COMPOSITIO MATHEMATICA

S. K. BOSE
DEVENDRA SHARMA

Integral functions of two complex variables

Compositio Mathematica, tome 15 (1962-1964), p. 210-226
<http://www.numdam.org/item?id=CM_1962-1964__15__210_0>

© Foundation Compositio Mathematica, 1962-1964, tous droits réservés.

L'accès aux archives de la revue « Compositio Mathematica » (http://http://www.compositio.nl/) implique l'accord avec les conditions générales d'utilisation (http://www.numdam.org/conditions). Toute utilisation commerciale ou impression systématique est constitutive d'une infraction pénale. Toute copie ou impression de ce fichier doit contenir la présente mention de copyright.

NUMDAM
Article numérisé dans le cadre du programme
Numérisation de documents anciens mathématiques
http://www.numdam.org/

Integral functions of two complex variables

by

S. K. Bose and Devendra Sharma [1] [2]

1. Let [3]

$$f(z_1, z_2) = \sum_{m_1, m_2=0}^{\infty} a_{m_1, m_2} z_1^{m_1} z_2^{m_2},$$

be a function of the two complex variables z_1 and z_2, regular for $|z_t| \leqq r_t$, $t = 1, 2$. If r_1 and r_2 are arbitrarily large, then $f(z_1, z_2)$ is an integral function of the two complex variables. We know that

$$M(r_1, r_2) = \max_{|z_t| \leqq r_t} |f(z_1, z_2)|, t = 1, 2,$$

is the maximum modulus of $f(z_1, z_2)$ for $|z_t| \leqq r_t$.

In this paper we have defined maximum term and the ranks of the maximum term, and have extended the method of systematic determination of these as in the case of one variable by Newton's polygon ((1), p. 28). Also we have obtained relations between these, and inequalities involving these and the maximum modulus. Further, we have defined order and have obtained necessary and sufficient conditions for the function to be of finite order, and also the same for functions of finite order and type τ.

2. Let [4])

$$f(z_1, z_2) = \sum_{m_1, m_2=0}^{\infty} a_{m_1, m_2} z_1^{m_1} z_2^{m_2}$$

be a function of the two complex variables z_1 and z_2, regular in $|z_1| \leqq r_1$ and $|z_2| \leqq r_2$. Writing $z_1 = r_1 e^{i\theta_1}$ and $z_2 = r_2 e^{i\theta_2}$,

$$f(z_1, z_2) = U(r_1, r_2, \theta_1, \theta_2) + iV(r_1, r_2, \theta_1, \theta_2)$$

---

1) I regret to announce the sudden and untimely death of Devendra Sharma on 18th June, 1957.

2) We are thankful to the referee for the valuable criticism.

3) We have considered only two variables for simplicity. The results can easily be extended to several variables.

4) If (z_1^0, z_2^0) is any given point, then by neighbourhood of this point we would mean a bicylinder $|z_1 - z_1^0| < r_1, |z_2 - z_2^0| < r_2, r_1 > 0, r_2 > 0$.

and

$$a_{m_1,m_2} = \alpha_{m_1,m_2} + i\beta_{m_1,m_2},$$

$$(2.1)\quad U(r_1, r_2, \theta_1, \theta_2) = \sum_{m_1,m_2=0}^{\infty} r_1^{m_1} r_2^{m_2}\{\alpha_{m_1,m_2}\cos(m_1\theta_1 + m_2\theta_2) - \beta_{m_1,m_2}\sin(m_1\theta_1 + m_2\theta_2)\}.$$

This series is convergent for all values of θ_1 and θ_2 because $\sum_{m_1,m_2=0}^{\infty}|a_{m_1,m_2}|r_1^{m_1}r_2^{m_2}$ is uniformly convergent, by hypothesis. We, therefore, multiply both sides by cos $(m_1\theta_1 + m_2\theta_2)$ or $\sin(m_1\theta_1 + m_2\theta_2)$ and integrate term by term between the limits *zero* and 2π. We thus have

$$\frac{(2\pi)^2}{2} r_1^{m_1} r_2^{m_2} \alpha_{m_1,m_2} = \int_0^{2\pi}\int_0^{2\pi} U \cos(m_1\theta_1 + m_2\theta_2) d\theta_1 d\theta_2$$

and

$$\frac{(2\pi)^2}{2} r_1^{m_1} r_2^{m_2} \beta_{m_1,m_2} = -\int_0^{2\pi}\int_0^{2\pi} U \sin(m_1\theta_1 + m_2\theta_2) d\theta_1 d\theta_2.$$

Multiplying the second by i and adding, we get

$$\frac{(2\pi)^2}{2} a_{m_1,m_2} r_1^{m_1} r_2^{m_2} = \int_0^{2\pi}\int_0^{2\pi} U e^{-i(m_1\theta_1 + m_2\theta_2)} d\theta_1 d\theta_2.$$

Hence

$$(2.2)\qquad \frac{(2\pi)^2}{2} |a_{m_1,m_2}| r_1^{m_1} r_2^{m_2} \leqq \int_0^{2\pi}\int_0^{2\pi} |U| d\theta_1 d\theta_2.$$

Again, if we integrate (2.1) with respect to θ_1 and θ_2 in the range *zero* to 2π, we get

$$(2.3)\qquad (2\pi)^2 \alpha_{0,0} = \int_0^{2\pi}\int_0^{2\pi} U d\theta_1 d\theta_2.$$

From (2.2) and (2.3) follows

$$\frac{(2\pi)^2}{2} \{|a_{m_1,m_2}| r_1^{m_1} r_2^{m_2} + 2\alpha_{0,0}\} \leqq \int_0^{2\pi}\int_0^{2\pi} \{|U| + U\} d\theta_1 d\theta_2.$$

Now, if U is positive, then the integrand is equal to $2U$ and if U is negative or zero, then the right hand side is zero. Hence we have

THEOREM 1. *If the function $f(z_1, z_2)$ is regular for $|z_1| \leqq r_1$ and $|z_2| \leqq r_2$ and if $A(r_1, r_2)$ is its maximum real part for $|z_1| \leqq r_1$ and $|z_2| \leqq r_2$, then for all positive values of m_1 and m_2, the number $|a_{m_1,m_2}| r_1^{m_1} r_2^{m_2}$ is less than or equal to the greatest of the two numbers $-2\alpha_{0,0}$ and $4A(r_1, r_2) - 2\alpha_{0,0}$.*

COROLLARY. *If $f(z_1, z_2)$ is an integral function, then*

$$|a_{m_1,m_2}|r_1^{m_1}r_2^{m_2} \leqq 4A(r_1, r_2) - 2\alpha_{0,0}, \tag{2.4}$$

for all positive values of m_1 and m_2, and for all $r_1 > r_1^0$, $r_2 > r_2^0$.

Similar results can be obtained for minimum of $U(r_1, r_2, \theta_1, \theta_2)$. Also for the maximum and minimum of $V(r_1, r_2, \theta_1, \theta_2)$.

