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Spectral Representations for Solutions of Certain Abstract Functional Equations*)

by

George Maltese

Introduction

The only non-constant, continuous, complex-valued functions which are solutions of the functional equation

$$F(s+t)+F(s-t) = 2F(s)F(t) \qquad s, t \text{ real} \tag{1}$$

are $\cos \lambda s$, where λ is an arbitrary complex number (see A. Cauchy [4] p. 98—105 for the case of real-valued solutions). The functional equation (1) and the system of functional equations

$$\text{(S)} \qquad \begin{aligned} G(s+t) &= G(s)G(t)-H(s)H(t) \\ H(s+t) &= G(s)H(t)+H(s)G(t) \end{aligned} \qquad s, t \text{ real}$$

are the starting points of our investigation. In this paper we shall study a generalized form of equation (1) as well as a generalized form of the system (S) conveniently expressed in terms of generalized convolution algebras. We shall be chiefly concerned with abstract solutions $F(s)$, $G(s)$, $H(s)$ where for each s, $F(s)$, $G(s)$, $H(s)$ are (bounded) normal operators on a Hilbert space. Under certain weak continuity hypothesis we shall obtain spectral representations of the abstract solutions of (1) and (S) and of the generalized forms of (1) and (S). For the equation (1) in particular when s, t belong to a locally compact Abelian group and $F(s)$ is a normal operator, we shall also study various relationships between measurability and continuity of the operator solutions (for the solutions of equation (1) we shall give more details concerning the spectral representation and concentration of the spectral measures).

In a recent paper published in Canadian Journal, S. Kurepa [21] has obtained, under certain supplementary conditions, the form of the solutions of (1) when s, t are real numbers and $F(s)$, $F(t)$ are normal operators on a separable Hilbert space. In particular our results extend, in various directions, the results of S. Kurepa. We wish to remark that the methods used here to

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obtain the spectral representations are suggested by those used by C. Ionescu Tulcea ([13], [15]; see also R. Phillips [29], A. Nussbaum [26], [27]), and are completely different from those of S. Kurepa. In fact even in the particular case when our generalized convolution algebra is the group algebra of a locally compact Abelian group, and when the functional equation and the system of functional equations have exactly the form (1) or (S), it does not seem possible to obtain our results using S. Kurepa's methods. The prototype for studies of this nature is the semi-group theory of E. Hille [11], [12], who studied the exponential functional equation (see also N. Dunford-E. Hille [8], J. Lee [22], M. Nagumo [24], D. Nathan [25] and K. Yosida [35]). Our theorems concerning relationships between measurability and continuity are also suggested by those in E. Hille-R. Phillips [12].

This paper is divided into three parts. The first part is introductory and contains various general definitions and results concerning generalized convolution algebras and spectral families. The main portion of the paper consists of Part II and Part III. In Part II we shall study a generalization of the functional equation (1). In Part III we study a generalization of the system (S).

Many of the theorems of this paper may be stated and proved also for the case of unbounded normal operators, but this extension will not be considered here.

PART 1

Preliminaries

1. Notation. We shall denote below by Z a locally compact space, by $K(Z)$ the vector space of continuous complex-valued functions f defined on Z and having compact support $S(f)$ and for each compact set $A \subset Z$, by $K(Z, A)$ the vector space of all $f \in K(Z)$ such that $S(f) \subset A$. We shall denote by $M(Z)$ the vector space of all complex Radon measures μ on Z having compact support $S(\mu)$ endowed with the norm

$$\mu \to ||\mu|| = \sup \{|\mu(f)| : f \in K(Z),\ ||f|| \leqq 1\}.$$

2. Generalized convolution algebras. Let Z be as above and for each $z \in Z$ let m_z be a real Radon measure on $Z \times Z$. For every $f, g \in K(Z)$ denote by $f*g$ the function [1]) $z \to m_z(f \otimes g)$. Suppose that:

[1]) The function $f \otimes g$ is defined on $Z \times Z$ by the equations: $f \otimes g((x, y)) = f(x)g(y)$ for all $(x, y) \in Z \times Z$.

(AI) For each compact set $A \subset Z$ there is a compact set $K_A \subset Z$ such that $f, g \in K(Z, A)$ implies $f^*g \in K(Z, K_A)$.

(AII) The multiplication $(f, g) \to f^*g$ defines on the vector space $K(Z)$ the structure of a commutative algebra.

(AIII) There is a Radon measure m defined on Z which is strictly positive on nonvoid open sets, such that $m(|f^*g|) \leqq m(|f|)m(|g|)$ for all $f, g \in K(Z)$.

(AIV) There is an involutive homeomorphism $z \to z^*$ of Z onto Z such that [2]) $m(\tilde{f}) = m(\bar{f})$ and $m((g^*\tilde{h})k) = m((k^*\tilde{h})g)$ for every $f, g, h, k \in K(Z)$.

(AV) There is $e \in Z$ such that $e^* = e$ and $m_e(f \otimes \tilde{g}) = m(f\tilde{g})$ for any $f, g \in K(Z)$.

The mappings $(f, g) \to f^*g$ and $f \to \tilde{f}$ extended by continuity to $L^1(Z, m)$, give $L^1(Z, m)$ the structure of a commutative Banach algebra with involution; we shall denote by $f \to ||f||_1$ the norm in $L^1(Z, m)$. Such an algebra $L^1(Z, m)$ will be called a generalized convolution algebra; sometimes to make our notations more precise, we shall say the generalized convolution algebra $\{Z, m_z, m, (f, g) \to f^*g\}$ instead of the generalized convolution algebra $L^1(Z, m)$.

Consider now a generalized convolution algebra $\{Z, m_z, m, (f, g) \to f^*g\}$. Define L to be the vector subspace of $M(Z)$ which consists of those measures ν having the form [3]) $\nu = g \cdot m$ where g is an m-integrable function with compact support. For any measures $\nu, \mu \in L$, $\nu = g \cdot m$ and $\mu = h \cdot m$ define $\nu\mu = (g^*h) \cdot m$; when endowed with this multiplication L may be obviously identified with a subalgebra of $L^1(Z, m)$. For every $\mu = h \cdot m \in L$ and $z \in Z$ let μ_z denote the mapping [4]) $f \to \mu_z(f) = \check{h}^*f(z)$. In the sequel we shall always suppose that the following condition is also satisfied:

(AVI) For given $\mu = h \cdot m \in L$ and compact set $A \subset Z$ the mappings $z \to \mu_z(g)$ with $g \in K(Z, A)$, $||g||_\infty \leqq 1$ are equicontinuous.

If we consider on L the topology defined by the norm $\mu \to ||\mu||$, then condition (AVI) means that $z \to \mu_z$ is a continuous mapping of Z into L.

[2]) The function $\tilde{f}$ is defined by the equations: $\tilde{f}(z) = \overline{f(z^*)}$ for all $z \in Z$.

[3]) The measure $\nu = g \cdot m$ is defined by the equations $\nu(f) = \int fg \, dm$ for $f \in K(Z)$.

[4]) The function $\check{f}$ is defined by the equations: $\check{f}(z) = f(z^*)$ for all $z \in Z$. Similarly for $\nu = f \cdot m \in L$, we define a measure $\check{\nu}$ by the equation $\check{\nu} = \check{f} \cdot m$.

