_{k=1}^{\infty} S_{m_1 \ldots m_k}(U)$$

and open coverings $\mathfrak{S}_{m_1 m_2 \ldots}$ by

$$\mathfrak{S}_{m_1 m_2 \ldots} = \{\mathfrak{S}_{m_1 m_2 \ldots}(U) | U \in \mathfrak{U}_{m_1}\}.$$

[5]) The proof of sufficiency in [4] should be read as follows: First, let us note that $\{BS_{1/2^i}(p) | p \in A\}$ is closure preserving in $B[\cup\{S_{1/2^i}(p) | p \in A\}] \ldots$ Hence dim $B[\cup\{S_{1/2^i}(p) | p \in A\}] \leqq n-1$ follows from dim $BS_{1/2^i}(p) \leqq n-1$, $p \in A$ by virtue of a theorem due to Nagami.

[6]) Let $\mathfrak{A}$, A, p be a covering, a set and a point of R respectively. Then $S(p, \mathfrak{A}) = \cup\{U | p \in U \in \mathfrak{A}\}$, $S(A, \mathfrak{A}) = \cup\{U | \mathfrak{A} \ni U \not\subset R-A\}$, $S^n(p, \mathfrak{A}) = S(S^{n-1}(p, \mathfrak{A}), \mathfrak{A})$, $S^n(A, \mathfrak{A}) = S(S^{n-1}(A, \mathfrak{A}), \mathfrak{A})$, $\mathfrak{A}^* = \{S(U, \mathfrak{A}) | U \in \mathfrak{A}\}$.

[7]) We proved in [6], [7] $\rho(x, y)$ satisfied another condition which also characterized the dimension of R. That condition was simplified in separable cases by [1].

Suppose

$$0 < \varepsilon = \frac{1}{2^{m_1}} + \frac{1}{2^{m_2}} + \dots$$

and

$$1 \leqq m_1 < m_2 < \dots;$$

then we can assert

$$\text{(A)} \qquad S_\varepsilon(p) = S(p, \mathfrak{S}_{m_1 m_2 \dots}).$$

For if $q \notin S(p, \mathfrak{S}_{m_1 \dots m_k})$, $k = 1, 2, \dots$, then $\rho(p, q) \geqq 1/2^{m_1} + +1/2^{m_2} + \dots$ which means $q \notin S_\varepsilon(p)$. Hence we get

$$S_\varepsilon(p) \subset S(p, \mathfrak{S}_{m_1 m_2 \dots})$$

from

$$\mathfrak{S}_{m_1 \dots m_k} < \mathfrak{S}_{m_1 m_2 \dots}.$$

Conversely, if $q \in S(p, \mathfrak{S}_{m_1 m_2 \dots})$, then there exists $U \in \mathfrak{U}_{m_1}$ such that $p, q \in S_{m_1 m_2 \dots}(U)$. In view of the definition of $S_{m_1 m_2 \dots}(U)$ we get $p, q \in S_{m_1 \dots m_k}(U)$ for some $k \geqq 1$. Hence $\rho(p, q) \leqq 1/2^{m_1} + + \dots + 1/2^{m_k} < \varepsilon$, which means $q \in S_\varepsilon(p)$, and hence

$$S(p, \mathfrak{S}_{m_1 m_2 \dots}) \subset S_\varepsilon(p).$$

Thus we can conclude

$$S_\varepsilon(p) = S(p, \mathfrak{S}_{m_1 m_2 \dots}).$$

To show $\dim B[S_\varepsilon(p)] \leqq n-1$ we shall prove

(B) $\operatorname{ord} \mathfrak{S}_{m_1 m_2 \dots} \leqq n+1$ for every $\mathfrak{S}_{m_1 m_2 \dots}$.

To this end we shall inductively prove

$$S^3(S_{m_1 \dots m_{k-1}}(U), \quad \mathfrak{U}_{m_k}) \subset S^3(U, \mathfrak{U}_{m_1+1}), \quad k \geqq 2.$$

This proposition is clearly valid for $k = 2$ since $\mathfrak{U}_{m_2} < \mathfrak{U}_{m_1+1}$ is implied by $m_2 \geqq m_1+1$.