THEOREM II. *If $f(z_1, z_2)$ is an integral function, and q_1 and q_2 are two fixed finite positive numbers such that $|f(z_1, z_2)|$ is algebraically less than $Kr_1^{q_1}r_2^{q_2}$ for $|z_1| = r_1$ and $|z_2| = r_2$, where r_1 and r_2 are arbitrary large numbers and K is a constant, then $f(z_1, z_2)$ is a polynomial of degree not greater than q_1+q_2.*

PROOF: Since $|f(z_1, z_2)| \leqq Kr_1^{q_1}r_2^{q_2}$ for $|z_1| = r_1$ and $|z_2| = r_2$, therefore, from (2.4) follows

$$|a_{m_1,m_2}| \leqq 4Kr_1^{q_1-m_1}r_2^{q_2-m_2} - 2\alpha_{0,0}r_1^{-m_1}r_2^{-m_2}.$$

If $m_1 > q_1$ or $m_2 > q_2$, then the right hand side vanishes as r_1 or r_2 respectively tends to infinity, and so a_{m_1,m_2} is zero for $m_1 > q_1$ or $m_2 > q_2$.

THEOREM III. *If $f(z_1, z_2)$ is an integral function of the two complex variables z_1 and z_2, then*

$$M(r_1, r_2) < \frac{R_1R_2}{(R_1-r_1)(R_2-r_2)}[4A(R_1, R_2) + 3|a_{0,0}|],$$

for all $R_1 > r_1 > r_1^0$ and $R_2 > r_2 > r_2^0$, where

$$M(r_1, r_2) = \max|f(z_1, z_2)|,$$

for $|z_1| \leqq r_1$ and $|z_2| \leqq r_2$.

PROOF: We can write (2.4) as

$$|a_{m_1,m_2}|r_1^{m_1}r_2^{m_2} \leqq [4A(R_1, R_2) - 2\alpha_{0,0}]\left(\frac{r_1}{R_1}\right)^{m_1}\left(\frac{r_2}{R_2}\right)^{m_2}$$

and taking $r_1 < R_1$ and $r_2 < R_2$, it follows that

$$\begin{aligned} M(r_1, r_2) &\leqq \sum_{m_1,m_2=0}^{\infty} |a_{m_1,m_2}|r_1^{m_1}r_2^{m_2} \\ &\leqq |a_{0,0}| + [4A(R_1, R_2) - 2\alpha_{0,0}]\sum_{m_1,m_2}{}' \left(\frac{r_1}{R_1}\right)^{m_1}\left(\frac{r_2}{R_2}\right)^{m_2} \\ &= |a_{0,0}| + [4A(R_1, R_2) - 2\alpha_{0,0}]\frac{r_1}{R_1-r_1}\cdot\frac{r_2}{R_2-r_2} \\ &< \frac{R_1R_2}{(R_1-r_1)(R_2-r_2)}[4A(R_1, R_2) + 3|a_{0,0}|]. \end{aligned}$$

COROLLARY: *If* $f(z_1, z_2)$ *is zero at* $z_1 = 0, z_2 = 0$, *then*

$$M(r_1, r_2) \leqq \frac{4r_1 r_2}{(R_1-r_1)(R_2-r_2)} A(R_1, R_2)$$

for $R_1 > r_1, R_2 > r_2$.

3. Consider the moduli of the terms of the double series in the expansion of the integral function[1])

$$f(z_1, z_2) = \sum_{m_1, m_2=0}^{\infty} c_{m_1, m_2} z_1^{m_1} z_2^{m_2}, \tag{3.1}$$

$$\begin{array}{llllll} C_{0,0} & C_{0,1}\, r_2 & C_{0,2}\, r_2^2 & \ldots C_{0,m_2}\, r_2^{m_2} & & \ldots \\ C_{1,0}\, r_1 & C_{1,1}\, r_1 r_2 & C_{1,2}\, r_1 r_2^2 & \ldots C_{1,m_2}\, r_1 r_2^{m_2} & & \ldots \\ C_{2,0}\, r_1^2 & C_{2,1}\, r_1^2 r_2 & C_{2,2}\, r_1^2 r_2^2 & \ldots C_{2,m_2}\, r_1^2 r_2^{m_2} & & \ldots \\ \ldots\ldots & & & & & \\ \ldots\ldots, & & & & & \\ C_{m_1,0}\, r_1^{m_1} & C_{m_1,1}\, r_1^{m_1} r_2 & C_{m_1,2}\, r_1^{m_1} r_2^2 & \ldots C_{m_1,m_2}\, r_1^{m_1} r_2^{m_2} & & \ldots \\ \ldots\ldots & & & & & \\ \ldots\ldots & & & & & \end{array} \tag{3.2}$$

where

$$C_{m_1, m_2} = |c_{m_1, m_2}|.$$

If we consider any column or row, then the sequence thus obtained tends to zero for all values of r_1 or r_2. Hence for every value of r_1, keeping m_2 and r_2 fixed, there is, therefore, one term of the sequence thus obtained which is greater than or equal to all the rest. This term will be the maximum term in that column and will be denoted by $\mu(m_2; r_1, r_2)$. If there are more than one such term, then the term of the highest rank will be regarded as the maximum term of this column and the rank will be denoted by $\nu_1(m_2; r_1)$. We next give different values to m_2, i.e. consider different columns and suppose the greatest term occurs in the ν_2th column, then the term $\mu(\nu_2; r_1, r_2)$ will be the maximum term with respect to columns and the rank of this term will be denoted by $\nu_1(r_1)$. If there are more than one column containing such term, then the term of the highest rank with regard to column will be regarded as maximum term, i.e. the column of highest rank which contains the maximum term is ν_2th. Similarly, if we consider m_1th row, keeping r_1, r_2 and m_1 fixed, then we shall have

[1]) Suffix 1 will indicate row, for example, m_1, p_1, ν_1 etc. and suffix 2 will indicate column, m_2, p_2, ν_2 etc.

$\mu(m_1; r_1, r_2)$ as the maximum term in that row and the rank as $\nu_2(m_1; r_2)$. Further, for different values of m_1, suppose the greatest term occurs in the ν_1th row, then the maximum term will be denoted by $\mu(\nu_1; r_1, r_2)$ and the rank of this term will be denoted by $\nu_2(r_2)$. If there are more than one such term, then the same convention as for columns is adopted. Hence we shall denote the *maximum term* for given values of r_1 and r_2 by $\mu(r_1, r_2)$ and the rank of this term will be denoted by $\nu(r_1, r_2)$.

For a systematic study of finding the maximum term we shall extend the method of Valiron ((1), p. 28) for one variable.

Let $\log C_{m_1, m_2} = -g_{m_1, m_2}$, then

$$\text{(3.3)} \qquad \underset{m_1 \to \infty}{Lt} \frac{g_{m_1, m_2}}{m_1} = +\infty$$

and

$$\underset{m_2 \to \infty}{Lt} \frac{g_{m_1, m_2}}{m_2} = +\infty.$$

Since $\sqrt[m_1]{C_{m_1, m_2}}$ and $\sqrt[m_2]{C_{m_1, m_2}}$ tend to zero as m_1 and m_2 respectively tend to infinity.

