COMPOSITIO MATHEMATICA

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Compositio Mathematica, tome 13 (1956-1958), p. 81-102
<http://www.numdam.org/item?id=CM_1956-1958__13__81_0>

Arithmetical Extensions of Relational Systems

by

Alfred Tarski and Robert L. Vaught

Introduction

We consider mathematical systems each of which is formed by an arbitrary set, some relations among the elements of the set, and possibly some operations on these elements. Two such systems are called *similar* if, roughly speaking, they have the same number of relations and the same number of operations, and corresponding relations and operations have the same ranks. Two similar systems $\mathfrak{R}$ and $\mathfrak{S}$ are called *arithmetically equivalent* if every sentence of the first-order predicate logic which is true in one of these systems is also true in the other; if, in addition $\mathfrak{R}$ is a subsystem of $\mathfrak{S}$, then $\mathfrak{S}$ may be called an *arithmetically equivalent extension* of $\mathfrak{R}$. We introduce in this paper a stronger notion, that of *arithmetical extension*. A system $\mathfrak{S}$ is said to be an arithmetical extension of a subsystem $\mathfrak{R}$ of $\mathfrak{S}$ if, whenever some elements of $\mathfrak{R}$ satisfy a formula (of the first-order predicate logic) in $\mathfrak{R}$, they also satisfy it in $\mathfrak{S}$, and conversely. The notion of arithmetical extension proves to be very useful in applications. As we shall see, in many cases it is easy to show that a given system $\mathfrak{S}$ is an arithmetical extension of a given system $\mathfrak{R}$, although a direct proof, by some other method, of the consequent fact that $\mathfrak{R}$ and $\mathfrak{S}$ are arithmetically equivalent is difficult or unknown. A simple explanation of this phenomenon is that, in establishing properties holding for all formulas, we can apply an inductive procedure, while, in general, no such procedure can be used to establish properties holding exclusively for sentences (i.e., formulas without free variables). For essentially the same reason it is much easier to define the truth of sentences in terms of the satisfaction of formulas — instead of defining the notion of truth directly; cf. [14], p. 307 (the numbers in brackets refer to the bibliography at the end of the paper).

The paper is divided into three sections. In § 1 we define, in a rigorous way, the main notions involved in our discussion and we study general properties of arithmetical extensions; in particular we establish some useful criteria (necessary and sufficient conditions) for one system to be an arithmetical extension of another system. § 2 deals with problems related to the well known Löwenheim-Skolem theorem. In its original form this theorem amounts to the statement that every infinite relational system is arithmetically equivalent to a denumerable system. Using some elementary properties of arithmetical extensions we give what seems to us a new, very simple, and natural proof of this theorem. Moreover, we obtain some improvements, not only of the original Löwenheim-Skolem theorem, but also of its generalizations which can be found in the literature. Indeed, we show that, for every infinite relational system $\mathfrak{R}$ of power $\mathfrak{a}$ and every infinite cardinal $\mathfrak{b}$, there is a system $\mathfrak{S}$ of power $\mathfrak{b}$ such that $\mathfrak{R}$ is an arithmetical extension of $\mathfrak{S}$ in case $\mathfrak{b} < \mathfrak{a}$, and $\mathfrak{S}$ is an arithmetical extension of $\mathfrak{R}$ in case $\mathfrak{a} \leqq \mathfrak{b}$.

In the first theorem of § 3 we state a purely algebraic condition which is sufficient (though not necessary) for a system $\mathfrak{S}$ to be an arithmetical extension of a system $\mathfrak{R}$. This condition is that, given any finite set A' of elements of $\mathfrak{R}$ and any single element b of $\mathfrak{S}$, there is an automorphism of $\mathfrak{S}$ which leaves all the elements of A' unchanged and carries b into an element of $\mathfrak{R}$. A number of examples are given in which this theorem is applied to establish the arithmetical equivalence of known mathematical systems. The most important consequence thus obtained is that any two free algebras with infinitely many generators over the same class of algebraic systems (e.g., the class of all groups or of all lattices replace by) are arithmetically equivalent. When applied to groups, this result presents a partial confirmation of a conjecture made by Tarski that any two free groups with at least two generators are arithmetically equivalent. This conjecture (which was the stimulus for the investigations in § 3) still remains an open question. It is closely related to another problem which also remains open, the decision problem for the elementary theory of free groups (see [18], p. 85).[1]