Assume the validity for $k = k$; then

$$\begin{aligned} S^3(S_{m_1 \dots m_k}(U), \mathfrak{U}_{m_{k+1}}) &= S^3(S^2(S_{m_1 \dots m_{k-1}}(U), \mathfrak{U}_{m_k}), \mathfrak{U}_{m_{k+1}}) \\ &\subset S^3(S_{m_1 \dots m_{k-1}}(U), \mathfrak{U}_{m_k}) \subset S^3(U, \mathfrak{U}_{m_1+1}) \end{aligned}$$

follows from $\mathfrak{U}^*_{m_{k+1}} < \mathfrak{U}_{m_k}$ combined with the inductive assumption. Hence we get

$$\text{(C)} \qquad S_{m_1 m_2 \dots}(U) \subset S^3(U, \mathfrak{U}_{m_1+1}).$$

Since by 3) each $S(p, \mathfrak{U}^*_{m_1+1})$ intersects at most $n+1$ sets of $\mathfrak{U}_{m_1}$, each point p of R is contained in at most $n+1$ of $S^3(U, \mathfrak{U}_{m_1+1})$, $U \in \mathfrak{U}_{m_1}$. This combined with (C) implies (B).

Now let us turn to the proof of $\dim B[S_\varepsilon(p)] \leqq n-1$. Let $q \in B[S_\varepsilon(p)]$; then we can express the positive number $\varepsilon(\leqq 1)$ in the form of

$$\varepsilon = \frac{1}{2^{m_1}} + \frac{1}{2^{m_2}} + \cdots$$

for some countable sequence $m_1, m_2, \ldots$ of integers with $1 \leqq m_1 < m_2 < \ldots$. We can prove

$$\operatorname{ord}_q \mathfrak{U}_{m_k} \leqq n \text{ [8]}, \quad k = 1, 2, \ldots.$$

For, if we suppose $q \in U_i \in \mathfrak{U}_{m_k}$, $i = 1, \ldots, n+1$, then by virtue of (A), there exists $U \in \mathfrak{U}_{m_k}$ such that

$$U \subset S_\varepsilon(p), \quad S_{m_k m_{k+1}} \ldots (U) \cap \left(\bigcap_{i=1}^{n+1} U_i\right) \neq \phi.$$

But this implies

$$\operatorname{ord} \mathfrak{S}_{m_k m_{k+1}} \ldots \geqq n+2$$

and hence it contradicts (B). Thus $\{\mathfrak{U}_{m_1}, \mathfrak{U}_{m_2}, \ldots\}$ can be regarded as a sequence of open coverings of $B[S_\varepsilon(p)]$ satisfying

$$\mathfrak{U}_{m_1} > \mathfrak{U}^*_{m_2} > \mathfrak{U}_{m_2} > \mathfrak{U}^*_{m_3} > \cdots$$
$$\{S(p, \mathfrak{U}_{m_k}) | k = 1, 2, \ldots\} \text{ is an nbd basis of } p,$$
$$\operatorname{ord} \mathfrak{U}_{m_k} \leqq n, \quad k = 1, 2, \ldots.$$

Therefore we can conclude

$$\dim B[S_\varepsilon(p)] \leqq n-1$$

by one of our n-dimensionality theorems[9]).

Finally, we shall show that $\{S_\varepsilon(p) | p \in R\}$ is closure preserving for any $\varepsilon > 0$. It follows from (A) and (B) that each $S_\varepsilon(p)$ is a finite sum of sets of $\mathfrak{S}_{m_1 m_2} \ldots$ if $\varepsilon = 1/m_1 + 1/m_2 + \ldots$. Hence closure preserving property of $\mathfrak{S}_{m_1 m_2} \ldots$ implies that of $\{S_\varepsilon(p) | p \in R\}$. To see the closure preserving of $\mathfrak{S}_{m_1 m_2} \ldots$ we should notice the condition (3) which implies that each set of $\mathfrak{U}_{m_1+1}$ intersects at most $n+1$ sets of $\{S^3(U, \mathfrak{U}_{m_1+1}) | U \in \mathfrak{U}_{m_1}\}$. Hence, in view of (C), we can conclude that each set of $\mathfrak{U}_{m_1+1}$ intersects at most $n+1$ sets

