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The Jordan-Hölder-Schreier Theorem for General Algebraic Systems

by

N. J. S. Hughes

Introduction

We wish to give an axiomatic proof of Zassenhaus' Lemma and the Jordan-Hölder-Schreier Refinement Theorem which will apply to such systems as groups, rings and lattices as well as to abstract algebraic systems.

A set G is an algebraic system with set of operators Ω (or an Ω-system) if every ω in Ω is an operator on G.

By a *type* of algebraic system (of which the systems mentioned above, each with given notation, are examples) we mean a class of algebraic systems, with a prescribed set of operators,[1]) so that, for example, we consider additive and multiplicative groups as distinct types.

We shall consider systems of some (unspecified) type T with set of operators $\Omega(T)$ and we require that, if G is a system of type T (or T-system), then any $\Omega(T)$-system isomorphic to G shall also be of type T.

We obtain for T-systems an analogue of normal subgroup or, more exactly, of the relation "H is a normal subgroup of K" by defining the binary relation N to be a T-normality relation if, whenever H and K are in the relation N (denoted by HNK), then H and K are T-systems, such that $H \leqq K$ (or H is a T-subsystem of K) and also there exists a T-system (the factor system), denoted by K/H, and a mapping $\psi_{H,K}$, single valued but not assumed to be a homomorphism, of K onto K/H.

N, K/H and $\psi_{H,K}$ are to be well-defined in the sense that, whenever H is a T-subsystem of K, whether or not HNK is determined and, if so, K/H and $\psi_{H,K}$ are uniquely determined.

In order to prove Zassenhaus' Lemma and the Jordan-Hölder-Schreier Theorem, we require further axioms. It is convenient to place these not on N but on a lattice, whose elements are T-subsystems of G. The proof of Zassenhaus' Lemma, which

[1]) This restriction is for convenience only.

generalises that of Jacobson (1) for groups, does not use homomorphisms (except isomorphisms) or homomorphic equivalences (Goldie (1)).

Finally, the general results are applied to the particular cases of groups, rings, quasi-groups and modular lattices.

1. Algebraic Systems

We say that (a set) G is an algebraic system with respect to (a set of operators) Ω (or an Ω-system), if every element ω of Ω is an operator of definite order (not necessarily finite) on G. That is, if ω has order r, for every $x_1, \ldots, x_r$ in G, there is defined a (possibly empty) subset of G, denoted by

$$\omega(x_1, \ldots, x_r).$$

Thus G is a groupoid when Ω consists of one single-valued binary operator.

If H is a subset of G (denoted by $H \leqq G$ or by $H < G$ if the subset is proper), it is a subsystem (or Ω-subsystem) of G if, for any ω in Ω of order r and $x_1, \ldots, x_r$ in H,

$$\omega(x_1, \ldots, x_r) \leqq H,$$

so that H is itself an Ω-system.

If G and G' are both Ω-systems, they are Ω-isomorphic (denoted by $G \cong G'$) if there exists a one-one mapping ψ of G onto G', such that, for any ω in Ω of order r and $x_1, \ldots, x_r$ in G,

$$(\omega(x_1, \ldots, x_r))\psi = \omega(x_1\psi, \ldots, x_r\psi).$$

Isomorphic systems are algebraically identical.

We shall write

$$G \cong G'; \ x \to x\psi \quad (x \in G)$$

to show that G and G' are isomorphic by a specific isomorphism ψ.

A *type* T of algebraic systems defines a set $\Omega(T)$ of operators and the algebraic systems of type T (T-systems) form a (well-defined) class of $\Omega(T)$-systems. We require that, if G is a T-system and G' an $\Omega(T)$-system, such that $G \cong G'$, then G' is also a T-system.

A T-subsystem of the T-system G is an $\Omega(T)$-subsystem, which is itself a T-system.

2. Normality relations and N-lattices

We shall say that the relation N is a T-normality relation if, whenever H and K are in the relation N (denoted by HNK), H is a T-subsystem of the T-system K and there is defined a T-system, denoted by K/H and called the factor system, and a single valued mapping $\psi_{H,K}$ of K onto K/H.

If G and G' are T-systems and

$$G \cong G'; \quad x \to x\varphi \quad (x \in G),$$

HNG and $H' = H\varphi$, then $H'NG'$ and

$$G/H \cong G'/H'; \quad x\psi_{H,G} \to (x\varphi)\psi_{H',G'} \quad (x \in G).$$

Let G be a T-system and L be a lattice whose elements are T-subsystems of G, the partial ordering being set inclusion, and, for A, $B \in L$, the glb. (which need not be the set intersection) be denoted by $A \cap_L B$ (or $A \cap B$) and the lub. by $A \cup_L B$ (or $A \cup B$ or, if multiplication is not otherwise required, AB).