Taking OX, OY, OZ as the axes of coordinates, if we plot the points A_{m_1, m_2} of coordinates (m_1, m_2, g_{m_1, m_2}), then, from (3.3), it follows that we can construct a surface with plane faces and every section of this surface by planes parallel to the XZ-plane and YZ-plane form a Newton's polygon, having certain of the points A_{m_1, m_2}, lying in this vertical plane. Out of these some of them coincide with the vertices of the polygon, whilst the remainder lie either on or above it. Let us denote this surface by $S(f)$ and call it *Newton's polyhedron.*

If ν_1 and ν_2 be the rank of a maximum term as defined above and $m_1 \neq \nu_1$ or $m_2 \neq \nu_2$, then, it easily follows:

$$g_{m_1, m_2} - m_1 \log r_1 - m_2 \log r_2 \geqq g_{\nu_1, \nu_2} - \nu_1 \log r_1 - \nu_2 \log r_2.$$

Now, let us consider the geometrical interpretation of this inequality. Let D_{r_1, r_2} denote a tangent plane, having direction cosines proportional to $-\log r_1$, $-\log r_2$ and 1, passing through the point A_{ν_1, m_2} or A_{m_1, ν_2}. If we now draw the plane parallel to the XZ-plane through A_{ν_1, m_2}, then those points A_{m_1, m_2} which lie in the plane do not lie below the line L_{r_1}, the line of intersection with the plane $D_{r_1 r_2}$, of slope $\log r_1$. Similarly, if we consider the plane parallel to the YZ-plane through A_{m_1, ν_2}, those points A_{m_1, m_2} which lie in that plane do not lie below the line L_{r_2}, the line of intersection with the plane $D_{r_1 r_2}$, of slope $\log r_2$. The point A_{ν_1, m_2} or A_{m_1, ν_2} is therefore a point of the polyhedron $S(f)$ and the plane $D_{r_1 r_2}$ is a tangent to this polyhedron.

Now, we take a vertical plane parallel to XZ-plane through A_{ν_1, m_2}, then m_2 is fixed, i.e. in (3.2) we are considering a column. The intersection of the plane with the surface will include a Newton's polygon and the tangent line L_{r_1} of slope $\log r_1$ will pass through A_{ν_1, m_2}. The point A_{ν_1, m_2} is uniquely determined in the plane, when $\log r_1$ is not equal to the slope of one of the sides of $\pi_1(f)$, and for such values of r_1, there is only one term in the sequence (column of (3.2) under consideration) equal to $\mu(m_2; r_1, r_2)$. When $\log r_1$ is equal to the slope of a side of $\pi_1(f)$, there are several such terms and their number is equal to the number of the points A_{ν_1, m_2} which lie on this side of this polygon. When more than one term are equal to $\mu(m_2; r_1, r_2)$, we shall take the term of highest rank amongst them as the maximum term, with respect to m_2 and r_1.

Thus having obtained the maximum term or terms with respect to m_2 and r_1, we must draw a plane parallel to YZ-plane through A_{m_1, ν_2}, then m_1 is fixed, i.e. in (3.2) we are considering m_1th row. Again the intersection will include a Newton's polygon and the tangent line L_{r_2} of slope $\log r_2$ will pass through A_{m_1, ν_2}. Here again the point A_{m_1, ν_2} is uniquely determined in the plane, when $\log r_2$ is not equal to the slope of one of the sides of $\pi_2(f)$, and for such values of r_2 there is only one term in the sequence of terms in the m_1th row equal to $\mu(m_1; r_1, r_2)$. When $\log r_2$ is equal to the slope of a side of $\pi_2(f)$, there are several terms and their number is equal to the number of points A_{m_1, ν_2} which lie on the side of the polygon. As in the case of m_2 and r_1, we shall take the term of highest rank amongst them as the maximum term, with respect to m_1 and r_2. Finally the greater of the two terms obtained above shall be denoted by $\mu(r_1, r_2)$.

We have thus, with this convention, obtained one term as maximum with respect to m_1, m_2, r_1 and r_2. Thus $\nu_1(m_2; r_1, r_2)$ or $\nu_1(r_1, r_2)$, r_2 fixed, $\nu_2(m_1; r_1, r_2)$ or $\nu_2(r_1, r_2)$, r_1 fixed, and $\nu(r_1, r_2)$ will be used to denote ranks of the maximum term of the double series. $\nu_1(m_2; r_1, r_2)$, $\nu_2(m_1; r_1, r_2)$ and $\nu(r_1, r_2)$ are unbounded non-decreasing functions of r_1 and r_2. Further, $\nu_1(m_2; r_1, r_2)$ and $\nu_2(m_1; r_1, r_2)$ have left hand discontinuity wherever r_1 and r_2 respectively pass through a value such that $\log r_1$ and $\log r_2$ respectively equal the slope of one of the sides of the polygons $\pi_1(f)$ and $\pi_2(f)$. Hence $\nu(r_1, r_2)$ has also discontinuity for such values of r_1 and r_2.

4. Two functions $f(z_1, z_2)$ and $g(z_1, z_2)$ having the same polyhedron will have the same maximum term and the rank.

Let us consider the function

$$W(r_1, r_2) = \sum_{m_1, m_2=0}^{\infty} e^{-G_{m_1, m_2}} r_1^{m_1} r_2^{m_2}, \tag{4.1}$$

where G_{m_1, m_2} is the Z-coordinate of the point, whose X and Y coordinates are m_1 and m_2.

The function $W(r_1, r_2)$ is a dominant function for $f(z_1, z_2)$ and has the same maximum term. Also it is the simplest function corresponding to the polyhedron $S(f)$. The ratio

$$R_{m_1}^{m_2} = e^{G_{m_1, m_2} - G_{m_1-1, m_2}}, \quad S_{m_2}^{m_1} = e^{G_{m_1, m_2} - G_{m_1, m_2-1}}$$

and

$$R_{m_1, m_2} = e^{G_{m_1, m_2} - G_{m_1-1, m_2-1}}.$$

of the coefficients in $W(r_1, r_2)$ corresponds to the ratio of a_{m_1, m_2} and a_{m_1-1, m_2}, a_{m_1, m_2} and a_{m_1, m_2-1}, and a_{m_1, m_2} and a_{m_1-1, m_2-1}. We shall call these as rectified ratios. The logarithm of $R_{m_1}^{m_2}$ is equal to the slope of the Newton's polygon obtained by plotting the points (m_1, G_{m_1, m_2}) in a plane parallel to XZ-plane at a distance m_2, and is therefore a non-decreasing function of m_1 tending to infinity. Similarly the logarithm of $S_{m_2}^{m_1}$ and R_{m_1, m_2} are non-decreasing functions of m_1, and m_1 and m_2 tending to infinity.