[8]) Let $\mathfrak{A}$ be a collection of sets of R and q a point of R. Then $\operatorname{ord}_q \mathfrak{A}$ denotes the number of elements of $\mathfrak{A}$ which contain q. Then $\operatorname{ord} \mathfrak{A} = \max\{\operatorname{ord}_q \mathfrak{A} | q \in R\}$.

[9]) [7], Theorem 3.

of $\mathfrak{S}_{m_1 m_2 \ldots}$. Hence $\mathfrak{S}_{m_1 m_2 \ldots}$ is locally finite, and accordingly closure preserving. Thus $\{S_\varepsilon(p) | p \in R\}$ is closure preserving, which completes the proof of this theorem.

The metric in this theorem is rather peculiar considering that the usual metric of Euclidean space does not satisfy the closure preserving condition, but the metric in the following corollary will be more reasonable.

COROLLARY 1. *A metric space* R *has* $\dim \leqq n$ *if and only if we can introduce a topology-preserving metric* ρ *into* R *such that*

$$\dim B[S_\varepsilon(F)] \leqq n-1, \ \varepsilon > 0$$

for any closed set F *of* R.

PROOF. We can easily deduce it from Theorem 1 as we have deduced (II) from (I). [10]).

COROLLARY 2. *A metric space* R *has* $\dim \leqq n$ *if and only if we can introduce a topology-preserving metric* ρ *into* R *such that*

$$\dim C_\varepsilon(p) \leqq n-1$$

for any irrational (or for almost all) $\varepsilon > 0$ *and for any point* p *of* R *and such that* $\{C_\varepsilon(p) | p \in R\}$ *is closure preserving for any irrational (or for almost all)* $\varepsilon > 0$, *where*

$$C_\varepsilon(p) = \{q | \rho(p, q) = \varepsilon\}.$$

PROOF. The sufficiency of condition is clear.

Referring to the necessity we can show the metric in the proof of Theorem 1 is the required one. To see this it suffices to prove

$$C_\varepsilon(p) = B[S_\varepsilon(p)]$$

for any irrational $\varepsilon > 0$. Since $B[S_\varepsilon(p)] \subset C_\varepsilon(p)$ is clear, we let q be a given point with $q \notin B[S_\varepsilon(p)]$ to establish the inverse. If $q \in S_\varepsilon(p)$, then $q \notin C_\varepsilon(p)$ is obvious, so we suppose $q \notin \overline{S_\varepsilon(p)}$. Let $\varepsilon = 1/2^{m_1} + 1/2^{m_2} + \ldots$; then by (A) in the proof of Theorem 1

$$S_\varepsilon(p) = S(p, \mathfrak{S}_{m_1 m_2 \ldots}).$$

Since ε is irrational, we can choose a sufficiently large m_i such that

$$S(q, \mathfrak{U}_{m_i}) \cap S(p, \mathfrak{S}_{m_1 m_2 \ldots}) = \phi$$
$$m_{i+1} \geqq m_i + 2.$$

[10]) See [4].

Then it is easily seen that

$$q \notin S(p, \mathfrak{S}_{m_1 \ldots m_i m_i+1}).$$

Hence

$$\rho(p, q) \geqq \frac{1}{2^{m_1}} + \ldots + \frac{1}{2^{m_i}} + \frac{1}{2^{m_i+1}} > \varepsilon,$$

which means $q \notin C_\varepsilon(p)$, and hence

$$C_\varepsilon(p) \subset B[S_\varepsilon(p)].$$

Thus $C_\varepsilon(p) = B[S_\varepsilon(p)]$ is proved for every irrational ε.

In view of this proof we see that

$$C_\varepsilon(p) = B[S_\varepsilon(p)]$$

holds not only for irrational numbers but for any positive number $\varepsilon = 1/2^{m_1}+1/2^{m_2}+ \ldots$ such that for any positive m there exists m_i satisfying $m \leqq m_i < m_{i+1}-2$.