Let us also state explicitly the following two results which we shall need below and which are valid in our generalized convolution algebra:

I) *If f is a complex-valued, m-measurable function which is bounded (almost everywhere) on each compact set $K \subset Z$, and if $\int f(z)d\mu(z) = 0$ for all $\mu \in L$, then $f(z) = 0$ locally almost everywhere; if f is continuous then the conclusion is that $f(z) = 0$ everywhere.*

II) *If f is a continuous complex-valued function on Z, then for every μ, $\nu \in L$ (we shall usually write $\int$ instead of $\int_Z$)*

$$\int f(z)d\nu\mu(z) = \int d\nu(z) \int f(s)d\mu_z(s).$$

Remarks. 1° Concerning this section see C. Ionescu Tulcea and A. Simon [17], [18] and Yu. Berezanski and S. Krein [1] (see also the reviews by R. Godement, Math. Review 12 (1951), p. 188—189). The reader is especially referred to the paper [17] for a brief survey of important properties of generalized convolution algebras. 2° For various examples of generalized convolution algebras see, for instance, Yu. Berezanski and S. Krein [1] (see especially example 4) and A. Povzner [31]. If Z is an Abelian group and m the Haar measure on Z, then $L^1(Z, m)$, for the usual convolution, is obviously a generalized convolution algebra. The Radon measures m_z, $z \in Z$ are defined in this case by the equations $m_z(f) = \int f(t, t^{-1}z)dm(t)$ for all $f \in K(Z \times Z)$.

3. A topology for L. Let $K \subset Z$ be a compact set and define $L(Z, K) \subset L$ to be the set of $\mu \in L$ such that $S(\mu) \subset K$: for the norm $\mu \to ||\mu||$ $L(Z, K)$ is a Banach space. On L which is the union of the directed family $L(Z, K)$, let us consider the inductive limit topology, (see N. Bourbaki [3], chap. I, p. 61), of the topologies of the subspaces $L(Z, K)$; L is a barelled space (espace tonnelé) for this topology (N. Bourbaki [3], chap. III, p. 2). Suppose that x' is a continuous linear form on L. By the properties of the inductive limit topology, x' is continuous on L if and only if the restriction of x' (which we shall also denote by x') to each $L(Z, K)$ is continuous on $L(Z, K)$. Using this remark it can be shown that for every linear continuous form x' on L, there is a complex-valued, m-measurable function φ bounded (almost everywhere) on every compact set, such that

$$x'(\mu) = \int \varphi(z)d\mu(z) = \int \varphi(z)h(z)dm(z)$$

for all $\mu = h \cdot m \in L$ (see J. Dieudonné [5] and C. Ionescu Tulcea [13]). We shall need below the following result which can be easily proved (for an analogous one see, for instance, C. Ionescu Tulcea [13], [15]):

III) *Let* $\mathscr{F}$ *be a directed set of m-measurable functions on* Z. *Suppose that for every compact set* $K \subset Z$ *there exists a constant* $c(K)$ *such that for* $f \in \mathscr{F}$, $m(\{z : |f(z)| \geqq c(K)\} \cap K) = 0$. *If* $\mathscr{F}$ *converges weakly to* g, *then for every* $\mu \in L$ *we have*

$$\lim_{f \in \mathscr{F}} \int f(s) d\mu_z(s) = \int g(s) d\mu_z(s)$$

uniformly for z *in any given arbitrary compact set* $K \subset Z$.

4. Spectral families. Let X be a Hilbert space and T a locally compact space. A family $\mathscr{F} = (\mu_{x,y})_{x \in X, y \in X}$ of bounded Radon measures defined on T is (hermitian) semi-spectral if

(HI) $x \to \mu_{x,y}$ is linear for all $y \in X$.

(HII) $\mu_{x,y} = \bar{\mu}_{y,x}$ for all $x \in X$, $y \in X$.

(HIII) There is a constant $M(\mathscr{F})$ satisfying the inequality $||\mu_{x,y}|| \leqq M(\mathscr{F}) \, ||x|| \, ||y||$, for all $x \in X$, $y \in X$.

If $\mathscr{F}$ is a semi-spectral family, then for every function f which is bounded and $\mu_{x,y}$-measurable for all $x \in X$, $y \in X$ there exists [5] $U_f \in \mathscr{L}(X, X)$ satisfying the equation

$$(U_f x | y) = \int_T f(t) d\mu_{x,y}(t).$$

If we denote by $B^\infty(T)$ the algebra of all bounded complex-valued $\mathscr{F}$-measurable functions defined on T, then $f \to U_f$ is a linear mapping of $B^\infty(T)$ into $\mathscr{L}(X, X)$. If we endow $B^\infty(T)$ with the norm $f \to ||f|| = \sup_{t \in T} |f(t)|$, then $f \to U_f$ is a continuous mapping of $B^\infty(T)$ into $\mathscr{L}(X, X)$ endowed with the usual norm. A semi-spectral family $\mathscr{F} = (\mu_{x,y})$ is called (hermitian) spectral if

(HIV) $g \cdot \mu_{x,y} = \mu_{U_g x, y}$ for all $g \in B^\infty(T)$, $x \in X$, $y \in X$.

Clearly (HIV) is satisfied if and only if $g \to U_g$ is an algebra representation.

If $\mathscr{F} = (\mu_{x,y})_{x \in X, y \in X}$ is a spectral family on T, then we shall denote by $T(\mathscr{F})$ the set of all $A \subset T$ such that the characteristic function $\varphi_A \in B^\infty(T)$; then $T(\mathscr{F})$ is a tribe. If we define $P_{\mathscr{F}}(A) = U\varphi_A$ then $P_{\mathscr{F}}$ is a strongly countably additive spectral measure. The spectral family $\mathscr{F}$ satisfies the equation $P_{\mathscr{F}}(T) = I$ (identity operator) if and only if $||\mu_{x,x}|| = ||x||^2$ for all $x \in X$; (see for spectral families J. Dixmier [7] and C. Ionescu Tulcea [15].

[5]) $\mathscr{L}(X, X)$ is the algebra of all linear continuous mappings of X into X.

PART II

An Abstract Functional Equation

1. The functional equation. The characters.

We consider here the generalized convolution algebra introduced in Part I and we suppose that the condition (AVI) is satisfied. Let us consider the following (formal) equation:

$$\text{(E)} \qquad aX_{\mu\nu^\alpha}+bX_{\mu\nu^\beta} = X_\mu \int g(s)X_{\varphi(s)}d\nu(s)$$

In this and the next paragraph we shall begin to study the equation (E) and certain particular forms of (E).

By a character we shall mean a (non-identically zero) continuous complex-valued function χ defined on Z and satisfying the equation

$$1) \quad a\int \chi(z)d\mu\nu^\alpha(z)+b\int \chi(z)d\mu\nu^\beta(z) = \int \chi(z)d\mu(z)\int \chi(\varphi(z))g(z)d\nu(z)$$

for all $\mu, \nu \in L$; here a, b are complex numbers, α, β, φ m-measure preserving homeomorphisms of Z onto Z and g is a continuous complex-valued function on Z such that $g(z) \neq 0$ for all $z \in Z$. The measure ν^α is defined by the relations $\nu^\alpha(f) = \int f(\alpha(z))d\nu(z)$ for $f \in K(Z)$.

Let E be the set of all characters. Define $\mathscr{M}$ to be the set of all locally bounded functions r defined on Z with $r(z) \geqq 0$ for all $z \in Z$. For every $r \in \mathscr{M}$ define the set $E(r)=\{\chi \in E:|\chi(z)|\leqq r(z)$ for $z \in Z\}$.