Suppose for simplicity that $G_{0,0} = 0$. Then we have

$$\mu(\nu_1; r_1, r_2) = \frac{r_1^{\nu_1} r_2^{\nu_2}}{(S_{\nu_2}^{\nu_1} S_{\nu_2-1}^{\nu_1} \dots S_1^{\nu_1})(R_{\nu_1}^0 R_{\nu_1-1}^0 \dots R_1^0)} \tag{4.2}$$

and since

$$\int_{R_i^0}^{R_{i+1}^0} \nu_1(0; x_1) \frac{dx_1}{x_1} = i(\log R_{i+1}^0 - \log R_i^0)$$

and

$$\int_{S_j^{\nu_1}}^{S_{j+1}^{\nu_1}} \nu_2(\nu_1; x_2) \frac{dx_2}{x_2} = j(\log S_{j+1}^{\nu_1} - \log S_j^{\nu_1}),$$

therefore,

$$\log \mu(\nu_1; r_1, r_2) = \int_0^{r_1} \nu_1(0; x_1) \frac{dx_1}{x_1} + \int_0^{r_2} \nu_2(\nu_1; x_2) \frac{dx_2}{x_2}. \tag{4.3}$$

We may also put (4.2) as

$$\mu(\nu_2; r_1, r_2) = \frac{r_1^{\nu_1} r_2^{\nu_2}}{(R_{\nu_1}^{\nu_2} R_{\nu_1-1}^{\nu_2} \dots R_1^{\nu_2})(S_{\nu_2}^0 S_{\nu_2-1}^0 \dots S_1^0)}$$

and hence

$$(4.4)\quad \log \mu(\nu_2; r_1, r_2) = \int_0^{r_2} \nu_2(0, x_2)\frac{dx_2}{x_2} + \int_0^{r_1} \nu_1(\nu_2; x_1)\frac{dx_1}{x_1}.$$

Since $\mu(r_1, r_2)$ is greater of $\mu(\nu_1; r_1, r_2)$ and $\mu(\nu_2; r_1, r_2)$, therefore, $\log \mu(r_1, r_2)$ will be given by either (4.3) or (4.4) or both.

We are now in a position to find a relation between $\mu(r_1, r_2)$ and $\nu_1(r_1, r_2)$, $\nu_2(r_1, r_2)$ and $\nu(r_1, r_2)$. In the first place

$$\mu(r_1, r_2) < M(r_1, r_2).$$

Also it is obvious that $M(r_1, r_2)$ does not exceed the value of the function $W(r_1, r_2)$. Suppose that p_1 and p_2 are integers greater than $\nu_1 = \nu_1(r_1)$ and $\nu_2 = \nu_2(r_2)$ and such that the rectified ratio $R_{p_1}^{p_2-1} > r_1$ and $S_{p_2}^{p_1-1} > r_2$. Then, for $q_1 \geqq p_1$ and $q_2 \geqq p_2$,

$$e^{-G_{q_1, q_2}} r_1^{q_1} r_2^{q_2} = e^{-G_{p_1-1, p_2-1}} r_1^{p_1-1} r_2^{p_2-1}$$

$$\times \frac{r_1^{q_1-p_1+1} r_2^{q_2-p_2+1}}{R_{q_1}^{q_2} R_{q_1-1}^{q_2} \cdots R_{p_1}^{q_2} S_{q_2}^{p_1-1} S_{q_2-1}^{p_1-1} \cdots S_{p_2}^{p_1-1}}$$

$$< \mu(r_1, r_2)\left(\frac{r_1}{R_{p_1}^{q_2}}\right)^{q_1-p_1+1}\left(\frac{r_2}{S_{p_2}^{p_1-1}}\right)^{q_2-p_2+1}$$

Hence

$$W(r_1, r_2) = \sum_{m_1, m_2=0}^{p_1-1, p_2-1} e^{-G_{m_1, m_2}} r_1^{m_1} r_2^{m_2} + \sum_{m_1=p_1, m_2=p_2}^{\infty} e^{-G_{m_1, m_2}} r_1^{m_1} r_2^{m_2}$$

$$+ \sum_{m_1=0}^{p_1-1} \sum_{m_2=p_2}^{\infty} e^{-G_{m_1, m_2}} r_1^{m_1} r_2^{m_2} + \sum_{m_2=0}^{p_2-1} \sum_{m_1=p_1}^{\infty} e^{-G_{m_1, m_2}} r_1^{m_1} r_2^{m_2}$$

$$< \mu(r_1, r_2)\Bigg[p_1 p_2 + \sum_{q_2=p_2}^{\infty} \sum_{q_1=p_1}^{\infty} \left(\frac{r_1}{R_{p_1}^{q_2}}\right)^{q_1-p_1+1}\left(\frac{r_2}{S_{p_2}^{p_1-1}}\right)^{q_2-p_2+1}$$

$$+ p_1 \sum_{q_2=p_2}^{\infty}\left(\frac{r_2}{S_{p_2}^{p_1-1}}\right)^{q_2-p_2+1} + p_2 \sum_{q_1=p_1}^{\infty}\left(\frac{r_1}{R_{p_1}^{p_2-1}}\right)^{q_1-p_1+1}\Bigg].$$

$$\leqq \mu(r_1, r_2)\Bigg[p_1 p_2 + \sum_{q_2=p_2}^{\infty}\left(\frac{r_2}{S_{p_2}^{p_1-1}}\right)^{q_2-p_2+1}\frac{r_1}{R_{p_1}^{q_2}-r_1}$$

$$+ p_1 \sum_{q_2=p_2}^{\infty}\left(\frac{r_2}{S_{p_2}^{p_1-1}}\right)^{q_2-p_2+1} + p_2 \sum_{q_1=p_1}^{\infty}\left(\frac{r_1}{R_{p_1}^{p_2-1}}\right)^{q_1-p_1+1}\Bigg]$$

$$\leqq \mu(r_1, r_2)\Bigg[p_1 p_2 + \frac{r_1}{R_{p_1}^{p_2-1}-r_1}\sum_{q_2=p_2}^{\infty}\left(\frac{r_2}{S_{p_2}^{p_1-1}}\right)^{q_2-p_2+1}$$

$$(4.5)\qquad + \frac{p_1 r_2}{S_{p_2}^{p_1-1}-r_2} + \frac{p_2 r_1}{R_{p_1}^{p_2-1}-r_1}\Bigg]$$

$$\leqq \mu(r_1, r_2)\Bigg[p_1 p_2 + \frac{r_1}{R_{p_1}^{p_2-1}-r_1}\cdot\frac{r_2}{S_{p_2}^{p_1-1}-r_2}$$

$$+ \frac{p_1 r_2}{S_{p_2}^{p_1-1}-r_2} + \frac{p_2 r_1}{R_{p_1}^{p_2-1}-r_1}\Bigg].$$