Corollary 3. *A metric space R has* dim $\leqq n$ *if and only if we can introduce a topology-preserving metric ρ into R such that for all irrational* (*or for almost all*) *positive numbers ε and for any closed set F of R,* dim $C_\varepsilon(F) \leqq n-1$, *where*

$$C_\varepsilon(F) = \{p | \rho(p, F) = \varepsilon\}.$$

Proof. The sufficiency is clear. Referring to the necessity we can easily see that the metric in the proof of Corollary 2 satisfies the desired condition.

2. Our next problem is to give a new type of condition for n-dimensionality by use of the new terminology '*rank*' of collection of sets.

Definition 1. Two subsets A and B of R are called *independent* if $A \not\subset B$ and $B \not\subset A$. A collection of subsets is called independent if any two members of it are independent.

Definition 2. Let $\mathfrak{U}$ be a collection of subsets of a space R and p a point of R. Then $\text{rank}_p \mathfrak{U}$ is the largest integer n such that there are n independent members of $\mathfrak{U}$ which contain p. Moreover rank $\mathfrak{U} = \max\{\text{rank}_p \mathfrak{U} | p \in R\}$.

In view of this definition we clearly see $\text{rank}_p \mathfrak{U} \leqq \text{ord}_p \mathfrak{U}$ for any point p and collection $\mathfrak{U}$ of subsets, and accordingly rank $\mathfrak{U} \leqq$ ord $\mathfrak{U}$.

Definition 3. Let A and B be two subsets of R. If A meets B as well as $R-B$, then we say A *overflows* B.

Now we can prove the following.

Theorem 2. A metric space has dim $\leqq n$ if and only if it has an open basis $\mathfrak{U}$ with rank $\mathfrak{U} \leqq n+1$.

Proof. To begin with, let us prove the if part by induction. Let $\mathfrak{U}$ be an open basis with rank $\leqq 1$. Suppose F and G are disjoint closed sets of R. Then we let

$$U = \cup \{U' | U' \in \mathfrak{U}, \quad U' \cap F \neq \phi, \quad U' \cap G = \phi\}.$$

Since $\mathfrak{U}$ is an open basis of R, U is an open set satisfying

$$F \subset U \subset R-G.$$

If $p \notin U$, then there exists $U' \in \mathfrak{U}$ such that $p \in U' \subset R-F$. If we assume $U' \cap U \neq \phi$, then $U' \cap U'' \neq \phi$ for some $U'' \in \mathfrak{U}$ with $U'' \cap F \neq \phi$. Since U' and U'' are clearly independent, we reach a contradiction to rank $\mathfrak{U} \leqq 1$. Hence $U' \cap U = \phi$, which means that the open set U is closed in R. Thus we get dim $R \leqq 0$.

Suppose we have proved that the existence of an open basis with rank $\leqq n$ implies dim $R \leqq n-1$. Then we suppose R has an open basis $\mathfrak{U}$ with rank $\mathfrak{U} \leqq n+1$. Let F and G be two disjoint closed sets of R. Then we define an open set U by

$$U = \cup \{U' | U' \in \mathfrak{U}, \quad U' \cap F \neq \phi, \quad U' \cap G = \phi\}.$$

U clearly satisfies

$$F \subset U \subset R-G.$$

We shall prove that $\mathfrak{U}' = \{U' | U' \in \mathfrak{U},\ U' \cap F = \phi\}$ restricted to $B[U]$ makes an open basis of $B[U]$ satisfying rank $\mathfrak{U}' \leqq n$. It is clear that $\mathfrak{U}'$ is an open basis of $B[U]$ if restricted to $B[U]$.