If N is a T-normality relation, we say that L is an N-lattice on G if $G \in L$ and, for any A, B, $C \in L$, the following conditions are satisfied.

1) If ANB and $A \leqq C \leqq B$, then
 i) ANC,
 ii) $x\psi_{A,C} = x\psi_{A,B} \quad (x \in C)$,
(we will write Ax for $x\psi_{A,B}$ when there is no risk of ambiguity: if $\cap_L$ is set intersection, then $x\psi_{A,B}$ is independent of B)
 iii) CNB if and only if $(C/A)N(B/A)$ and then
$$B/C \cong (B/A)/(C/A); \quad Cx \to (C/A)(Ax) \quad (x \in B).$$

2) If ANB and CNB, then $(A \cup C)NB$.

3) If ANB and $C \leqq B$, then
 i) $(A \cap C)NC$,
 ii) $C\psi_{A,B} = (A \cup C)/A$,
 iii) $C/(A \cap C) \cong (A \cup C)/A; \quad (A \cap C)x \to Ax \quad (x \in C)$.

3. Zassenhaus Lemma

THEOREM 1. *If G is a T-system, N a T-normality relation, L an N-lattice on G and A, B, C, D are in L and such that ANB and CND, then*
 i) $A \cup (B \cap C)NA \cup (B \cap D)$,
 ii) $C \cup (D \cap A)NC \cup (D \cap B)$,
 iii) $(A \cap D) \cup (C \cap B)N(B \cap D)$,
 iv) $$\frac{A \cup (B \cap D)}{A \cup (B \cap C)} \cong \frac{B \cap D}{(A \cap D) \cup (C \cap B)} \cong \frac{C \cup (D \cap B)}{C \cup (D \cap A)}.$$

In §§ 3, 4 and 5, we shall write HK for $H \cup K$.

By symmetry, we need only prove i), iii) and the left hand isomorphism in iv).

Since

$$A \leqq A(B \cap C) \leqq A(B \cap D) \leqq B,$$

i) is satisfied, by 1), provided that

$$\frac{A(B \cap C)}{A} N \frac{A(B \cap D)}{A}$$

and then

4) $$\frac{A(B \cap D)/A}{A(B \cap C)/A} \cong \frac{A(B \cap D)}{A(B \cap C)}.$$

Now $A \cap (B \cap D) = A \cap D$ and, by 3),

5) $$(A \cap D)N(B \cap D)$$

and we have an isomorphism φ

$$\frac{B \cap D}{A \cap D} \cong \frac{A(B \cap D)}{A}; \ (A \cap D)x \to Ax \quad (x \in B \cap D).$$

By symmetry, $(C \cap B)N(B \cap D)$ and, if $E = (A \cap D)(C \cap B)$, by 5) and 2), $EN(B \cap D)$ and iii) is satisfied.

Since $A \cap D \leqq E \leqq B \cap D$, by 5) and 1), $E/(A \cap D)N(B \cap D)/(A \cap D)$ and also

6) $$\frac{B \cap D}{E} \cong \frac{(B \cap D)/(A \cap D)}{E/(A \cap D)}.$$

Now

$$\begin{aligned} \varphi(E/(A \cap D)) &= (Ax; \quad x \in E) \\ &= E\psi_{A,B} \\ &= AE/A, \quad \text{(by 3))} \\ &= A(B \cap C)/A \end{aligned}$$

and therefore, since φ is an isomorphism and isomorphic systems are algebraically identical,

$$A(B \cap C)/ANA(B \cap D)/A$$

and i) is satisfied. We also have

7) $$\frac{(B \cap D)/(A \cap D)}{E/(A \cap D)} \cong \frac{A(B \cap D)/A}{A(B \cap C)/A}.$$

The left hand isomorphism in iv) now follows from 6), 7) and 4).

4. (L, N)-series and refinements

If G is a T-system, N a T-normality relation and L an N-lattice on G, we say that the series

$$H_0, H_1, \ldots, H_m$$

is an (L, N)-series for G if $H_0 = G$ and, for $i = 1, \ldots, m$, $H_i \in L$ and $H_i N H_{i-1}$.

The T-systems H_{i-1}/H_i are the *factors* of the series and two series are *equivalent* if their factors (arranged in suitable order) are isomorphic in pairs.

THEOREM 2. Any two (L, N)-series for G have equivalent refinements.