In order that the terms in the bracket may be substantially equivalent, we choose

$$p_1 = \nu_1\left(r_1 + \frac{r_1}{\nu_1(r_1, r_2)}, r_2 + \frac{r_2}{\nu_2(r_1, r_2)}\right) + 1$$

and

$$p_2 = \nu_2\left(r_1 + \frac{r_1}{\nu_1(r_1, r_2)}, r_2 + \frac{r_2}{\nu_2(r_1, r_2)}\right) + 1,$$

which implies that

$$R_{p_1}^{p_2-1} > r_1 + \frac{r_1}{\nu_1(r_1, r_2)}$$

and

$$S_{p_2}^{p_1-1} > r_2 + \frac{r_2}{\nu_2(r_1, r_2)}.$$

Hence we have the following result:

$$\begin{aligned} \mu(r_1, r_2) &< M(r_1, r_2) \\ &< \mu(r_1, r_2)\left[3\nu_1\left(r_1 + \frac{r_1}{\nu_1(r_1, r_2)}, r_2 + \frac{r_2}{\nu_2(r_1, r_2)}\right)\right. \\ &\qquad \left. \times \nu_2\left(r_1 + \frac{r_1}{\nu_1(r_1, r_2)}, r_2 + \frac{r_2}{\nu_2(r_1, r_2)}\right) + 3\right] \\ &\leqq \mu(r_1, r_2)\left[3\nu\left(r_1 + \frac{r_1}{\nu_1(r_1, r_2)}, r_2 + \frac{r_2}{\nu_2(r_1, r_2)}\right) + 3\right]. \end{aligned} \tag{4.6}$$

5. We shall now define *order*. Here we shall restrict to integral functions of *finite order*.

Let

$$\overline{Lt}_{r_2\to\infty}\ \overline{Lt}_{r_1\to\infty} \frac{\log \nu_1(r_1, r_2)}{\log r_1} = \rho_1, \quad \overline{Lt}_{r_1\to\infty}\ \overline{Lt}_{r_2\to\infty} \frac{\log \nu_2(r_1, r_2)}{\log r_2} = \rho_2,$$

and let

$$\rho = \max.\ (\rho_1, \rho_2), \quad 0 \leqq \rho < \infty,$$

then

$$\overline{Lt}_{r_1, r_2\to\infty} \frac{\log \nu(r_1, r_2)}{\log (r_1 r_2)} = \rho, \tag{5.1}$$

and $f(z_1, z_2)$ is said to be an integral function of *finite order* ρ. Further, $\rho_1(r_2)$ and $\rho_2(r_1)$ be defined as *proximate orders*.

Now, for those functions which satisfy (5.1), the relation (4.6) appears in an especially simple form, and it may be written

$$\log M(r_1, r_2)$$
$$= \left[1 + \theta \frac{\log\left\{3\nu\left(r_1 + \frac{r_1}{\nu_1(r_1, r_2)}, r_2 + \frac{r_2}{\nu_2(r_1, r_2)}\right) + 3\right\}}{\log \mu(r_1, r_2)}\right] \log \mu(r_1, r_2),$$
$$0 < \theta < 1.$$

Hence

$$\underset{r_1, r_2 \to \infty}{Lt} \frac{\log M(r_1, r_2)}{\log \mu(r_1, r_2)} = 1.$$

We may also define order as

$$\text{(5.2)} \qquad \underset{r_1, r_2 \to \infty}{\overline{Lt}} \frac{\log\log M(r_1, r_2)}{\log(r_1 r_2)} = \rho.$$

We will now prove the equivalence of the two definitions. Let us start with the definition (5.1). From (4.3) and (4.4) and the definitions of ρ_1, ρ_2 and ρ, we have

$$\log \mu(r_1, r_2) < \log \mu(r_{\varepsilon_1}, r_{\varepsilon_2}) + k_1 r_1^{\rho+\varepsilon} \int_{r_{\varepsilon_2}}^{r_2} x_2^{\rho-1+\varepsilon}\, dx_2,$$

or

$$\log \mu(r_1, r_2) < \log \mu(r_{\varepsilon_1}, r_{\varepsilon_2}) + k_2 r_2^{\rho+\varepsilon} \int_{r_{\varepsilon_1}}^{r_1} x_1^{\rho-1+\varepsilon}\, dx_1,$$

where $\varepsilon = \max.(\varepsilon_1, \varepsilon_2)$ and $k_1 > 1$, $k_2 > 1$, and so

$$\log \mu(r_1, r_2) < \frac{k}{\rho + \varepsilon}(r_1 r_2)^{\rho+\varepsilon} + K, \quad k = \max(k_1, k_2),$$

whence, in virtue of (4.6)

$$\text{(5.3)} \quad \underset{r_1, r_2 \to \infty}{\overline{Lt}} \frac{\log\log M(r_1, r_2)}{\log(r_1 r_2)} = \underset{r_1, r_2 \to \infty}{\overline{Lt}} \frac{\log\log \mu(r_1, r_2)}{\log(r_1 r_2)} \leqq \rho.$$

Next, if we suppose that

$$\log M(r_1, r_2) < (r_1 r_2)^{\rho+\varepsilon}, \quad \text{for} \quad r_1 > r_\varepsilon, r_2 > r_\varepsilon,$$

then

$$\{\nu_2(0; r_2) + \nu_1(\nu_2; r_1, r_2)\} \log 2 < \int_{r_2}^{2r_2} \frac{\nu_2(0; x_2)}{x_2}\, dx_2 + \int_{r_1}^{2r_1} \frac{\nu_1(\nu_2; x_1)}{x_1}\, dx_1,$$
$$\leqq \log \mu(2r_1, 2r_2)$$
$$< (r_1 r_2)^{\rho+\varepsilon},$$

or

$$\{\nu_1(0; r_1) + \nu_2(\nu_1; r_1, r_2)\} \log 2 < \log \mu(2r_1, 2r_2) < (r_1 r_2)^{\rho+\varepsilon}.$$

Hence

$$\overline{Lt}_{r_1, r_2 \to \infty} \frac{\log \nu(r_1, r_2)}{\log(r_1 r_2)} \leqq \overline{Lt}_{r_1, r_2 \to \infty} \frac{\log \log \mu(r_1, r_2)}{\log (r_1 r_2)}$$

$$(5.4) \qquad = \overline{Lt}_{r_1, r_2 \to \infty} \frac{\log \log M(r_1, r_2)}{\log (r_1 r_2)}.$$

Thus, from (5.3) and (5.4), the equivalence of the two definitions follows.

6. We shall now come to the necessary and sufficient condition for an integral function $f(z_1, z_2)$ to be of finite order ρ. The result is as follows:

THEOREM IV. *The integral function*

$$f(z_1, z_2) = \sum_{m_1, m_2 = 0}^{\infty} a_{m_1, m_2} z_1^{m_1} z_2^{m_2}$$

is of finite order, if and only if

$$(6.1) \qquad \mu = \overline{Lt}_{m_1, m_2 \to \infty} \frac{\log(m_1^{m_1} m_2^{m_2})}{\log(1/|a_{m_1, m_2}|)}$$

is finite; and then the order ρ of $f(z_1, z_2)$ is equal to μ.

PROOF: We first prove that $\rho \geqq \mu$. We may note that in case $\mu = \infty$ the above statement is to be interpreted as meaning that the order is infinite, or else $f(z_1, z_2)$ is not an integral function.