[1]) The notion of arithmetical extension and most of the results in § 1 and § 2 are due to Tarski and were discussed in his seminar at the University of California, Berkeley, during the academic year 1952—53; however, Theorems 1.11 and 2.2 were obtained by both authors independently, and Theorem 1.12 and the examples following it were found by Vaught. The results in § 3 are due to Vaught. Most of them were included in Chapter 3 of his doctoral dissertation [19], which was prepared under Tarski's guidance, and submitted to the University of California in June, 1954; they were summarized in [21]. This paper was prepared for publi-

§ 1. General properties of arithmetical extensions

The meaning of notions and symbols to be used will, for the most part, be explained; for more detailed information the reader is referred to [16], [17], and [18].

Greek letters $\alpha, \beta, \ldots, \xi, \eta, \ldots$ will represent arbitrary ordinals; finite ordinals, i.e., natural numbers, will be represented by the letters $k, l, m, n, \ldots$. Given an ordinal α, an α-termed sequence will be represented by $x = \langle x_0, \ldots, x_\xi, \ldots \rangle_{\xi<\alpha}$ or sometimes simply by $\langle x_0, \ldots, x_\xi, \ldots \rangle$. Ordinary infinite sequences, with $\alpha = \omega$, and finite n-termed sequences will be represented by $\langle x_0, \ldots, x_n, \ldots \rangle$ and $\langle x_0, \ldots, x_{n-1} \rangle$, respectively. Given a sequence $x = \langle x_0, \ldots, x_n, \ldots \rangle$ of members of some set, a natural number k, and an element a of the set, we denote by $x(k/a)$, or by $\langle x_0, \ldots x_{k-1}, a, x_{k+1}, \ldots \rangle$, the sequence $y = \langle y_0, \ldots, y_n, \ldots \rangle$ such that $y_k = a$, and $y_n = x_n$ for $n \neq k$. For any given set A, A^α denotes the set of all α-termed sequences of which all the terms belong to A; by $A^{(\omega)}$ we denote the set of all sequences $x \in A^\omega$ which are eventually constant, i.e., such that, for some m, $x_n = x_m$ for all $n \geqq m$.

By a *(finitary) relation*, and, specifically, by a *relation of rank* n, among the elements of A, we understand an arbitrary subset of A^n. A *relational system* is a sequence $\mathfrak{R} = \langle A, R_0, \ldots, R_\xi, \ldots \rangle_{\xi<\alpha}$ in which A is a non-empty set and each R_ξ is a relation among the elements of A; α is called the order of $\mathfrak{R}$. Instead of "relational system" we shall sometimes say simply "system". Elements of the set A are referred to as elements of the system $\mathfrak{R}$; the system $\mathfrak{R}$ is called infinite if the set A is infinite, and we speak of the power of $\mathfrak{R}$ meaning the power of A. In most discussions it is tacitly assumed that all the systems $\langle A, R_0, \ldots, R_\xi, \ldots \rangle_{\xi<\alpha}$ involved are *similar*, i.e., that they all have the same order α and, for each $\xi < \alpha$, all relations R_ξ have the same rank n_ξ. By means of well known devices, mathematical systems formed by a non-empty set A, some relations among its elements, some operations on its elements, and some distinguished members of it can also be treated as relational systems. However, at two places it will be convenient to use systems with distinguished elements,

$$\langle A, R_0, \ldots, R_\xi, \ldots; a_0, \ldots, a_\eta, \ldots \rangle_{\xi<\alpha, \eta<\beta}$$

explicitly; the necessary modifications in earlier definitions and theorems which would justify this use are clear and will be omitted.