THEOREM 1. *For every $r \in \mathscr{M}$ the space $E(r)$ is locally compact for the topology of uniform convergence on the compact sets of Z.*

PROOF: Every $\chi \in E(r)$ is continuous (and hence locally bounded). Therefore the equation

$$x'_\chi(\mu) = \int \chi(z)d\mu(z), \quad \mu \in L$$

defines an element $x'_\chi \in L'$. The correspondence $\chi \to x'_\chi$ embeds $E(r)$ as a subset of L' (we use here I) of Part I to conclude that the mapping $\chi \to x'_\chi$ is one-to-one.) Since for all $\mu \in L$

$$|x'_\chi(\mu)| \leqq \sup_{z\in S(\mu)} r(z)||\mu||$$

we see that $E(r)$ is a weakly bounded subset of L'. A weakly bounded subset of the dual of a barrelled space is weakly relatively compact (N. Bourbaki [3] chap. III, p. 65). Suppose now that $f_\infty \not\equiv 0$ (locally almost everywhere) is an element of the weak closure of $E(r)$ and let $\mathscr{F} \subset E(r)$ be a directed set converging weakly to f_∞:

$$\lim_{f \in \mathscr{F}} \int f(z) d\mu(z) = \int f_\infty(z) d\mu(z)$$

for every $\mu \in L$. By I) of Part I there is $\mu \in L$ such that $\int f_\infty(z) d\mu(z) \neq 0$. It is also clear that we may suppose that $\int f(z) d\mu(z) \neq 0$ for all $f \in \mathscr{F}$. For any compact set $K \subset Z$ if we choose $c(K) = \sup_{z \in K} r(z)$ then the condition of III) of Part I is satisfied. Therefore

$$\lim_{f \in \mathscr{F}} \int f(s) d\mu_{\alpha(z)}(s) = \int f_\infty(s) d\mu_{\alpha(z)}(s)$$

and

$$\lim_{f \in \mathscr{F}} \int f(s) d\mu_{\beta(z)}(s) = \int f_\infty(s) d\mu_{\beta(z)}(s)$$

uniformly on compact sets of Z. Recall that for every $f \in \mathscr{F}$, and $\nu \in L$.

$$a \int f(z) d\mu\nu^\alpha(z) + b \int f(z) d\mu\nu^\beta(z) = \int f(z) d\mu(z) \int f(\varphi(z)) g(z) d\nu(z).$$

Using I) and II) of Part I we find that

$$a \int f(s) d\mu_{\alpha(z)}(s) + b \int f(s) d\mu_{\beta(z)}(s) = f(\varphi(z)) g(z) \int f(s) d\mu(s)$$

hence

$$\lim_{f \in \mathscr{F}} f(\varphi(z))$$
$$= \left[a \int f_\infty(s) d\mu_{\alpha(z)}(s) + b \int f_\infty(s) d\mu_{\beta(z)}(s) \right] \left[g(z) \int f_\infty(s) d\mu(s) \right]^{-1}$$

where the convergence is uniform on compact sets. Since φ is a homeomorphism of Z onto Z, we deduce that $\mathscr{F}$ converges uniformly on every compact set of Z to a continuous limit which must therefore be identical (locally almost everywhere) to f_∞ (we identify f_∞ with this continuous limit). We conclude that

$$f_\infty(\varphi(z)) g(z) \int f_\infty(s) d\mu(s) = a \int f_\infty(s) d\mu_{\alpha(z)}(s) + b \int f_\infty(s) d\mu_{\beta(z)}(s)$$

and from this it is immediate upon integration with respect to any $\nu \in L$ that f_∞ is indeed a character and belongs to $E(r)$. This shows that every element of the weak closure of $E(r)$ is either a character or identically zero. This means that $E(r) \cup (0)$ is (weakly) compact; hence $E(r)$ is (weakly) locally compact. Since the above proof has shown that on $E(r)$ the weak topology is finer (or stronger) than the topology of uniform convergence on every compact set of Z (and hence coincides with it), the theorem is proved.

2. The spectral representation of the solutions of the functional equation.

Let H be a Hilbert space and $\mathscr{L}(H, H)$ the set of bounded linear transformations (opertaors) of H into H. Let $z \to U_z$ be a mapping of Z into $\mathscr{L}(H, H)$ such that

(CI) $z \to (U_z x|y)$ is continuous for every $x, y \in H$.

For every $\mu \in L$ define an operator U_μ by the relations

$$(U_\mu x|y) = \int (U_z x|y) d\mu(z) \quad \text{for } x, y \in H.$$

We shall frequently use the notation $U_\mu = \int U_z d\mu(z)$. Let us now assume that the mapping $z \to U_z$ also satisfies the following conditions:

(CII) $aU_{\mu\nu\alpha} + bU_{\mu\nu\beta} = U_\mu \int g(s) U_{\varphi(s)} d\nu(s)$ for all $\mu, \nu \in L$ [6])

(CIII) $U_e = I$ (identity operator)

(CIV) $\{U_z : z \in Z\}$ is a commuting family of normal operators.

We shall denote by $\mathfrak{N}_1$ the set of objects $\{H, U_z\}$ satisfying the conditions (CI), (CII), (CIII) and (CIV).

Remark. Suppose that Z is a locally compact Abelian group and that our generalized convolution algebra is the convolution algebra of the group Z (in this case m is the Haar measure). It is immediate (using I) of Part I) in this case that the mapping $z \to U_z$ satisfies the conditions (CI), (CII), (CIII) and (CIV) if and only if it satisfies the conditions (CI), (CIII), (CIV) and

(CII)* $aU_{s+\alpha(t)} + bU_{s+\beta(t)} = g(t) U_s U_{\varphi(t)} \qquad s, t \in Z.$

By the same method we can show that in this case the condition 1) in the definition of the characters is equivalent to the following:

1)* $a\chi(s+\alpha(t)) + b\chi(s+\beta(t)) = g(t)\chi(s)\chi(\varphi(t)) \qquad s, t \in Z.$

We are now ready to prove the main result concerning the functional equation (E).

Theorem 2. *Let $\{H, U_z\} \in \mathfrak{N}_1$ and let $r(z) = \|U_z\|$ for all $z \in Z$. There exists then a spectral family $\mathscr{F} = (\mu_{x,y})_{x \in H, y \in H}$ defined on $E(r)$ such that*

$$(U_z x|y) = \int_{E(r)} \chi(z) d\mu_{x,y}(\chi) \quad \text{for all } z \in Z, \ x, y \in H.$$

[6]) Here a, b, α, β, φ and g are as in paragraph 1. For each $\nu \in L$ the operator $\int g(s) U_{\varphi(s)} d\nu(s)$ is defined by the relations $(\int g(s) U_{\varphi(s)} d\nu(s) x|y) = \int g(s)(U_{\varphi(s)} x|y) d\nu(s)$ for $x, y \in H$. We remark that for every compact set $K \subset Z$ there exists a constant c_K such that $\|U_\mu\| \leq c_K \|\mu\|$ whenever $S(\mu) \subset K$.

Proof: 1) Let $\mathfrak{A}$ be the von Neumann algebra spanned by $\{U_z : z \in Z\}$ and let W be the spectrum of $\mathfrak{A}$; there exists a spectral family $\mathscr{F}'' = (\mu''_{x,y})_{x \in H,\, y \in H}$ such that

$$(U_z x|y) = \int_W w(U_z) d\mu''_{x,y}(w) \text{ for all } z \in Z,\ x,\ y \in H.$$

2) Let $T = \bigcup_{\mu \in L} \{w : w(U_\mu) \neq 0\}$. The mapping $w \to w(U_\mu)$ is continuous for each $\mu \in L$, hence T is open. Let us now show that $W - T$ is $\mathscr{F}''$-negligible (that is; $W-T$ is negligible for each measure $\mu''_{x,y} \in \mathscr{F}''$). Since $W-T$ is closed, its characteristic function ψ_{W-T} is measurable (and of course bounded). Hence ψ_{W-T} is equal to a continuous function (except possibly on an $\mathscr{F}''$-negligible set N). We may therefore choose $A \in \mathfrak{A}$ such that $w(A) = \psi_{W-T}(w)$ for $w \notin N$. Let $\mu \in L$ and define a function θ by the relation $\theta(w) = w(AU_\mu) = w(A)w(U_\mu)$. If $w \notin T$ then $w(U_\mu) = 0$ so that $\theta(w) = 0$. On the other hand if $w \in T-N$, then $w(A) = \psi_{W-T}(w) = 0$, so that $\theta(w) = 0$. Hence $\theta(w) = 0$ if $w \notin N$. Since θ is continuous and N is nowhere dense [7]) we obtain $\theta(w) \equiv 0$ and hence $AU_\mu = 0$ for all $\mu \in L$. Therefore $0 = (AU_\mu x|y) = \int (AU_z x|y) d\mu(z)$ for all $\mu \in L$. Applying I) of Part I we conclude that $AU_z = 0$ for all $z \in Z$. In particular $0 = AU_e = A$ (see condition (CIII)). Finally we remark that $w(A) = 0$ implies that $\psi_{W-T}(w) = 0$ for all $w \notin N$, which implies that $W-T$ is contained in N. Hence $W-T$ is $\mathscr{F}''$-negligible.