We know that

$$|a_{m_1, m_2}| = \frac{1}{m_1!\, m_2!} \left| \frac{\partial^{m_1+m_2} f}{\partial z_1^{m_1} \partial z_2^{m_2}} \right|$$

$$(6.2) \qquad = \left| \left(\frac{1}{2\pi i}\right)^2 \int_{|z_1|=r_1} \int_{|z_2|=r_2} \frac{f(z_1, z_2)}{z_1^{m_1+1} z_2^{m_2+1}} dz_1 dz_2 \right|$$

$$\leqq \frac{M(r_1, r_2)}{r_1^{m_1} r_2^{m_2}}.$$

(i) If $\mu = 0$, $\rho \geqq \mu$, since ρ is not negative. Let us suppose that $0 < \varepsilon < \mu < \infty$. Then, from (6.1), we have

$$(\mu - \varepsilon) \log (1/|a_{m_1, m_2}|) \leqq \log (m_1^{m_1} m_2^{m_2})$$

i.e.,

$$(6.3) \qquad \log |a_{m_1, m_2}| \geqq -(\mu - \varepsilon)^{-1}(m_1 \log m_1 + m_2 \log m_2),$$

for an infinite sequence of values of m_1 and m_2.

Also (6.2) may be written as

$$\begin{aligned}\log M(r_1, r_2) &\geqq \log |a_{m_1,m_2}| + \log(r_1^{m_1} r_2^{m_2}) \\ &\geqq -(\mu-\varepsilon)^{-1}(m_1 \log m_1 + m_2 \log m_2) \\ &\qquad\qquad + m_1 \log r_1 + m_2 \log r_2 \\ &= m_1\left(\log r_1 - \frac{1}{\mu-\varepsilon}\log m_1\right) + m_2\left(\log r_2 - \frac{1}{\mu-\varepsilon}\log m_2\right).\end{aligned}$$

Let $r_1 = (em_1)^{1/\mu-\varepsilon}$ and $r_2 = (em_2)^{1/\mu-\varepsilon}$. Then

$$\log M(r_1, r_2) \geqq \frac{m_1}{\mu-\varepsilon} + \frac{m_2}{\mu-\varepsilon} = \frac{r_1^{\mu-\varepsilon} + r_2^{\mu-\varepsilon}}{e(\mu-\varepsilon)}.$$

Since $\mu-\varepsilon$ is independent of r_1 and r_2, therefore,

$$\rho = \overline{Lt}_{r_1, r_2 \to \infty} \frac{\log \log M(r_1, r_2)}{\log (r_1 r_2)} \geqq \mu - \varepsilon.$$

Further, ε is arbitrary and so $\rho \geqq \mu$.

(ii) Next we prove that $\rho \leqq \mu$. We note that if $\mu = \infty$, the result is obvious. So we suppose that $\mu < \infty$. Let $\varepsilon > 0$. Then from (6.1) follows

$$0 \leqq \frac{\log (m_1^{m_1} m_2^{m_2})}{\log(1/|a_{m_1,m_2}|)} \leqq \mu + \varepsilon,$$

for $m_1 > m_1^0$ and $m_2 > m_2^0$,

i.e.

$$\log m_1^{-\frac{m_1}{\mu+\varepsilon}} + \log m_2^{-\frac{m_2}{\mu+\varepsilon}} \geqq \log |a_{m_1,m_2}|,$$

or

$$|a_{m_1,m_2}| \leqq m_1^{-\frac{m_1}{\mu+\varepsilon}} m_2^{-\frac{m_2}{\mu+\varepsilon}}.$$

Hence

$$\begin{aligned}M(r_1, r_2) &< A r_1^{m_1^0} r_2^{m_2^0} + \sum_{m_1=m_1^0+1,\, m_2=m_2^0+1}^{\infty} |a_{m_1,m_2}| r_1^{m_1} r_2^{m_2} \\ &\quad + \sum_{m_1=0}^{m_1^0} \sum_{m_2=m_2^0+1}^{\infty} |a_{m_1,m_2}| r_1^{m_1} r_2^{m_2} + \sum_{m_2=0}^{m_2^0} \sum_{m_1=m_1^0+1}^{\infty} |a_{m_1,m_2}| r_1^{m_1} r_2^{m_2}\end{aligned}$$

$$\begin{aligned}(6.4) \qquad &\leqq A r_1^{m_1^0} r_2^{m_2^0} + \sum_{m_1, m_2}^{\infty} m_1^{-\frac{m_1}{\mu+\varepsilon}} m_2^{-\frac{m_2}{\mu+\varepsilon}} r_1^{m_1} r_2^{m_2} \\ &\quad + B r_1^{m_1^0} \sum_{m_2=m_2^0+1}^{\infty} m_2^{-\frac{m_2}{\mu+\varepsilon}} r_2^{m_2} + C r_2^{m_2^0} \sum_{m_1=m_1^0+1}^{\infty} m_1^{-\frac{m_1}{\mu+\varepsilon}} r_1^{m_1}.\end{aligned}$$

Let Σ_1 be the part of the double series in (6.4) for which $m_1 < (2r_1)^{\mu+\varepsilon} < m_2 < (2r_2)^{\mu+\varepsilon}$. We estimate Σ_1 by taking the largest value of $r_1^{m_1} r_2^{m_2}$. Then

$$\begin{aligned} \Sigma_1 &= \sum_{m_1<(2r_1)^{\mu+\varepsilon}} \sum_{m_2<(2r_2)^{\mu+\varepsilon}} m_1^{-\frac{m_1}{\mu+\varepsilon}} m_2^{-\frac{m_2}{\mu+\varepsilon}} r_1^{m_1} r_2^{m_2} \\ &\leqq r_1^{(2r_1)^{\mu+\varepsilon}} r_2^{(2r_2)^{\mu+\varepsilon}} \sum_{m_1, m_2} m_1^{-\frac{m_1}{\mu+\varepsilon}} m_2^{-\frac{m_2}{\mu+\varepsilon}} \qquad (6.5) \\ &= 0\left(e^{(2r_1)^{\mu+2\varepsilon}+(2r_2)^{\mu+2\varepsilon}}\right), \end{aligned}$$

since the series in (6.5) is convergent and is independent of r_1 and r_2.

Let Σ_2 contain the terms for which $m_1 \geqq (2r_1)^{\mu+\varepsilon}$ and $m_2 \geqq (2r_2)^{\mu+\varepsilon}$ and so in Σ_2, we have $r_1 m_1^{-1/\mu+\varepsilon} \leqq \frac{1}{2}$ and $r_2 m_2^{-1/\mu+\varepsilon} \leqq \frac{1}{2}$, and hence

$$\begin{aligned} \Sigma_2 &= \sum_{m_1\geqq(2r_1)^{\mu+\varepsilon}} \sum_{m_2\geqq(2r_2)^{\mu+\varepsilon}} m_1^{-\frac{m_1}{\mu+\varepsilon}} m_2^{-\frac{m_2}{\mu+\varepsilon}} r_1^{m_1} r_2^{m_2} \\ &\leqq \sum_{m_1, m_2} (\tfrac{1}{2})^{m_1} (\tfrac{1}{2})^{m_2} \leqq 1. \end{aligned}$$