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cation while Tarski was working on a research project in the foundations of mathematics sponsored by the National Science Foundation, U.S.A.

In order to simplify the notation we shall henceforth restrict ourselves to an explicit discussion of relational systems $\langle A, R\rangle$ formed by a non-empty set A and a single ternary relation R. However, with two exceptions concerning systems with non-denumerable orders, all our results extend, with virtually no changes in the proofs, to arbitrary relational systems; the two exceptions will be pointed out and discussed.

The notions of *isomorphism* of two relational systems and of *automorphism* of a system are assumed to be known. The relational system $\mathfrak{R} = \langle A, R\rangle$ is said to be a *subsystem* of the system $\mathfrak{S} = \langle B, S\rangle$ if A is a subset of B ($A \subseteq B$) and $R = S \cap A^3$. Under the same conditions $\mathfrak{S}$ is called an *extension* of $\mathfrak{R}$; it is called a *proper extension* if, in addition, $A \neq B$.

Given a non-empty class $\boldsymbol{K}$ of systems, by the *union* of $\boldsymbol{K}$ we understand the system $\langle A, R\rangle$ in which A is the set-theoretical union of all sets B and R is the set theoretical union of all relations S occurring in the systems $\langle B, S\rangle$ of $\boldsymbol{K}$. In general, the union of $\boldsymbol{K}$ need not be an extension of each system belonging to $\boldsymbol{K}$; this is the case, however, under some additional assumptions, e.g., if $\boldsymbol{K}$ is a *directed* class of systems, i.e., if any two systems in $\boldsymbol{K}$ have a common extension which is also in $\boldsymbol{K}$.

We construct a formalized theory T of relational systems $\langle A, R\rangle$. T is simply the first-order predicate logic, with an infinite sequence $\langle v_0, \ldots, v_n, \ldots\rangle$ of *variables*; with four *logical constants*, the *negation symbol* $\sim$, the *conjunction symbol* $\wedge$, the *existential quantifier* $\bigvee$, and the *identity symbol* $\equiv$; and with a single non-logical constant, the *three-placed predicate* P. The *formulas* of T are the members of the smallest class which contains all the *atomic formulas*, $v_m \equiv v_n$ and $P(v_m, v_n, v_p)$ $(m, n, p = 0, 1, 2, \ldots)$, and is closed under the operations of forming the *negation* $\sim \phi$ of an expression ϕ, the *conjunction* $\phi \wedge \psi$ of two expressions ϕ and ψ, and the *existential quantification* $\bigvee v_k \phi$ of an expression ϕ under a variable v_k $(k = 0, 1, 2, \ldots)$. *Sentences* of T are formulas without free variables. When speaking of formulas and sentences without further qualification, we shall mean formulas and sentences of the theory T.

The notions of *satisfaction* and *truth* will play an essential part in our discussion and therefore will be defined here in a formal way. The definition of satisfaction is given in a recursive form (cf. [14], p. 311):

Definition 1.1. *We say that x* ***satisfies ϕ in a relational system*** $\mathfrak{R} = \langle A, R\rangle$ *if $x \in A^{(\omega)}$, ϕ is a formula, and one of the following five conditions holds*:

(i) *ϕ is of the form $v_m \equiv v_n$, where m and n are natural numbers, and $x_m = x_n$;*

(ii) ϕ is of the form $P(v_m, v_n, v_p)$, *where m, n, and p are natural numbers, and $\langle x_m, x_n, x_p\rangle \in R$;*

(iii) *ϕ is of the form $\sim \psi$, where ψ is a formula which is not satisfied by x;*

(iv) *ϕ is of the form $\psi' \wedge \psi''$, where ψ' and ψ'' are formulas which are both satisfied by x;*

(v) *ϕ is of the form $\bigvee v_k \psi$, where k is a natural number, ψ is a formula, and there is an element $a \in A$ such that $x(k/a)$ satisfies ψ.*