Let the two series be $H_0, \ldots, H_m$ and $K_0, \ldots, K_n$ and let

$$\begin{aligned}
H_{ij} &= H_i(H_{i-1} \cap K_j) && (i = 1, \ldots, m;\ j = 0, \ldots, n),\\
H_{m+1,j} &= H_m \cap K_j && (j = 0, \ldots, n),\\
K_{ji} &= K_j(H_{j-1} \cap H_i) && (i = 0, \ldots, m;\ j = 1, \ldots, n),\\
K_{n+1,i} &= K_n \cap H_i && (i = 0, \ldots, m).
\end{aligned}$$

We clearly have

$$\begin{aligned}
&H_{i-1} = H_{i0} \geqq H_{ij} \geqq H_i && (i = 1, \ldots, m;\ j = 1, \ldots, n),\\
&H_m = H_{m+1,0} \geqq H_{m+1,j} && (j = 1, \ldots, n),\\
&K_{j-1} = K_{j0} \geqq K_{ji} \geqq K_j && (i = 1, \ldots, m;\ j = 1, \ldots, n),\\
&K_n = K_{n+1,0} \geqq K_{n+1,i} && (i = 1, \ldots, m).
\end{aligned}$$

By Theorem 1, for $i = 1, \ldots, m$, $j = 1, \ldots, n$, we have $H_{ij} N H_{i,j-1}$, $K_{ji} N K_{j,i-1}$, $H_{i,j-1}/H_{ij} \cong K_{j,i-1}/K_{ji}$.

For $j = 1, \ldots, n$, $H_{m+1,j-1} \leqq K_{j-1}$ and $H_{m+1,j-1} \cap K_j = H_{m+1,j}$ and also $K_j H_{m+1,j-1} = K_{jm}$, so that, by 3),

$H_{m+1,j}\, N H_{m+1,j-1}$ and $H_{m+1,j-1}/H_{m+1,j} \cong K_{jm}/K_j$.

Similarly, for $i = 1, \ldots, m$,

$K_{n+1,i}\, N K_{n+1,i-1}$ and $K_{n+1,i-1}/K_{n+1,i} \cong H_{in}/H_i$.

If we interpolate $H_{i1}, \ldots, H_{in}$ between H_{i-1} and H_i, for $i = 1, \ldots, m$, and add $H_{m+1,1}, \ldots, H_{m+1,n}$ after H_m, with the corresponding operations for the series $K_0, \ldots, K_n$, we obtain the required equivalent refinements.

5. Jordan-Hölder-Schreier Theorem

We say that the (L, N)-series $H_0, \ldots, H_m$ is non-repetitive if, for $i = 1, \ldots, m$, $H_i < H_{i-1}$.

We wish to be able to drop repeated terms from two equivalent series and obtain non-repetitive equivalent series.

If A, $B \in L$ and ANB, by 1), ANA and $(A/A)N(B/A)$ and also

$$B/A \cong (B/A)/(A/A); \quad Ax \rightarrow (A/A)(Ax) \quad (x \in B).$$

If ANA and $C \leqq A$, by 3), CNC and $C/C \cong A/A$.

By taking $C = A \cap B$, we deduce that, whenever A, $B \in L$, ANA and BNB, $A/A \cong B/B$.

This T-system therefore is determined to an isomorphism (except when the relation ANB, A, $B \in L$, is never satisfied) and may be denoted by $U = U(L, N)$. Every repetition in an (L, N)-series has U as factor system.

If we assume that, whenever A, $B \in L$, ANB and $A < B$, B/A is not isomorphic to U, then two equivalent series must have the same number of repetitions and, by omitting the repeated terms, we obtain equivalent non-repetitive series.

The following results may now be deduced from Theorem 2.

THEOREM 3. (Jordan-Hölder-Schreier). *If G is a T-system, N a T-normality relation and L an N-lattice on G, such that, if A, $B \in L$ and ANB, then $B/A \cong U(L, N)$ only if $A = B$, then any two non-repetitive (L, N)-series for G have equivalent non-repetitive refinements.*

COROLLARY 1. *Any two (L, N)-composition series (non-repetitive series without proper refinement) for G are equivalent.*

COROLLARY 2. *If G has at least one composition series, then any non-repetitive (L, N)-series for G may be refined to a composition series.*

6. Applications

To prove a Jordan-Hölder-Schreier Theorem for an algebraic system G of definite type, it is sufficient to define a suitable normality relation N and an N-lattice L and to verify that conditions 1), 2) and 3) are satisfied.[2])

Groups and Rings.