Let Σ_3 be the part of the series for which $m_1 < (2r_1)^{\mu+\varepsilon}$ and $m_2 \geqq (2r_2)^{\mu+\varepsilon}$, then

$$\Sigma_3 = \sum_{m_1<(2r_1)^{\mu+\varepsilon}} m_1^{-\frac{m_1}{\mu+\varepsilon}} r_1^{m_1} \sum_{m_2\geqq(2r_2)^{\mu+\varepsilon}} m_2^{-\frac{m_2}{\mu+\varepsilon}} r_2^{m_2}.$$

Since

$$\sum_{m_2\geqq(2r_2)^{\mu+\varepsilon}} m_2^{-\frac{m_2}{\mu+\varepsilon}} r_2^{m_2} \leqq 1$$

and

$$\sum_{m_1<(2r_1)^{\mu+\varepsilon}} m_1^{-\frac{m_1}{\mu+\varepsilon}} r_1^{m_1} \leqq 0\{e^{(2r_1)^{\mu+2\varepsilon}}\},$$

therefore,

$$\Sigma_3 \leqq 0\{e^{(2r_1)^{\mu+2\varepsilon}}\}.$$

Let the remaining part of the double series in (6.4) be denoted by Σ_4 i.e. for $m_1 \geqq (2r_1)^{\mu+\varepsilon}$ and $m_2 < (2r_2)^{\mu+\varepsilon}$, then

$$\Sigma_4 \leqq 0\{e^{(2r_2)^{\mu+2\varepsilon}}\}.$$

Further,

$$Br_1^{m_1^0} \sum_{m_2} m_2^{-\frac{m_2}{\mu+\varepsilon}} r_2^{m_2} \leqq 0\{e^{(2r_2)^{\mu+2\varepsilon}}\}$$

$$Cr_2^{m_2^0} \sum_{m_1} m_1^{-\frac{m_1}{\mu+\varepsilon}} r_1^{m_1} \leqq 0\{e^{(2r_1)^{\mu+2\varepsilon}}\}.$$

Hence, substituting these values in (6.4), we get

$$M(r_1, r_2) < \Sigma_1 + \Sigma_2 + \Sigma_3 + \Sigma_4 + 0\{e^{(2r_1)^{\mu+2\varepsilon}}\} + 0\{e^{(2r_2)^{\mu+2\varepsilon}}\}$$
$$\leqq 0\{e^{(2r_1)^{\mu+2\varepsilon} + (2r_2)^{\mu+2\varepsilon}}\}$$
$$\leqq 0\{e^{(4r_1r_2)^{\mu+2\varepsilon}}\},$$

and hence

$$(6.6) \qquad \overline{Lt}_{r_1, r_2 \to \infty} \frac{\log \log M(r_1, r_2)}{\log (r_1 r_2)} \leqq \mu,$$

since ε is arbitrary and independent of r_1 and r_2.

7. Suppose $0 < \rho < \infty$ and let us define

$$(7.1) \qquad \alpha = \overline{Lt}_{m_1, m_2 \to \infty} \{m_1^{m_1} m_2^{m_2} |a_{m_1, m_2}|^{\rho}\}^{\frac{1}{m_1+m_2}}$$

and

$$\tau = \overline{Lt}_{r_1, r_2 \to \infty} \frac{\log M(r_1, r_2)}{r_1^{\rho} + r_2^{\rho}}.$$

The functions which satisfy the latter equality are said to be functions of exponential type τ.

If $\alpha = e\tau\rho$ and using the Sterling's formula

$$m_1! \sim m_1^{m_1} e^{-m_1} (2\pi m_1)^{\frac{1}{2}} e^{\delta/(12m_1)},$$

(7.1) takes the form

$$\tau\rho = \overline{Lt}_{m_1, m_2 \to \infty} \frac{1}{e} \{m_1^{m_1} m_2^{m_2} |a_{m_1, m_2}|^{\rho}\}^{\frac{1}{m_1+m_2}}$$

$$(7.2) \qquad = \overline{Lt}_{m_1, m_2 \to \infty} \frac{1}{e} \left\{ (m_1^{m_1} m_2^{m_2})^{\frac{1}{m_1+m_2}} \left(\frac{1}{m_1!\, m_2!} \left| \frac{\partial^{m_1+m_2}}{\partial z_1^{m_1} \partial z_2^{m_2}} f(z_1, z_2) \right| \right)^{\frac{\rho}{m_1+m_2}} \right\},$$
$$= \overline{Lt}_{m_1, m_2 \to \infty} \left\{ \left(\frac{m_1^{m_1} m_2^{m_2}}{e^{m_1+m_2}} \right)^{\frac{1-\rho}{m_1+m_2}} \left| \frac{\partial^{m_1+m_2}}{\partial z_1^{m_1} \partial z_2^{m_2}} f(z_1, z_2) \right|^{\frac{\rho}{m_1+m_2}} \right\},$$

where z_1, z_2 are any (fixed) complex numbers.

If in this we put $\rho = 1$, then

$$\tau = \overline{Lt}_{m_1, m_2 \to \infty} \left\{ \left| \frac{\partial^{m_1+m_2}}{\partial z_1^{m_1} \partial z_2^{m_2}} f(z_1, z_2) \right|^{\frac{1}{m_1+m_2}} \right\}.$$

We now deduce the following result:

Theorem V. *If* $0 < \alpha < \infty$, *the function*

$$f(z_1, z_2) = \sum_{m_1, m_2 = 0}^{\infty} a_{m_1, m_2} z_1^{m_1} z_2^{m_2}$$

is of order ρ and type τ, if and only if $\alpha = e\tau\rho$.

PROOF: From (7.1), we have

$$|a_{m_1, m_2}| \leqq (\alpha+\varepsilon)^{\frac{m_1+m_2}{\rho}} m_1^{-\frac{m_1}{\rho}} m_2^{-\frac{m_2}{\rho}}, \tag{7.3}$$

for $\varepsilon > 0$ and m_1, m_2 large.

We shall first prove that $\tau \leqq \alpha/e\rho$. Since we may add a polynomial to $f(z_1, z_2)$ without affecting its type, we may suppose that (7.3) holds for all m_1 and m_2, interpreting its right hand side as 1 for $m_1 = 0, m_2 = 0$. Then

$$|f(z_1, z_2)| \leqq \sum_{m_1, m_2=0}^{\infty} |a_{m_1, m_2}| r_1^{m_1} r_2^{m_2}$$
$$\leqq \sum_{m_1, m_2=0}^{\infty} (\alpha+\varepsilon)^{\frac{m_1+m_2}{\rho}} m_1^{-\frac{m_1}{\rho}} m_2^{-\frac{m_2}{\rho}} r_1^{m_1} r_2^{m_2}.$$

The general term of the right hand side does not exceed its maximum.