As an easy consequence of this definition we obtain

Theorem 1.2. *Suppose $\mathfrak{R} = \langle A, R\rangle$* and *$\mathfrak{S} = \langle B, S\rangle$ are relational systems, the function h maps $\mathfrak{R}$ isomorphically onto $\mathfrak{S}$, x is any sequence in $A^{(\omega)}$, and ϕ is a formula. Then $\langle x_0, \ldots, x_n, \ldots\rangle$ satisfies ϕ in $\mathfrak{R}$ if and only if $\langle h(x_0), \ldots, h(x_n), \ldots\rangle$ satisfies ϕ in $\mathfrak{S}$.*

Definition 1.3. *A sentence σ is said to be* ***true in the relational system*** *$\mathfrak{R} = \langle A, R\rangle$ if every sequence $x \in A^{(\omega)}$ satisfies σ in $\mathfrak{R}$. (Under the same conditions we say that $\mathfrak{R}$ is a* ***model of*** *σ.)*

As is readily seen, the word "*every*" can be replaced in 1.3 by "*some*".

The notions of *arithmetical equivalence* and *arithmetical extension* will now be formally defined.

Definition 1.4. *The systems $\mathfrak{R}$ and $\mathfrak{S}$ are said to be* ***arithmetically*** *(or* ***elementarily****)* ***equivalent*** *if every sentence which is true in $\mathfrak{R}$ is also true in $\mathfrak{S}$, and conversely.*

Corollary 1.5. *Any two systems which are isomorphic are arithmetically equivalent.*

Proof: by 1.2, 1.3, and 1.4.

Definition 1.6. *The system $\mathfrak{S} = \langle B, S\rangle$ is called an* ***arithmetical extension of the system*** *$\mathfrak{R} = \langle A, R\rangle$ if the following two conditions are satisfied:*

(i) *$\mathfrak{S}$ is an extension of $\mathfrak{R}$;*

(ii) *for every formula ϕ and every sequence $x \in A^{(\omega)}$, if x satisfies ϕ in $\mathfrak{R}$, it also satisfies ϕ in $\mathfrak{S}$, and conversely.*

By considering negations of formulas and making use of 1.1(iii), one easily sees that an equivalent formulation of Definition 1.6 is obtained if in condition 1.6(ii) the implication in both directions is replaced by the implication in either of the two directions. An analogous remark applies to Definition 1.4.

Corollary 1.7. *If $\mathfrak{S}$ is an arithmetical extension of $\mathfrak{R}$, then $\mathfrak{R}$ and $\mathfrak{S}$ are arithmetically equivalent.*

Proof: by 1.3, 1.4, and 1.6.

Theorem 1.8. (i) *Every system is an arithmetical extension of itself.*

(ii) *If $\mathfrak{R}'$ is an arithmetical extension of $\mathfrak{R}$ and $\mathfrak{R}''$ is an arithmetical extension of $\mathfrak{R}'$, then $\mathfrak{R}''$ is an arithmetical extension of $\mathfrak{R}$.*

(iii) *If a system $\mathfrak{R}$ and an extension $\mathfrak{R}'$ of $\mathfrak{R}$ have a common arithmetical extension $\mathfrak{R}''$, then $\mathfrak{R}'$ is an arithmetical extension of $\mathfrak{R}$.*

Proof: obvious, using 1.6.

Another general property of arithmetical extensions is stated in the following theorem:

Theorem 1.9. *If $\boldsymbol{K}$ is a non-empty family of systems such that any two systems in $\boldsymbol{K}$ have a common arithmetical extension which is also in $\boldsymbol{K}$, then the union of $\boldsymbol{K}$ is a common arithmetical extension of all members of $\boldsymbol{K}$.*

Proof: Let $\mathfrak{R} = \langle A, R\rangle$ be the union of $\boldsymbol{K}$ and consider all formulas ϕ for which the following condition holds:

(1) if $\mathfrak{S} = \langle B, S\rangle$ is any system in $\boldsymbol{K}$ and $x \in B^{(\omega)}$, then x satisfies ϕ in $\mathfrak{R}$ if and only if x satisfies ϕ in $\mathfrak{S}$.