Thus all we have to show is that $\text{rank}_p \mathfrak{U}' \leqq n$ for a given point $p \in B[U]$. Suppose the contrary, i. e. $U_1, \ldots, U_{n+1}$ are independent sets of $\mathfrak{U}'$ which contain p. Since $p \in B[U]$, we get

$$q \in U_1 \cap \ldots \cap U_{n+1} \cap U \neq \phi.$$

Thus

$$q \in U_1 \cap \ldots \cap U_{n+1} \cap U'$$

for some $U' \in \mathfrak{U}$ with $U' \cap F \neq \phi$, $U' \subset U$. Since $U_i \cap F = \phi$, $U_i \cap (R-U') \neq \phi$, $i = 1, \ldots, n+1$, $U_1, \ldots, U_{n+1}$ and U' are independent contradicting rank $\mathfrak{U} \leqq n+1$. Thus we get $\text{rank}_p \mathfrak{U}'$ $\leqq n$, and hence dim $B[U] \leqq n-1$ follows from the inductive assumption. Therefore dim $R \leqq n$ is proved.

To prove the only if part we suppose R is a metric space with

$\dim R \leqq n$. R can be decomposed into $n+1$ zero-dimensional subspaces A_i, $i = 1, \ldots, n+1$. Let us apply one of our previous results [11]) to the present problem to get a locally finite open covering $\mathfrak{U}_1$ with mesh $\mathfrak{U}_1 < 1$ such that

$$\mathrm{ord}_p B[\mathfrak{U}_1] \leqq i-1 \text{ [12])} \quad \text{for every } p \in A_i.$$

Let

$$B_k = \{p | \mathrm{ord}_p B[\mathfrak{U}_1] \geqq k\}, \quad k = 0, 1, \ldots, n;$$

then it follows from $B_k \subset A_{k+1} \cup \ldots \cup A_{n+1}$ that

$$\dim B_k \leqq n-k, \quad k = 0, 1, \ldots, n.$$

Each B_k is closed since $B[\mathfrak{U}_1]$ is locally finite. Moreover $B_k \subset B_{k-1}$ is clear from the definition of B_k. Let $\mathfrak{S}$ be an open covering with mesh $< \frac{1}{2}$. For every point p of $B_k - B_{k+1}$ we choose an open nbd $U(p)$ of p such that $U(p)$ overflows just k sets of $\mathfrak{U}_1$. We see the existence of such an nbd in view of the definition of B_k. Then

$$\mathfrak{W}_k = \{U(p) | p \in B_k - B_{k+1}\}$$

is a collection of open sets which covers $B_k - B_{k+1}$. Now we can define a locally finite open covering $\mathfrak{P} < \mathfrak{S}$ such that $\mathfrak{P} = \bigcup_{k=0}^{n} \mathfrak{P}_k$, $\mathfrak{P}_k \supset \mathfrak{P}_{k-1}$, $\mathrm{ord}\, \mathfrak{P}_k \leqq k+1$, $\mathfrak{P}_k - \mathfrak{P}_{k-1} < \mathfrak{W}_{n-k}$ [13]) and $\mathfrak{P}_k$ covers B_{n-k}. To realize it we shall show, by induction, that for any m with $0 \leqq m \leqq n$ we can define locally finite open collections $\mathfrak{P}_m$ of R such that

$$\mathfrak{P}_m = \bigcup_{k=0}^{m} \mathfrak{P}_k, \quad \mathfrak{P}_k \supset \mathfrak{P}_{k-1}, \quad \mathrm{ord}\, \mathfrak{P}_k \leqq k+1, \quad \mathfrak{P}_k - \mathfrak{P}_{k-1} < \mathfrak{W}_{n-k},$$
$$\mathfrak{P}_k < \mathfrak{S}$$

and such that $\mathfrak{P}_k$ covers B_{n-k}.

For $m = 0$ we choose, by use of $\dim B_n \leqq 0$, an open covering $\mathfrak{Q}$ of B_n with $\mathrm{ord}\, \mathfrak{Q} \leqq 0$, $\mathfrak{Q} < \mathfrak{W}_n \wedge \mathfrak{S}$. It is easy to see that $\mathfrak{Q}$ can be extended to a locally finite collection $\mathfrak{P}_0$ of open sets of R such that

$$\mathrm{ord}\ \mathfrak{P}_0 \leqq 1, \quad \mathfrak{P}_0 < \mathfrak{W}_n \wedge \mathfrak{S}$$

and such that

$$\{P \cap B_n | P \in \mathfrak{P}_0\} = \mathfrak{Q}.$$

[11]) [5] Lemma 2.1.