Let

$$\phi(m_1, m_2) = \frac{m_1+m_2}{\rho} \log(\alpha+\varepsilon) - \frac{m_1}{\rho} \log m_1 + m_1 \log r_1$$
$$- \frac{m_2}{\rho} \log m_2 + m_2 \log r_2.$$

Then for ϕ to be maximum,

$$\frac{\partial \phi}{\partial m_1} = 0 \quad \text{and} \quad \frac{\partial \phi}{\partial m_2} = 0,$$

i.e.

$$m_1 = \frac{(\alpha+\varepsilon) r_1^\rho}{e} \quad \text{and} \quad m_2 = \frac{(\alpha+\varepsilon) r_2^\rho}{e}.$$

Therefore,

$$\text{max. term} = \exp\left\{\frac{m_1+m_2}{\rho} \log(\alpha+\varepsilon) - \frac{m_1}{\rho} \log m_1 \right.$$
$$\left. - \frac{m_2}{\rho} \log m_2 + m_1 \log r_1 + m_2 \log r_2 \right\}$$
$$= \exp\left\{\frac{\alpha+\varepsilon}{e\rho} (r_1^\rho + r_2^\rho)\right\}.$$

Thus the maximum term is $\exp\{(r_1^\rho + r_2^\rho)(\alpha+\varepsilon)/e\rho\}$, attained for

$m_1 = r_1^\rho(\alpha+\varepsilon)/e$ and $m_2 = r_2^\rho(\alpha+\varepsilon)/e$. Let Σ_1 denote the part of the series for which $m_1 \leqq (\alpha+2\varepsilon)r_1^\rho$ and $m_2 \leqq (\alpha+2\varepsilon)r_2^\rho$. Then

$$\Sigma_1 \leqq (\alpha+2\varepsilon)^2(r_1r_2)^\rho \exp\left\{\frac{\alpha+\varepsilon}{\rho e}(r_1^\rho+r_2^\rho)\right\}$$
$$= 0\left[\exp\left\{\frac{\alpha+\varepsilon}{e\rho}(r_1^\rho+r_2^\rho)\right\}\right].$$

Let Σ_2 denote the part of the series for which $r_1^\rho < m_1/(\alpha+2\varepsilon)$ and $r_2^\rho < m_2/(\alpha+2\varepsilon)$. Then

$$\Sigma_2 < \sum_{m_1>(\alpha+2\varepsilon)r_1{}^\rho}\ \sum_{m_2>(\alpha+2\varepsilon)r_2{}^\rho}(\alpha+\varepsilon)^{\frac{m_1+m_2}{\rho}}m_1^{-\frac{m_1}{\rho}}m_2^{-\frac{m_2}{\rho}}$$
$$\times\left(\frac{m_1}{\alpha+2\varepsilon}\right)^{\frac{m_1}{\rho}}\left(\frac{m_2}{\alpha+2\varepsilon}\right)^{\frac{m_2}{\rho}}$$
$$= \sum_{m_1>(\alpha+2\varepsilon)r_1{}^\rho}\left(\frac{\alpha+\varepsilon}{\alpha+2\varepsilon}\right)^{\frac{m_1}{\rho}}\sum_{m_2>(\alpha+2\varepsilon)r_2{}^\rho}\left(\frac{\alpha+\varepsilon}{\alpha+2\varepsilon}\right)^{\frac{m_2}{\rho}}$$
$$= 0(1).$$

Let Σ_3 denote the part of the series for which $m_1 \leqq (\alpha+2\varepsilon)r_1^\rho$ and $m_2 > (\alpha+2\varepsilon)r_2^\rho$. Then

$$\Sigma_3 = \sum_{m_1\leqq(\alpha+2\varepsilon)r_1{}^\rho}(\alpha+\varepsilon)^{\frac{m_1}{\rho}}m_1^{-\frac{m_1}{\rho}}r_1^{m_1}\sum_{m_2>(\alpha+2\varepsilon)r_2{}^\rho}(\alpha+\varepsilon)^{\frac{m_2}{\rho}}m_2^{-\frac{m_2}{\rho}}r_2^{m_2}$$
$$\leqq K\sum_{m_1\leqq(\alpha+2\varepsilon)r_1{}^\rho}(\alpha+\varepsilon)^{\frac{m_1}{\rho}}m_1^{-\frac{m_1}{\rho}}r_1^{m_1},$$

since

$$\sum_{m_2>(\alpha+2\varepsilon)r_2{}^\rho} = 0(1).$$

Therefore

$$\Sigma_3 \leqq 0\{e^{\frac{\alpha+\varepsilon}{e\rho}r_1{}^\rho}\}.$$

Let the remaining part of the series be denoted by Σ_4, i.e. for $m_1 > (\alpha+2\varepsilon)r_1^\rho$ and $m_2 \leqq (\alpha+2\varepsilon)r_2^\rho$. Then, as in Σ_3, we obtain

$$\Sigma_4 \leqq 0\{e^{\frac{\alpha+\varepsilon}{e\rho}r_2{}^\rho}\}.$$

Thus

$$|f(z_1, z_2)| \leqq \Sigma_1+\Sigma_2+\Sigma_3+\Sigma_4$$
$$\leqq 0\{e^{\frac{\alpha+\varepsilon}{e\rho}(r_1{}^\rho+r_2{}^\rho)}\}.$$

Hence

$$\tau \leqq \frac{\alpha}{e\rho},$$

since $f(z_1, z_2)$ is of exponential type τ.

Next, to show that $\tau \geqq \alpha/e\rho$, we have again from (7.1), for an infinite sequence of values of m_1 and m_2,

$$|a_{m_1,m_2}| \geqq (\alpha-\varepsilon)^{\frac{m_1+m_2}{\rho}} m_1^{-\frac{m_1}{\rho}} m_2^{-\frac{m_2}{\rho}}, \; 0 < \varepsilon < \alpha.$$

If we take r_1 and r_2 such that

$$r_1^\rho = \frac{m_1 e}{\alpha-\varepsilon} \quad \text{and} \quad r_2^\rho = \frac{m_2 e}{\alpha-\varepsilon},$$

for these values of m_1 and m_2, we have

$$M(r_1, r_2) \geqq |a_{m_1,m_2}| r_1^{m_1} r_2^{m_2} \geqq \exp\left\{\frac{\alpha-\varepsilon}{e\rho}(r_1^\rho+r_2^\rho)\right\}$$

for a sequence of values of r_1 and r_2, tending to infinity. Hence

$$\tau \geqq \frac{\alpha}{e\rho}.$$

It remains now to show that $f(z_1, z_2)$ is of order ρ at most if $\alpha < \infty$, and is of order ρ at least if $\alpha > 0$.

For large m_1, m_2, we have from (7.1)

$$\alpha+\varepsilon \geqq \{m_1^{m_1} m_2^{m_2} |a_{m_1,m_2}|^\rho\}^{\frac{1}{m_1+m_2}}, \quad \varepsilon > 0,$$

and hence

$$\frac{\log(m_1^{m_1} m_2^{m_2})}{\log(1/|a_{m_1,m_2}|)} \leqq \frac{\rho}{1-\dfrac{(m_1+m_2)\log(\alpha+\varepsilon)}{m_1\log m_1+m_2\log m_2}}.$$

By (6.1), the order of $f(z_1, z_2)$ is ρ at most. Similarly if $\alpha > 0$ the order of $f(z_1, z_2)$ is at least ρ.

REFERENCE

G. Valiron,
[1] *Theory of Integral Functions* (1923).

(Oblatum 8-6-60).