For each $x,\ y \in H$ let $\mu'_{x,y}$ be the restriction of $\mu''_{x,y}$ to the open (hence locally compact for the induced topology) set T. The family $\mathscr{F}' = (\mu'_{x,y})_{x \in H,\, y \in H}$ is a spectral family and for all $z \in Z,\ x,\ y \in H$

$$(U_z x|y) = \int_T w(U_z) d\mu'_{x,y}(w)$$

3) For every $w \in T$ let $\chi_w(z) = w(U_z)$. Then we have $|\chi_w(z)| = |w(U_z)| \leqq \|U_z\| = r(z)$ for all $z \in Z$. Let us now show that for each $w \in T$, the function χ_w is continuous. First we remark that for each $\mu \in L$ the mappings $(z, w) \to w(U_{\mu_{\alpha(z)}})$ and $(z, w) \to w(U_{\mu_{\beta(z)}})$ are continuous. In fact if (z, w) converges to (z', w') then

$$|w(U_{\mu_{\alpha(z)}}) - w'(U_{\mu_{\alpha(z')}})| \leqq M \, \|\mu_{\alpha(z)} - \mu_{\alpha(z')}\| + |w(U_{\mu_{\alpha(z')}}) - w'(U_{\mu_{\alpha(z')}})|$$

for some constant M (we use here the facts that $\|U_\mu\| \leqq M_K \|\mu\|$ whenever $S(\mu) \subset K$, that α is a homeomorphism, and that the

[7]) To show that $\theta(w) \equiv 0$ it is enough to remark that if $f, g \in C(W)$ and $f(w) = g(w)$ for $w \in W-N$, then $U_f = U_g$, which implies that $f = g$.

mapping $z \to \mu_{\alpha(z)}$ is continuous). Therefore $(z, w) \to w(U_{\mu_{\alpha(z)}})$ is continuous at (z', w') and hence on $T \times W$ since (z', w') is arbitrary. In the same manner we show that $(z, w) \to w(U_{\mu_{\beta(z)}})$ is continuous. Now the functional equation (CII) is equivalent with the equations

$$\int aU_z d\mu\nu^{\alpha}(z) + \int bU_z d\mu\nu^{\beta}(z) = U_{\mu} \int g(s) U_{\varphi(s)} d\nu(s)$$

that is with

$$\int d\nu(z) \int aU_s d\mu_{\alpha(z)}(s) + \int d\nu(z) \int bU_s d\mu_{\beta(z)}(s) = U_{\mu} \int g(s) U_{\varphi(s)} d\nu(s).$$

By I) of Part I this implies that

$$aU_{\mu_{\alpha(z)}} + bU_{\mu_{\beta(z)}} = g(z) U_{\varphi(z)} U_{\mu}$$

Now take $w_0 \in T$ and choose $\mu \in L$ such that $w_0(U_{\mu}) \neq 0$. Applying w to the last obtained relation we have (for w close enough to w_0)

$$\chi_w(\varphi(z)) = [g(z) w(U_{\mu})]^{-1} [aw(U_{\mu_{\alpha(z)}}) + bw(U_{\mu_{\beta(z)}})].$$

Since φ is a homeomorphism of Z onto Z, we deduce that the mapping $(z, w) \to \chi_w(z)$ is continuous at each point (z, w_0) and hence on $T \times W$. So for each $w \in T$, χ_w is continuous. Also $w \to \chi_w$ is a continuous mapping of W into the space of all continuous complex-valued functions on Z, endowed with the topology of uniform convergence on the compact sets of Z.

Finally we show that for each $w \in T$, χ_w is a character of our functional equation (E). For $w \in T$ choose $\mu \in L$ such that $w(U_{\mu}) \neq 0$. Then for every $\nu \in L$ we have

$$\int g(s) \chi_w(\varphi(s)) d\nu(s) = [w(U_{\mu})]^{-1} \int [aw(U_{\mu_{\alpha(s)}}) + bw(U_{\mu_{\beta(s)}})] d\nu(s) = {}^{8)}$$
$$= [w(U_{\mu})]^{-1} w\Big[\int (aU_{\mu_{\alpha(s)}} + bU_{\mu_{\beta(s)}}) d\nu(s)\Big]$$
$$= [w(U_{\mu})]^{-1} w(U_{\mu}) w\Big(\int g(s) U_{\varphi(s)} d\nu(s)\Big) = w\Big(\int g(s) U_{\varphi(s)} d\nu(s)\Big).$$

From this we deduce for $\mu = f \cdot m \in L$ that (take $\rho = (f \circ \varphi) g^{-1} \cdot m$)

$$\int \chi_w(z) d\mu(z) = \int \chi_w(\varphi(z)) f(\varphi(z)) dm(z) = \int g(z) \chi_w(\varphi(z)) d\rho(z) = {}^{8)}$$
$$= w\Big(\int g(z) U_{\varphi(z)} d\rho(z)\Big) = w\Big(\int f(\varphi(z)) U_{\varphi(z)} dm(z)\Big)$$
$$= w\Big(\int U_z d\mu(z)\Big) = w(U_{\mu}).$$

[8]) We use the fact that w is continuous in the uniform topology and that $z \to U_{\mu_{\alpha(z)}}$ and $z \to U_{\mu_{\beta(z)}}$ are also continuous in the uniform topology. (see for instance E. Hille- R. Phillips [12] p. 66).

Now we may write

$$a\int \chi_w(z)d\mu\nu^{\alpha}(z)+b\int \chi_w(z)d\mu\nu^{\beta}(z)=aw(U_{\mu\nu^\alpha})+bw(U_{\mu\nu^\beta})$$
$$=w\Big(U_\mu\int g(z)U_{\varphi(z)}d\nu(z)\Big)=\int \chi_w(z)d\mu(z)\int g(z)\chi_w(\varphi(z))\,d\nu(z);$$

hence $\chi_w \in E(r)$.

4) For every x, $y \in H$ define a measure $\mu_{x,y}$ on $E(r)$ by the equations

$$\mu_{x,y}(F)=\int_T F(\chi_w)d\mu'_{x,y}(w) \quad \text{for } F\in K(E(r)).$$

The family $\mathscr{F}=(\mu_{x,y})_{x\in H,\, y\in H}$ so defined is a spectral family on $E(r)$. Finally let $F(\chi)=\chi(z)$, then

$$\int_{E(r)}\chi(z)d\mu_{x,y}(\chi)=\int_T \chi_w(z)d\mu'_{x,y}(w)=(U_z x|y) \text{ for all } z\in Z,\ x,\ y\in H.$$

Hence the spectral representation theorem is proved.

Remark. For the particular case that (CII) becomes the cosine functional equation (see below) it can be shown that the spectral family given in Theorem 2 is unique in the following sense: If $\mathscr{D}=(\nu_{x,y})$ is a second spectral family defined on $\tilde{E}$ (the Stone-Cech compactification of the space of all characters) and concentrated on $E(r)$ and if $(U_z x|y)=\int_{E(r)}\chi(z)d\nu_{x,y}(\chi)$ for $z\in Z$, x, $y\in H$, then $\mathscr{D}=\mathscr{F}$.

3. The cosine functional equation.[9])

Let H be a Hilbert space and $\mathscr{L}(H, H)$ the set of operators of H into H. Let $z\to U_z$ be a mapping of Z into $\mathscr{L}(H, H)$ such that

(CI) $z\to(U_z x|y)$ is continuous for every x, $y\in H$.

