COMPOSITIO MATHEMATICA

CIPRIAN FOIAŞ

# On Hille's spectral theory and operational calculus for semi-groups of operators in Hilbert space

*Compositio Mathematica*, tome 14 (1959-1960), p. 71-73

<http://www.numdam.org/item?id=CM_1959-1960__14__71_0>

# On Hille's spectral theory and operational calculus for semi-groups of operators in Hilbert space.

by

Ciprian Foiaş

1. Let $H$ be a Hilbert space, $L(H)$ the algebra of all bounded linear operators on $H$, and $A$ a linear operator, bounded or not, with domain $D_A$, satisfying the condition: (*) *There is a real number r, such that* $\xi \in \rho(A)$ *and* $\|(A-\xi I)^{-1}\| \leqq (\xi-r)^{-1}$ *if* $\xi > r$. [1])

Let $\mathfrak{O}_r$ be the algebra of all functions $\varphi(\lambda)$ bounded and continuous on $Re\lambda \leqq r$ and holomorphic on $Re\lambda < r$.

In this paper we shall define $\varphi(A)$ for every $\varphi(\lambda) \in \mathfrak{O}_r$ and prove some properties concerning the mapping $\varphi(\lambda) \to \varphi(A)$. [2]) The operational calculus given here is a consequence of the calculus established by the author in [2]. For completeness we shall state some of the results proved in [2], which are needed here. Let $T \in L(H)$ and $S$ a spectral set of $T$ (see [3]), bounded by a closed Jordan curve. Denote by $\mathfrak{O}_e[S; T]$ the algebra of functions $\varphi(\lambda)$ defined on $S \cap CE_{\varphi(\lambda)}$ where $E_{\varphi(\lambda)} \subset FrS \cap CP\gamma(T)$ is a finite set, bounded, continuous and holomorphic in Int$S$. Then there is an isomorphic mapping of $\mathfrak{O}_e[S; T]$ into $L(H)$ such that: (i) $1 \to I$, $\lambda \to T$; (ii) $\|\varphi(T)\| \leqq \text{l.u.b.}_{\lambda \in S} |\varphi(\lambda)|$; (iii) if $\varphi_n(\lambda) \in \mathfrak{O}_e[S; T]$ is a bounded sequence which converges uniformly to $\varphi(\lambda) \in \mathfrak{O}_e[S; T]$, excepting a finite number of points which belong to $FrS \cap CP\sigma(T)$, then $\varphi_n(T)$ converges strongly to $\varphi(T)$.

2. For every $y \in D_A$ put $x = [A-(r+1)I]y$ and $Tx = [A-(r-1)I]y$.

**Theorem 1.** $1 \notin P\sigma(T)$, $T \in L(H)$ *and* $\|T\| \leqq 1$.

**Proof.** $T$ is uniquely defined on $H$. If $Tx = x$, then $y = 0$; hence $x = 0$. It follows that $1 \notin P\sigma(T)$. To show that $\|T\| \leqq 1$ is enough to show that $S = \{\lambda : |\lambda| \leqq 1\}$ is a spectral set of $T$. From (*) it follows that $S_\xi = \{\lambda : |\lambda| < (\xi-r)^{-1}\}$ is a spectral set of $R_\xi = (A-\xi I)^{-1}$, and that

$$T = [(\xi-r+1)R_\xi+I][(\xi-r-1)R_\xi+I]^{-1} = \omega_\xi(R_\xi)$$

---

[1]) Then $A$ is a semi-group generator. It is not supposed a priori that $D_A$ is dense in $H$ (see also [2], § 5).

[2]) The condition (*) is more restrictive than Hille's condition (see [1], p. 303) on which is based his operational calculus, but the class $\mathfrak{O}_r$ is larger than his corresponding class.

where

$$\omega_\xi(\lambda) = [(\xi-r+1)\lambda+1][(\xi-r+1)\lambda+1]^{-1}.$$

Hence ([4], p. 436) $\omega_\xi(S_\xi)$ is a spectral set of $T$. But for $\xi < \xi'$ we have $\omega_{\xi'}(S_{\xi'}) \subset \omega_\xi(S_\xi)$, and by a theorem due to J. v. Neumann ([3], p. 262), $S = \{\lambda : |\lambda| \leqq 1\} = \bigcap_{\xi > r+1} \omega_\xi(S_\xi)$ is a spectral set of $T$.

Now put $\mu = v(\lambda) = (\lambda-(r-1))/(\lambda-(r+1))$. According to theorem 1, $\varphi \circ v^{-1}(\mu) \in \bar{O}_e[S; T]$ for every $\varphi(\lambda) \in \bar{O}_r$.

Definition. *For every $\varphi \in \bar{O}_r$, we define $\varphi(A) = \varphi \circ v^{-1}(T)$, where $\varphi \circ v^{-1}(T)$ is defined as in* [2].

Theorem 2. *The mapping $\varphi(\lambda) \rightarrow \varphi(A)$ of $\bar{O}_r$ into $L(H)$ is an isomorphism, and has the properties*: (j) $||\varphi(A)|| \leqq \text{l.u.b.}_{Re\lambda \leqq r} |\varphi(\lambda)|$, (jj) *if $(\varphi_n(\lambda))$ is a bounded (on $Re\lambda \leqq r$) sequence in $\bar{O}_r$ and $\varphi_n(\lambda) \rightarrow \varphi(A)$ uniformly on every compact contained in $Re\lambda \leqq r$, then $\varphi_n(A) \rightarrow \varphi(A)$ strongly.*