[12]) Let $\mathfrak{A}$ be a collection of subsets of R; then mesh $\mathfrak{A} = \sup\{\text{diameter } U | U \in \mathfrak{A}\}$, $B[\mathfrak{A}] = \{B[U] | U \in \mathfrak{A}\}$.

[13]) We suppose $\mathfrak{W}_n = \{U(p) | p \in B_n\}$, $\mathfrak{P}_{-1} = \phi$.

Now let us suppose we have defined $\mathfrak{P}_m$ at our desire. Then let

$$\mathfrak{P}_k = \{P_\alpha | \alpha < \alpha_{k+1}\}, \quad k = 0, 1, \ldots, m.$$

Since $\dim B_{n-m-1} \leqq m+1$, we can find a locally finite open covering $\mathfrak{N}$ of B_{n-m-1} satisfying

$$\operatorname{ord} \mathfrak{N} \leqq m+2, \quad \mathfrak{N} < \mathfrak{P}_m \cup \mathfrak{W}_{n-m-1}, \quad \mathfrak{N} < \mathfrak{S}.$$

It is easy to see that $\mathfrak{N}$ can be extended to a locally finite collection $\mathfrak{M}$ of open sets of R such that

$$\operatorname{ord} \mathfrak{M} \leqq m+2, \quad \mathfrak{M} < \mathfrak{P}_m \cup \mathfrak{W}_{n-m-1}, \quad \mathfrak{M} < \mathfrak{S}.$$

We let

$$P'_\alpha = \cup\{M | M \in \mathfrak{M}, M \subset P_\alpha, M \not\subset P_\beta \text{ for any } \beta < \alpha\},$$
$$\mathfrak{P}'_k = \{P'_\alpha | \alpha < \alpha_{k+1}\}, \quad k = 0, 1, \ldots, m,$$
$$\mathfrak{P}'_{m+1} = \mathfrak{P}'_m \cup \{M | M \not\subset P_\alpha \text{ for any } \alpha < \alpha_{m+1}\}.$$

Then $\mathfrak{P}'_{m+1} = \bigcup_{k=0}^{m+1} \mathfrak{P}'_k$ is the desired locally finite open collection which covers B_{n-m-1}. The only problem is to show that $\mathfrak{P}'_k$ covers B_{n-k} but this can be easily deduced from the fact that each element of $\mathfrak{P}_m - \mathfrak{P}_k$ does not meet B_{n-k} since

$$\mathfrak{P}_m - \mathfrak{P}_k < \mathfrak{W}_{n-k-1} \cup \ldots \cup \mathfrak{W}_{n-m}$$

and each element of $\mathfrak{W}_{n-k-1} \cup \ldots \cup \mathfrak{W}_{n-m}$ does not meet B_{n-k} by the definition of $\mathfrak{W}_i$. Each element of $\mathfrak{W}_{n-m-1}$, of course, does not meet B_{n-k}, either. Let p be a given point of B_{n-k}; then $p \in M$ for some $M \in \mathfrak{M}$. Since $\mathfrak{M} < \mathfrak{P}_m \cup \mathfrak{W}_{n-m-1}$, it follows from the above remark that $p \in M \subset P$ for some $P \in \mathfrak{P}_k$, and hence $M \subset P'$ for some $P' \in \mathfrak{P}'_k$. Thus we can define the desired locally finite open covering $\mathfrak{P}$ of R. Let $\mathfrak{P} = \{P_\gamma | \gamma \in \Gamma\}$, $\mathfrak{P}_k = \{P_\gamma | \gamma \in \Gamma_k\}$, $k = 0, \ldots, n$; then there exists an open covering $\mathfrak{V} = \{V_\gamma | \gamma \in \Gamma\}$ of R such that $\bar{V}_\gamma \subset P_\gamma$, $\gamma \in \Gamma$. Now again by use of the lemma in [5], we can define an open covering $\mathfrak{U}_2 = \{U_\gamma | \gamma \in \Gamma\}$ of R satisfying $\bar{V}_\gamma \subset U_\gamma \subset P_\gamma$, $\gamma \in \Gamma$ and

$$\operatorname{ord}_p B[\mathfrak{U}_1 \cup \mathfrak{U}_2] \leqq i-1 \quad \text{for every} \quad p \in A_i.$$