(CII) $U_{\mu\nu}+U_{\mu\check{\nu}}=2U_\mu U_\nu$ for all μ, $\nu\in L$.

(CIII) $U_e=I$.

(CIV) U_z is a normal operator for each $z\in Z$.

We shall denote by $\mathfrak{N}_1$ the set of objects $\{H, U_z\}$ satisfying conditions (CI), (CII), (CIII) and (CIV). It is obvious that the equation given here in (CII) is a particular form of equation (E). It is enough to take $\alpha(z)\equiv z$, $\beta(z)\equiv z^*$, $\varphi(z)\equiv z$, $g(z)\equiv 2$ and $a=b=1$. The conditions (CI), (CII), (CIII) and (CIV) given here are identical with the corresponding conditions formulated in paragraph 2, as it follows from:

[9]) Specializing the functional equation (E) we obtain various equations studied in S. Kaczmarz [19] and E. VanVleck [33] and the exponential functional equation.

PROPOSITION 1. *If the object* $\{H, U_z\}$ *belongs to* $\mathfrak{R}_1$, *then we have the following*: a) *if* $(f_V)_{V\in\mathscr{V}(e)}$ *is an approximate identity* [10]) *for the generalized convolution algebra, then* $\lim_{V\in\mathscr{V}(e)} U_{(\mu^V)} = I$ *weakly, where* $\mu^V = f_V \cdot m$ *for* $V \in \mathscr{V}(e)$. b) $U_\mu = U_{\check{\mu}}$ *and* $U_z = U_{z^*}$ *for all* $\mu \in L$ *and all* $z \in Z$. c) $U_\mu U_\nu = U_\nu U_\mu$ *and* $U_z U_s = U_s U_z$ *for all* μ, $\nu \in L$ *and all* z, $s \in Z$.

PROOF: a) follows immediately from the fact that the mapping $z \to (U_z x|y)$ is continuous for every $x, y \in H$, and from the properties of the approximate identity sequence. b) Let $\mu = f \cdot m \in L$, then $\lim\ (U_{(\mu^V)\mu} x|y) = \lim \int (U_z x|y) f_V * f(z) dm(z) = \int (U_z x|y) f(z) dm(z) = (U_\mu x|y)$ weakly and hence $U_{\mu^V)\mu} \to U_\mu$. Since $\mu \in L$ is arbitrary, we obtain, for all x, $y \in H$,

$$(U_\mu x|y) + (U_{\check{\mu}} x|y) = \lim (U_{(\mu^V)} U_\mu x|y) + \lim (U_{(\mu^V)} U_{\check{\mu}} x|y)$$
$$= \lim 2(U_{(\mu^V)} U_\mu x|y) = 2(U_\mu x|y) \text{ and so } U_\mu + U_{\check{\mu}} = 2U_\mu.$$

Hence the first assertion in b) is proved; the second follows from this. c) The first relation follows from b) and the fact that $\mu\nu = \nu\mu$ for all μ, $\nu \in L$. To prove the second assertion let μ, $\nu \in L$ then we have

$$\int d\nu(z) \int (U_z U_s x|y) d\mu(s) = \int (U_\mu x|U_z^* y) d\nu(z) = (U_\nu U_\mu x|y).$$

On the other hand

$$\int d\nu(z) \int (U_s U_z x|y) d\mu(s) = \int (U_\mu U_z x|y) d\nu(z) = (U_\mu U_\nu x|y).$$

Since μ, ν are arbitrary we conclude that $U_z U_s = U_s U_z$ for all z, $s \in Z$. Hence the lemma is completely proved.

4. S. Kurepa's Results

In this paragraph we consider more closely various aspects of the case when $Z =$ the real line R and the functional equation (E) is the cosine functional equation. We shall use below the fact that: *) Every cosine character χ can be written under the form $\chi(s) = \cos \lambda s$ for an arbitrary (non-zero) complex number λ.[11]) On its basis we shall prove that S. Kurepa's results [21] can be deduced immediately from our results.

Define the following set T of points in the complex plane C:

$$T = \{\lambda : \mathscr{R}(\lambda) > 0\} \cup \{\lambda : \mathscr{R}(\lambda) = 0, \quad \mathscr{I}(\lambda) \geqq 0\}$$

[10]) We use here the following result: For every $V \in \mathscr{V}(e)$ (= the neighborhood system at e) let f_V be a function in $K(Z)$ such that $f_V \geqq 0$, $\int f_V(z) dm(z) = 1$, and $S(f_V) \subset V$. Then if $f \in L^1(Z, m)$ we have weakly $\lim_{V\in\mathscr{V}(e)} f * f_V = f$ (See C. Ionescu Tulcea and A. Simon [18]).

[11]) See A. Cauchy [4] and E. Picard [30] who consider the case of real-valued solutions.

(here $\mathscr{R}(\lambda)$ and $\mathscr{I}(\lambda)$ indicate respectively the real and imaginary parts of the complex number λ). If $\lambda', \lambda'' \in T$ and $\cos \lambda' s = \cos \lambda'' s$ for all $s \in R$, then $\lambda' = \lambda''$. Hence for any given cosine character χ there is only one $\lambda_\chi \in T$ such that $\chi(s) = \cos \lambda_\chi s$ for all $s \in R$. Denote by u the mapping $\chi \to \lambda_\chi$ of E into C.

PROPOSITION 2. *u is a Borel measurable mapping of E into C.*

PROOF: We have to show that for every open set $U \subset C$, $u^{-1}(U)$ is a Borel measurable set in E. For this it is enough to show that $u^{-1}(K)$ is Borel measurable for every compact set $K \subset T$. We know that if $\lambda_n \in T$ and $\cos \lambda_n s$ converges to $\cos \lambda_\infty s$, $\lambda_\infty \in T$, uniformly in s, on every compact subset of R, then $\lambda_n \to \pm\lambda_\infty$. Let $\chi \in u^{-1}(K)$ and suppose $\chi \to \chi_\infty$ (we remark that E is metrizable since $Z = R$). Then $u(\chi) \in K$ and $u(\chi) \to \pm u(\chi_\infty)$. But $u(\chi) \in K$ and $\lim u(\chi) \in K \subset T$, so that $u(\chi) \to +u(\chi_\infty)$. Hence $\chi_\infty \in u^{-1}(K)$ which implies that $u^{-1}(K)$ is closed i.e; Borel measurable, so the lemma is proved.

We prove now a corollary to the spectral representation theorem, which contains the main result of S. Kurepa [21].

THEOREM 3. *Let $Z = R$ and let $\{H, U_z\} \in \mathfrak{N}_1$. There exists then a spectral family $\mathscr{G} = (\nu_{x,y})_{x \in H,\, y \in H}$ defined on the complex plane C such that*

$$(U_s x|y) = \int_C \cos \lambda s \, d\nu_{x,y}(\lambda) \text{ for all } s \in R,\ x,\ y \in H.$$

The (unique) spectral measure $\mathscr{G}$ is concentrated in a cylinder (with bounded bases) parallel to the OX axis of the complex plane.