Proof. If $\varphi_1, \varphi_2 \in \bar{O}_r$, then

$$(\lambda_1\varphi_1+\lambda_2\varphi_2) \circ v^{-1}(\mu) = \lambda_1\varphi_1 \circ v^{-1}(\mu)+\lambda\varphi_2 \circ v^{-1}(\mu),$$

$$(\varphi_1\varphi_2) \circ v^{-1}(\mu) = \varphi_1 \circ v^{-1}(\mu) \cdot \varphi_2 \circ v^{-1}(\mu),$$

and by using the operational calculus with functions of $\bar{O}_e[S; T]$ it follows that

$$(\lambda_1\varphi_1+\lambda_2\varphi_2)(A) = \lambda_1\varphi_1(A)+\lambda_2\varphi_2(A), \qquad (\varphi_1\varphi_2)(A) = \varphi_1(A)\varphi_2(A).$$

The property (j) can be proved in the same manner. To prove (jj), we remark that $\varphi(\lambda) \in \bar{O}_r$, and that $\varphi_n \circ v^{-1}(\mu) \rightarrow \varphi \circ v^{-1}(\mu)$ in the sense precised in (ii); hence $\varphi_n \circ v^{-1}(T) \rightarrow \varphi \circ v^{-1}(T)$. It follows that $\varphi_n(A) \rightarrow \varphi(A)$ strongly.

Theorem 3. *$e^{tA}$ is a strongly continuous semi-group and $A$ is its generator.*

Proof. According to theorem 1, from $e^{t\lambda} \in \bar{O}_r$ (for $t \geqq 0$), $|e^{t\lambda}| \leqq e^{tr}$ for $Re\lambda \leqq r$, and $e^{(t_1+t_2)\lambda} = e^{t_1\lambda}\, e^{t_2\lambda}$ it follows that $e^{tA}$ is a semi-group of operators on $H$. The fact that $e^{tA} \rightarrow I$ strongly for $t \rightarrow +0$ follows from (jj). Let $A'$ be the generator of $e^{tA}$. We have $Ay = [(r+1)T-r+1]x$ for $y = (T-I)x$, $x \in H$. Hence

$$\frac{1}{\varepsilon}(e^{\varepsilon A}-I)y = \frac{1}{\varepsilon}[e^{\varepsilon v^{-1}(\mu)}-1](\mu-1)\Big|_{\mu=T} x$$

$$= \frac{1}{\varepsilon}[(r+1)\mu-(r-1)]\int_0^\varepsilon e^{tv^{-1}(\mu)}dt\Big|_{\mu=T} x$$

$$=\{[(r+1)T-(r-1)I]\frac{1}{\varepsilon}\int_0^\varepsilon e^{tA}\,dt\}x \rightarrow [(r+1)T-(r-1)I]x = Ay.$$

Thus $A \subseteqq A'$. But $||e^{tA}|| \leqq e^{tr}$, so that using a method due to B. Sz.–Nagy ([4], p. 400), we get that $(A'-\xi I)^{-1}$ exists for $\xi > r$. By (*), it follows $(A'-\xi I)^{-1} = (A-\xi I)^{-1}$ and hence $A' = A$.

THEOREM 4. *The mapping $\varphi(\lambda) \to \varphi(A)$ is uniquely determined by the properties stated in theorems* 2 *and* 3.

PROOF. Let $\varphi(\lambda) \to \tilde{\varphi}(A)$ be a mapping satisfying the properties formulated in theorems 2 and 3. If $T_t = \widetilde{\exp}\,(tA)$, then by Hille's exponential representation ([1], p. 189) $T_t$ and $e^{tA}$, having the same generator, are identical. Thus if

$$v_\varepsilon(\lambda) = [e^{\varepsilon\lambda}-(\varepsilon r+1-\varepsilon)][e^{\varepsilon\lambda}-(\varepsilon r+1+\varepsilon)]^{-1},$$

then

$$\begin{aligned} v_\varepsilon(A) &= [e^{\varepsilon A}-(\varepsilon r+1-\varepsilon)I][e^{\varepsilon A}-(\varepsilon r+1+\varepsilon)I]^{-1} \\ &= [T_\varepsilon-(\varepsilon r+1-\varepsilon)I][T_\varepsilon-(\varepsilon r+1+\varepsilon)I]^{-1} = \tilde{v}_\varepsilon(A). \end{aligned}$$

But $v_\varepsilon(\lambda) \to v(\lambda)$ in the sense given in theorem 2. Consequently, $v_\varepsilon(A) \to v(A)$, and $\tilde{v}_\varepsilon(A) \to \tilde{v}(A)$ strongly for $\varepsilon \to +0$; hence $v(A) = T = \tilde{v}(A)$. Thus $v^n(A) = \tilde{v}^n(A)$ for every $n = 0, 1, 2, \ldots$ It follows that the mapping $\varphi(\lambda) \to \tilde{\varphi}(A)$ and that given in the definition are equal for $\varphi = v^n$, $n = 0, 1, \ldots$. But the algebra generated by these functions is dense (in the sense precised in theorem 2) in $\bar{O}_r$; hence $\tilde{\varphi}(A) = \varphi(A)$ for any $\varphi \in \bar{O}_r$.

## BIBLIOGRAPHY

E. HILLE

[1] Functional Analysis and Semi-groups. New York, 1948.

C. FOIAŞ

[2] La mesure harmonique spectrale et la théorie spectrale des opérateurs généraux d'un espace de Hilbert. Bull. de la Soc. Math. de France 85, (1957).

J. v. NEUMANN

[3] Eine Spektraltheorie für allgemeine Operatoren eines unitären Raumes, Math. Nachr. 4, (1951).

F. RIESZ and B. SZ.-NAGY

[4] Leçons d'analyse fonctionnelle, Budapest, 1953.

(Oblatum 6-3-58).

Institutul de Matematica,
Bucureşti.