In view of the process of definition it is clear that

$$\mathfrak{U}_2 < \mathfrak{S}, \quad \operatorname{ord} \mathfrak{U}^k \leqq k+1, \quad \mathfrak{U}^k - \mathfrak{U}^{k-1} < \mathfrak{W}_{n-k},$$

where $\mathfrak{U}^k = \{U_\gamma | \gamma \in \Gamma_k\}$.

Let us finally show rank $\mathfrak{U}_1 \cup \mathfrak{U}_2 \leqq n+1$.

Suppose

$$p \in U_1 \cap \ldots \cap U_k \cap U_{k+1} \cap \ldots \cap U_{n+2}$$

for $n+2$ independent sets

$$U_1, \ldots, U_k \in \mathfrak{U}_1 \quad \text{and} \quad U_{k+1}, \ldots, U_{n+2} \in \mathfrak{U}_2.$$

Then, since $\operatorname{ord} \mathfrak{U}^{n-k} \leqq n-k+1$, at most one of $U_{k+1}, \ldots, U_{n+2}$ does not belong to $\mathfrak{U}^{n-k}$. For example, let

$$U_{k+1} \in \mathfrak{U}^{l+1} - \mathfrak{U}^l \quad \text{for some} \quad l \geqq n-k.$$

Since $\mathfrak{U}^{l+1} - \mathfrak{U}^l < \mathfrak{W}_{n-l-1}$ and each member of $\mathfrak{W}_{n-l-1}$ overflows just $n-l-1$ sets of $\mathfrak{U}_1$, U_{k+1} overflows at most $n-l-1$ sets of $\mathfrak{U}_1$. Since $n-l-1 \leqq k-1$, U_{k+1} overflows at most $k-1$ sets of $\mathfrak{U}_1$. On the other hand, since $U_1, \ldots, U_k, U_{k+1}$ are independent and have a common point p, U_{k+1} must overflow k sets $U_1, \ldots, U_k$ of $\mathfrak{U}_1$, which is a contradiction. Thus we can conclude $\operatorname{rank} \mathfrak{U}_1 \cup \mathfrak{U}_2 \leqq n+1$.

By repeating this process again we can define the third locally finite open covering $\mathfrak{U}_3$ of R such that

$$\operatorname{mesh} \mathfrak{U}_3 < \tfrac{1}{3}, \quad \operatorname{rank} \mathfrak{U}_1 \cup \mathfrak{U}_2 \cup \mathfrak{U}_3 \leqq n+1$$

and

$$\operatorname{ord}_p B[\mathfrak{U}_1 \cup \mathfrak{U}_2 \cup \mathfrak{U}_3] \leqq i-1 \quad \text{for every } p \in A_i.$$

Eventually, by repeating this process, we get a sequence $\mathfrak{U}_1, \mathfrak{U}_2, \mathfrak{U}_3, \ldots$ of open coverings of R satisfying

$$\operatorname{mesh} \mathfrak{U}_i < \frac{1}{i}, \quad i = 1, 2, \ldots, \quad \operatorname{rank} \bigcup_{i=1}^{\infty} \mathfrak{U}_i \leqq n+1.$$

Thus $\mathfrak{U} = \bigcup_{i=1}^{\infty} \mathfrak{U}_i$ is the desired open basis of R with $\operatorname{rank} \mathfrak{U} \leqq n+1$.

The following is a direct consequence of this theorem.

Corollary 4. *A metric space R has an open basis $\mathfrak{U}$ with $\operatorname{rank}_p \mathfrak{U} < +\infty$ at every point p of R if and only if R is strongly countable-dimensional* [14]), *i.e. it is the countable sum of finite-dimensional closed sets.*

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