PROOF: By the spectral representation Theorem 2

$$(U_s x|y) = \int_{E(r)} \chi(s) d\mu_{x,y}(\chi) \text{ for all } s \in R,\ x,\ y \in H$$

where $\mathscr{F} = (\mu_{x,y})_{x \in H,\, y \in H}$ is a spectral family on $E(r)$. Define the family $\mathscr{G} = (\nu_{x,y})_{x \in H,\, y \in H}$ of bounded Radon measures on C by the relations

$$\int_C f(\lambda) d\nu_{x,y}(\lambda) = \int_{E(r)} f(u(\chi))\, d\mu_{x,y}(\chi) \text{ for } f \in K(C)$$

(see proposition 2 for the definition of u). Now $\mathscr{G} = (\nu_{x,y})_{x \in H,\, y \in H}$ is a spectral family on C (we have $U_{\mathscr{G},f} = U_{\mathscr{F},f \circ u}$ and from this follows immediately that $\mathscr{G}$ is a spectral family). For all $s \in R$ and $x,\ y \in H$ we may write,

$$\int_C \cos \lambda s d\nu_{x,y}(\lambda) = \int_{E(r)} \cos u(\chi) s \, d\mu_{x,y}(\chi)$$

$$= \int_{E(r)} \chi(s) d\mu_{x,y}(\chi) = (U_s x|y)$$

hence the first half of the theorem is proved. For the second half we remark that for each $s \in R$, $||U_s|| = \sup_{\lambda \in D} |\cos \lambda s|$ where D is the closure of the union of the supports of the measures $(\nu_{x,x})_{x \in H}$. We shall now show that D is contained in a cylinder of the complex plane as mentioned in the statement of the theorem. For this it is enough to show that $\sup_{\lambda \in D} |\mathscr{I}(\lambda)| < \infty$. But since $|\cos \lambda| \leqq ||U_1||$ for all $\lambda \in D$, this implies that the set $\{\mathscr{I}(\lambda) : \lambda \in D\}$ is bounded and completes the proof (uniqueness is a consequence of the remark following Theorem 2).

For more details concerning this paragraph see G. Maltese [23].

5. Measurable Solutions of the Cosine Functional Equation

In this paragraph we shall suppose that Z is a locally compact Abelian group, so that the generalized convolution algebra is the familiar group algebra. We shall discuss measurable solutions of the equation

$$U_{s+t} + U_{s-t} = 2U_s U_t \qquad s,\ t \in Z$$

where for each $s \in Z$, U_s is a normal operator on a Hilbert space H. We show that under various conditions, strong continuity of the solutions is a consequence of weak measurability. We shall suppose in this paragraph that the mapping $z \to U_z$ of Z into $\mathscr{L}(H, H)$ satisfies the above cosine functional equation and the following condition:

(MI) $z \to (U_z x|y)$ is measurable for each $x, y \in H$.

THEOREM 4. *Suppose the mapping $z \to ||U_z||$ is locally bounded and let H_0 the linear subspace spanned by the set $\{U_\mu x : \mu \in L, x \in H\}$ be dense in H. Then the mapping $z \to U_z$ is continuous in the strong operator topology.*

PROOF: Let $U_\mu x \in H_0$, $\mu = f \cdot m$ and let $s, t \in Z$; then we have $(M_f = \sup_{z \in S(\mu)} ||U_z x||)$

$$||2U_s U_\mu x - 2U_t U_\mu x|| = \sup_{||y|| \leq 1} \left| \int [(2U_s U_z x|y) - (2U_t U_z x|y)] d\mu(z) \right|$$

$$= \sup_{||y|| \leq 1} \left| \int [(U_{s+z} x|y) + (U_{s-z} x|y) - (U_{t+z} x|y) - U_{t-z} x|y)] f(z) dm(z) \right|$$

$$= \sup_{||y|| \leq 1} \left| \int (U_z x|y) [f(z-s) + f(s-z) - f(z-t) - f(t-z)] dm(z) \right|$$

$$\leqq M_f \left[\int |f(z-s) - f(z-t)| dm(z) + \int |f(s-z) - f(t-z)| dm(z) \right].$$

But the last two integrals which we have written converge to zero as s converges to t. Hence we have shown that $z \to U_z U_\mu x$ is continuous for all $U_\mu x \in H_0$. Since H_0 is dense, and since the mapping $z \to ||U_z||$ is locally bounded, we conclude that $z \to U_z x$ is continuous for every $x \in H$. Hence the theorem is proved.

Remark. The hypothesis that U_z, $z \in Z$ are normal operators is not used for the proof of this theorem.

Proposition 3. *Let H be a separable Hilbert space and let the mapping $z \to ||U_z||$ be locally bounded. Suppose that, if $U_z x = 0$ locally almost everywhere, then $x = 0$. Then the mapping $z \to U_z$ is continuous in the strong operator topology.*

Proof: By the previous theorem we have only to show that the linear span H_0 of the set $\{U_\mu y : \mu \in L,\ y \in H\}$ is dense in H. For this purpose let $h \in H$ and suppose that h is orthogonal to H_0, so that $0 = (h|U_\mu y) = \int (U_z y|h) d\mu(z)$ for all $\mu \in L$ and all $y \in H$. For fixed but arbitrary y, I) of Part I implies that σ) $(U_z^* h|y) = 0$ for all $z \notin N_y$, where N_y is locally negligible. Let $Y = \{y_1, y_2, \ldots\}$ be a countable set which is dense in H and let $N = \bigcup_1^\infty N_{y_n}$. By σ) we have $(U_z^* h|y_n) = 0$ for all $z \notin N$, that is to say $U_z^* h = 0$ locally almost everywhere, that is; $U_z h = 0$ locally almost everywhere since $||U_z^* h|| = ||U_z h||$. Consequently $h = 0$ and therefore H_0 is dense in H. Hence the proof is complete.

In the case that $Z = R^n$ it can be shown that under certain conditions weak measurability implies local boundedness. In fact a method of proof essentially similar to that of S. Kurepa [21] yields the following:

Proposition 4. *Let H be a Hilbert space and let the mapping $s \to U_s$ of R^n into $\mathscr{L}(H, H)$ satisfy the cosine equation. If $s \to ||U_s||$ is measurable, then $s \to ||U_s||$ is locally bounded.*

PART III

A System of Abstract Functional Equations

1. The System (S)

We consider here a generalized convolution algebra as in Part I. Let a, b, c, d, a', b', c', d', be complex numbers and consider the following (formal) system: (for μ, $\nu \in L$)

$$\text{(S)} \qquad \begin{aligned} X_{\mu\nu} &= aX_\mu X_\nu + bX_\mu Y_\nu + cY_\mu X_\nu + dY_\mu Y_\nu \\ Y_{\mu\nu} &= a'X_\mu X_\nu + b'X_\mu Y_\nu + c'Y_\mu X_\nu + d'Y_\mu Y_\nu. \end{aligned}$$

We shall say that the pair (χ, ρ) of continuous, complex-valued functions defined on Z satisfy the system (S) or is a solution of (S) if

$$\text{(S)}\quad \begin{aligned}\chi(\mu\nu) &= a\chi(\mu)\chi(\nu)+b\chi(\mu)\rho(\nu)+c\rho(\mu)\chi(\nu)+d\rho(\mu)\rho(\nu)\\ \rho(\mu\nu) &= a'\chi(\mu)\chi(\nu)+b'\chi(\mu)\rho(\nu)+c'\rho(\mu)\chi(\nu)+d'\rho(\mu)\rho(\nu).\end{aligned}$$

for all μ, $\nu \in L$. (For any continuous, complex-valued function f defined on Z, and $\nu \in L$ we write $f(\nu) = \int f(z)d\nu(z)$.) We shall denote by $\mathscr{E}$ the set of all pairs (χ, ρ) of functions χ, $\rho \in \mathscr{C}$ (the set of all continuous complex-valued functions on Z) which are not both identically zero and which are solutions of the system (S). We denote by $\mathscr{A}$ the first projection of $\mathscr{E}$ into $\mathscr{C}$ and by $\mathscr{B}$ the second projection of $\mathscr{E}$ into $\mathscr{C}$. By definition it follows that $\chi \in \mathscr{A}$ if there exists $\rho \in \mathscr{C}$ such that (χ, ρ) is a solution of (S); this implies in particular that $\rho \in \mathscr{B}$ (the characterization for $\mathscr{B}$ is similar). We shall sometimes call the functions $\chi \in \mathscr{A}$ and $\rho \in \mathscr{B}$ characters of the system (S). We consider on $\mathscr{C}$ the topology of uniform convergence on the compact subsets of Z. Define $\mathscr{M}$ to be the set of all locally bounded functions r defined on Z with $r(z) \geqq 0$ for all $z \in Z$. For every $r \in \mathscr{M}$ define the sets

$$\mathscr{A}(r) = \{\chi \in \mathscr{A}: |\chi(z)| \leqq r(z) \text{ for all } z \in Z\}$$
$$\mathscr{B}(r) = \{\rho \in \mathscr{B}: |\rho(z)| \leqq r(z) \text{ for all } z \in Z\}.$$

We consider $\mathscr{A}(r)$ and $\mathscr{B}(r)$ as subspaces of $\mathscr{C}$ and we denote by $\tilde{\mathscr{A}}(r)$ and $\tilde{\mathscr{B}}(r)$ their respective Stone-Cech compactifications.