Let G be a group (operation $+$).

We define ANB if and only if A is a normal subgroup of B, B/A is the usual factor (difference) group and $\psi_{A,B}$ given by

8) $$x\psi_{A,B} = A + x \quad (x \in B).$$

We may take L to be the lattice of subgroups of G which are invariant under any set of endomorphisms of G.

[2]) In each of the systems concerned below, K/H has exactly one element if and only if $H = K$.

For rings (not necessarily associative but with operations $+$ and $\times$) we define ANB to mean that A is an ideal of B, B/A is the difference ring and $\psi_{A,B}$ given by 8).

For a ring G, we may take L to be the lattice of all subrings or any sublattice of this.

If A is an ideal of B and C is a subring (ideal) of B, then $A+C$ is a subring (ideal) of B and we need only verify that the isomorphisms in 1) and 3) are ring (and not merely group) isomorphisms.

Quasigroups

By a right quasigroup we mean a (multiplicative) groupoid G, such that, for any a, $b \in G$, the equation

$$xa = b$$

has a unique solution, denoted by b/a, in G.

An equivalent condition is that the right multiplications of G shall be permutations.

We shall denote a/a by e_a and the set of all e_a $(a \in G)$ by E_G.

We call G a right loop if it has a left identity e.

A subset H of G is a subright quasigroup if, for any a, b in H, both ab and b/a are in H.

A subright quasigroup H of K is normal in K (HNK) if, for any h_1, $h_2 \in H$ and x, $y \in K$, we have

i) $(h_1x)(h_2y) = h(xy)$ (for some $h \in H$),

ii) $h_1(xy) = (hx)y$ (for some $h \in H$),

iii) $E_K \leqq H$.

The relation $y/x \in H$ (or $y \in Hx$) is easily seen to be a homographic equivalence on K, the equivalence classes (H-cosets) form a right loop K/H, with identity H, when multiplication is defined by

$$(Hx)(Hy) = H(xy) \quad (x,\ y \in K),$$

and then the mapping $\psi_{H,K}$, defined by

$$x\psi_{H,K} = Hx \quad (x \in K),$$

is a homomorphism.

Conversely, if ψ is a homomorphism of the right quasi-group G onto the right loop G' (with left identity e') and H is the inverse image of e', then HNG and

$$G/H \cong G'; \quad Hx \to x\psi \quad (x \in G).$$

For any right quasigroup G, we may take L to be the lattice of all subright quasigroups containing some fixed element of G.

If A, B, $C \in L$, ANB and $C \leqq B$, we may verify that

$$A \cup_L C = AC = (ac;\ a \in A,\ b \in B)$$

and that, if also CNB, then $(AC)NB$.

Conditions 1) and 3) are also satisfied and the Jordan-Hölder-Schreier Theorem for right quasigroups follows.

The Theorem for quasigroups may now be deduced, H being normal in K (considered as a right and left quasigroup simultaneously) if and only if it is the inverse image of the identity in a homomorphism of K onto a loop.[3])

We easily verify that the right and left cosets, Hx and xH $(x \in K)$, coincide.

Modular Lattices

Let G be a modular lattice with operations $\cup$ and $\cap$ and with a greatest element.

For any $a \in G$, $[a]$ denotes the principal ideal consisting of all x, such that $x \leqq a$ and, for any b, $b \geqq a$, $[a, b]$ denotes the interval consisting of all x, such that $a \leqq x \leqq b$.

We define a sublattice H of K to be normal in K (HNK) if and only if H is a principal ideal $[a]$ of K and then

$$x\psi_{H,K} = a \cup x \quad (x \in K).$$

We take L to be the lattice of all principal ideals of G and may identify L with G by defining

$$[a] \cup_L [b] = [a \cup b],\ [a] \cap_L [b] = [a \cap b],$$

so that aNb if and only if $a \leqq b$, $b/a = [a, b]$ and

$$x\psi_{a,b} = a \cup x \quad (x \leqq b).$$

The conditions 1), 2) and 3) are easily verified, the isomorphism in 3), for a, $c \leqq b$, being given by

$$x \rightarrow a \cup x \quad (x \in [a \cap c,\ c])$$

with the inverse isomorphism

$$x \rightarrow c \cap x \quad (x \in [a,\ b]).$$

The Jordan-Hölder-Schreier Theorem in this system is equivalent to the theorem that two chains with the same greatest and least elements have refinements, whose intervals are lattice isomorphic in pairs.[4])

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[3]) Albert 1), 2).

[4]) Birkhoff 1), pp. 72—73.

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