Let H be a Hilbert space and let $\mathscr{L}(H, H)$ be the set of all bounded linear transformations (operators) of H into H. Let $z \to U_z$ and $z \to V_z$ be two mappings of $Z \to \mathscr{L}(H, H)$ such that

$$\text{(CI)}\quad \begin{aligned}&z \to (U_z x|y) \text{ is continuous for all } x,\ y \in H.\\ &z \to (V_z x|y) \text{ is continuous for all } x,\ y \in H.\end{aligned}$$

For every $\mu \in L$ define the operators U_μ and V_μ again by the following equations:

$$(U_\mu x|y) = \int (U_z x|y)d\mu(z) \text{ and } (V_\mu x|y) = \int (V_z x|y)d\mu(z)$$

for all $x, y \in H$. For every compact set $K \subset Z$ there exists a constant c_K such that $\|U_\mu\| \leqq c_K\|\mu\|$ and $\|V_\mu\| \leqq c_K\|\mu\|$ whenever $S(\mu) \subset K$. Let us assume further that the mappings $z \to U_z$ and $z \to V_z$ satisfy the following:

$$\text{(CII)}\quad \begin{aligned}U_{\mu\nu} &= aU_\mu U_\nu+bU_\mu V_\nu+cV_\mu U_\nu+dV_\mu V_\nu\\ V_{\mu\nu} &= a'U_\mu U_\nu+b'U_\mu V_\nu+c'V_\mu U_\nu+d'V_\mu V_\nu\end{aligned}\qquad \mu,\ \nu \in L$$

For every $z \in Z$, $\mu \in L$ let us write

$$D_z = [aU_z+cV_z][b'U_z+d'V_z]-[bU_z+dV_z][a'U_z+c'V_z]$$

and

$$D_\mu = [aU_\mu+cV_\mu][b'U_\mu+d'V_\mu]-[bU_\mu+dV_\mu][a'U_\mu+c'V_\mu].$$

Finally we shall suppose that the following two conditions also hold:

(CIII) $D_e = I$ (Identity operator)
(CIV) $\{U_z : z \in Z\} \cup \{V_z : z \in Z\}$ is a set of commuting normal operators.

Denote by $\mathfrak{N}$ the set of all objects $\{H, A_z\}$ where H is a Hilbert space and $z \to A_z$ is a mapping of Z into $\mathscr{L}(H, H)$. We shall denote by $\mathfrak{N}(\mathscr{E})$ the set of all pairs of objects $(\{H, U_z\}, \{H, V_z\})$ having the conditions (CI), (CII), (CIII), and (CIV); $\mathfrak{N}(\mathscr{A})$ denotes the first projection of $\mathfrak{N}(\mathscr{E})$ into $\mathfrak{N}$ and $\mathfrak{N}(\mathscr{B})$ denotes the second projection of $\mathfrak{N}(\mathscr{E})$ into $\mathfrak{N}$.

2. The Spectral Representation for the Solutions of the System (S)

We are now ready to prove our main result concerning the system (S).

THEOREM 5. *Let* $(\{H, U_z\}, \{H, V_z\}) \in \mathfrak{N}(\mathscr{E})$ *and let* $r(z) = \sup(\|U_z\|, \|V_z\|)$ *for all* $z \in Z$. *There exist then two spectral families* $\mathscr{F} = (\mu_{x,y})_{x\in H, y\in H}$ and $\mathscr{D} = (\nu_{x,y})_{x\in H, y\in H}$ *defined respectively on* $\tilde{\mathscr{A}}(r)$ $\tilde{\mathscr{B}}(r)$ *and concentrated respectively on* $\mathscr{A}(r)$ *and* $\mathscr{B}(r)$, *such that*

$$(U_z x|y) = \int_{\tilde{\mathscr{A}}(r)} \chi(z) d\mu_{x,y}(\chi) \text{ and } (V_z x|y) = \int_{\tilde{\mathscr{B}}(r)} \rho(z) d\nu_{x,y}(\rho)$$

for all $z \in Z$ *and* $x, y \in H$.

PROOF:[12]) 1. Let $\mathfrak{A}$ be the von Neumann algebra spanned by the set $\{U_z : z \in Z\} \cup \{V_z : z \in Z\}$ and let W be the spectrum of $\mathfrak{A}$. There exists then a spectral family $\mathscr{G}' = (\delta'_{x,y})_{x\in H, y\in H}$ such that for every $z \in Z$ and $x, y \in H$,

$$(U_z x|y) = \int_W w(U_z) d\delta'_{x,y}(w) \text{ and } (V_z x|y) = \int_W w(V_z) d\delta'_{x,y}(w).$$

2. Let [13]) $T = \bigcup_{\mu \in L} \{w : w(D_\mu) \neq 0\}$. Applying I) of Part I successively and using the assumption that $D_e = I$, it can be shown that $W - T$ is $\mathscr{G}'$-negligible. For each $x, y \in H$ let $\delta_{x,y}$ be the restriction of $\delta'_{x,y}$ to the open set T. Then $\mathscr{G} = (\delta_{x,y})_{x,y\in H}$ is a spectral family such that

$$(U_z x|y) = \int_T w(U_z) d\delta_{x,y}(w) \text{ and } (V_z x|y) = \int_T w(V_z) d\delta_{x,y}(w)$$

for all $z \in Z$, $x, y \in H$.

[12]) Since the method of proof is parallel to that of Theorem 2 of Part I, we shall indicate here only the main ideas.

[13]) It is easily shown that V_μ, $U_\mu \in \mathfrak{A}$ and hence $D_\mu \in \mathfrak{A}$ for every $\mu \in L$.

3. For every $w \in T$ define $\chi_w(z) = w(U_z)$ and $\rho_w(z) = w(V_z)$ so that $|\chi_w(z)| \leqq r(z)$ and $|\rho_w(z)| \leqq r(z)$ for all $z \in Z$. Making use of the fact that the mappings $(z, w) \to w(U_{\mu_z})$ and $(z, w) \to w(V_{\mu_z})$ are continuous for each $\mu \in L$, and using the relations

$$D_\mu U_z = (b'U_\mu + d'V_\mu)U_{\mu_z} - (bU_\mu + dV_\mu)V_{\mu_z},$$
$$D_\mu V_z = (aU_\mu + cV_\mu)V_{\mu_z} - (a'U_\mu + c'V_\mu)U_{\mu_z}$$

obtained by an application of I) of Part I to the system (CII), we can show that for each $w \in T$, χ_w and ρ_w are continuous functions. We remark also that the mappings $w \to \chi_w$ and $w \to \rho_w$ are continuous mappings of W into the space $\mathscr{C}$ endowed with the topology of uniform convergence on the compact sets of Z.

Finally we show that for each $w \in T$ the pair $(\chi_w, \rho_w) \in \mathscr{E}$. For this we first show that for every $\nu \in L$ we have

$$(3.1) \qquad \int \chi_w(z)d\nu(z) = w(U_\nu) \text{ and } \int \rho_w(z)d\nu(z) = w(V_\nu).$$

In fact for $w \in T$ choose $\mu \in L$ such that $w(D_\mu) \neq 0$, then

$$\int \chi_w(z)d\nu(z) = [w(D_\mu)]^{-1}\int w(D_\mu U_z)d\nu(z)$$
$$= [w(D_\mu)]^{-1}w\Big(\int [(b'U_\mu + d'V_\mu)U_{\mu_z} - (bU_\mu + dV_\mu)V_{\mu_z}]d\nu(z)\Big)$$
$$= [w(D_\mu)]^{-1}w(D_\mu U_\nu) = w(U_\nu).$$

Hence the first relation in (3.1) is proved and the second may be proved in a similar fashion. From (3.1) it is now easy to show that for each $w \in T$, $(\chi_w, \rho_w) \in \mathscr{E}$ (obviously $|\chi_w| + |\rho_w| \not\equiv 0$ and $\chi_w \in \mathscr{A}(r)$, $\rho_w \in \mathscr{B}(r)$ for every $w \in T$).

4. For every $x, y \in H$ define two measures $\mu_{x,y}$ and $\nu_{x,y}$ on $\tilde{\mathscr{A}}(r)$ and $\tilde{\mathscr{B}}(r)$ respectively, in the following manner:

$$\mu_{x,y}(F) = \int_T F(\chi_w)d\delta_{x,y}(w) \text{ and } \nu_{x,y}(G) = \int_T G(\rho_w)d\delta_{x,y}(w)$$

for any continuous functions F, G defined on $\tilde{\mathscr{A}}(r)$ and $\tilde{\mathscr{B}}(r)$ respectively and having compact support. The families $\mathscr{F} = (\mu_{x,y})_{x\in H, y\in H}$ and $\mathscr{D} = (\nu_{x,y})_{x\in H, y\in H}$ are spectral families on $\tilde{\mathscr{A}}(r)$ and $\tilde{\mathscr{B}}(r)$ respectively (and concentrated on $\mathscr{A}(r)$ and $\mathscr{B}(r)$ respectively). Now let $F(\chi) = \chi(z)$ so that $F(\chi_w) = \chi_w(z) = w(U_z)$. Then we have

$$\int_{\tilde{\mathscr{A}}(r)} \chi(z)d\mu_{x,y}(\chi) = \int_T w(U_z)d\delta_{x,y}(w) = (U_z x|y).$$

Similarly letting $G(\rho) = \rho(z)$ so that $G(\rho_w) = w(V_z)$, we obtain

$$\int_{\tilde{\mathscr{B}}(r)} \rho(z)d\nu_{x,y}(\rho) = \int_T w(V_z)d\delta_{x,y}(w) = (V_z x|y).$$

Hence the spectral representation is demonstrated.

Remark. If we suppose that for each $z \in Z$, the operators U_z and V_z are hermitian, then χ_w and ρ_w are real-valued functions since $\chi_w(z) = w(U_z) = w(U_z^*) = \overline{w(U_z)} = \overline{\chi_w(z)}$ and $\rho_w(z) = \overline{\rho_w(z)}$ for all $z \in Z$. When we consider only real-valued functions on Z, we shall use the notation $\mathscr{E}_{\mathfrak{R}}$, $\mathscr{A}_{\mathfrak{R}}$, $\mathscr{B}_{\mathfrak{R}}$ etc.

Remark. In the case when our generalized convolution algebra is the convolution algebra of a locally compact Abelian group Z (here the measure m is the Haar measure), it is immediate that the mappings $z \to U_z$ and $z \to V_z$ of Z into $\mathscr{L}(H, H)$ satisfy (CI) and (CII) if and only if they satisfy (CI) and

$$\text{(CII)}' \qquad \begin{aligned} U_{s+t} &= aU_sU_t+bU_sV_t+cV_sU_t+dV_sV_t \\ V_{s+t} &= a'U_sU_t+b'U_sV_t+c'V_sU_t+d'V_sV_t \end{aligned} \qquad s,\ t \in Z.$$

By the same method we can show that the system (S) for characters is equivalent to the following system:

$$\text{(S)}' \qquad \begin{aligned} \chi(s+t) &= a\chi(s)\chi(t)+b\chi(s)\rho(t)+c\rho(s)\chi(t)+d\rho(s)\rho(t) \\ \rho(s+t) &= a'\chi(s)\chi(t)+b'\chi(s)\rho(t)+c'\rho(s)\chi(t)+d'\rho(s)\rho(t). \end{aligned}$$

3. An Example for the Case $Z = R$

In this paragraph we consider more closely a special case of the system (S) when $Z =$ the real line R. In particular we shall suppose that the mappings $z \to U_z$ and $z \to V_z$ satisfy the conditions (CI), (CII)$'$ with $a = 1$, $d = -1$, $b = c = 0$, $a' = d' = 0$, $c' = b' = 1$, (CIII) and (CIV) with the restriction that for every $z \in R$, the operators U_z and V_z are hermitian. The system therefore becomes

$$(\text{S}_{\mathfrak{R}}) \qquad \begin{aligned} U_{s+t} &= U_sU_t-V_sV_t \\ V_{s+t} &= U_sV_t+V_sU_t \end{aligned} \qquad s,\ t \in R$$

The set $\mathscr{E}_{\mathfrak{R}}$ corresponding to the system $(\text{S}_{\mathfrak{R}})$ consists of all pairs (χ, ρ) of continuous real-valued functions χ, ρ defined on R which are not both identically zero, and which satisfy the following:

$$(\text{S}_{\mathfrak{R}}) \qquad \begin{aligned} \chi(s+t) &= \chi(s)\chi(t)-\rho(s)\rho(t) \\ \rho(s+t) &= \chi(s)\rho(t)+\rho(s)\chi(t) \end{aligned} \qquad s,\ t \in R$$

It is well-known (W. Osgood [28] p. 608) that the elements $(\chi, \rho) \in \mathscr{E}_{\mathfrak{R}}$ are $\chi(s) = e^{\alpha s}\cos\beta s$ and $\rho(s) = \pm e^{\alpha s}\sin\beta s$ where α and β are arbitrary real constants. Denote by R_+ the set of non-negative real numbers. It is easy to see that for any given $(\chi, \rho) \in \mathscr{E}_{\mathfrak{R}}$ there exists only one pair $(\alpha_\chi, \beta_\chi) \in R \times R_+$ such that $\chi(s) = e^{\alpha_\chi s} \cos \beta_\chi s$ and (if $\rho \not\equiv 0$) only one pair $(\alpha_\rho, \beta_\rho) \in R \times R_+$ such that $\rho(s) = e^{\alpha_\rho s} \sin \beta_\rho s$.

Denote by u the mapping $\chi \to (\alpha_\chi, \beta_\chi)$ of $\mathscr{A}_{\mathfrak{R}}$ into $R \times R_+$ and by v the mapping $\rho \to (\alpha_\rho, \beta_\rho)$ of $\mathscr{B}_{\mathfrak{R}} - \{0\}$ into $R \times R_+$. It is directly verified that the mappings u and v are continuous (the topology in $\mathscr{A}_{\mathfrak{R}}$ and $\mathscr{B}_{\mathfrak{R}}$ is the topology of uniform convergence on compact sets of R). With this we state now (without proof) the following corollary to the spectral representation theorem of the preceding paragraph:

Theorem 6. *Let $Z = R$ (we consider now the system $(S_{\mathfrak{R}})$) and let $(\{H, U_s\}, \{H, V_s\}) \in \mathfrak{N}(\mathscr{E}_{\mathfrak{R}})$. There exist then two spectral families $\mathscr{F} = (\mu_{x,y})_{x \in H, y \in H}$ and $\mathscr{D} = (\nu_{x,y})_{x \in H, y \in H}$ defined on $R \times R$ such that $(U_s x|y) = \int_{R \times R} e^{\alpha s} \cos \beta s \, d\mu_{x,y}(\alpha, \beta)$ and $(V_s x|y) = \int_{R \times R} e^{\alpha s} \sin \beta s \, d\nu_{x,y}(\alpha, \beta)$ for all $s \in R$ and $x, y \in H$.*

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