$\mathfrak{R}$ is obviously an extension of every member of $\boldsymbol{K}$. Hence it follows directly from 1.1(i), (ii) that (1) holds for all atomic formulas. Also, by 1.1(iii), (iv), if (1) holds for each of two formulas ϕ and ψ, then (1) holds as well for $\sim \phi$ and for $\phi \wedge \psi$.

Let now ϕ be any formula for which (1) holds and v_k be any variable. Let $\mathfrak{S} = \langle B, S\rangle$ be any system in $\boldsymbol{K}$ and x any sequence in $B^{(\omega)}$. If x satisfies $\bigvee v_k\phi$ in $\mathfrak{R}$, then, by 1.1(v), there is an element $a \in A$ such that the sequence $x(k/a)$ satisfies ϕ in $\mathfrak{R}$. By the definition of the union of a class of systems, there is a system $\mathfrak{S}' = \langle B', S'\rangle \in \boldsymbol{K}$ for which $a \in B'$. By hypothesis $\mathfrak{S}$ and $\mathfrak{S}'$ have a common arithmetical extension $\mathfrak{S}'' = \langle B'', S''\rangle$ in $\boldsymbol{K}$. Clearly, $x(k/a) \in B''^{(\omega)}$. Therefore, by (1), $x(k/a)$ satisfies ϕ in $\mathfrak{S}''$; hence, by 1.1(v), x satisfies $\bigvee v_k\phi$ in $\mathfrak{S}''$, and consequently by 1.6, x satisfies $\bigvee v_k\phi$ in $\mathfrak{S}$. In a similar (though simpler) way one shows that, if x satisfies $\bigvee v_k\phi$ in $\mathfrak{S}$, it also satisfies $\bigvee v_k\phi$ in $\mathfrak{R}$. Thus, whenever (1) holds for a formula ϕ, it holds for the formula $\bigvee v_k\phi$ as well.

By now applying the principle of induction for formulas (or the definition of a formula) of the theory T, we arrive at the conclusion that (1) holds for all formulas. Hence, by Definition 1.6,

$\mathfrak{R}$ is an arithmetical extension of every system belonging to $\boldsymbol{K}$, and this is what was to be proved.

It can easily be shown by means of examples that 1.9 (as opposed to 1.8) expresses a specific property of arithmetical extensions which applies neither to isomorphic extensions nor to arithmetically equivalent extensions.

In the next theorem we give a new characterization of arithmetical extension, which will prove very useful in applications. This characterization is simpler than the one originally given in 1.6 inasmuch as it avoids any semantical reference to the smaller of the two systems involved.

THEOREM 1.10. *The following two conditions are (severally) necessary and (jointly) sufficient for a system* $\mathfrak{S} = \langle B, S\rangle$ *to be an arithmetical extension of a system* $\mathfrak{R} = \langle A, R\rangle$:

(i) $\mathfrak{S}$ *is an extension of* $\mathfrak{R}$;

(ii) *for every formula* ϕ, *every natural number* k, *and every sequence* $x \in A^{(\omega)}$, *if* x *satisfies* $\bigvee v_k\phi$ *in* $\mathfrak{S}$, *then there is an element* $a \in A$ *such that* $x(k/a)$ *satisfies* ϕ *in* $\mathfrak{S}$.

PROOF: The necessity of the conditions clearly follows from 1.1(v) and 1.6.

To show the sufficiency, we assume that conditions (i) and (ii) hold and we derive from them condition 1.6(ii) by induction on the formula ϕ involved in this condition. From 1.1(i)—(iv), it obviously follows that 1.6(ii) holds for every atomic formula (and every sequence $x \in A^{(\omega)}$) and that 1.6(ii) holds for $\sim\phi$ and $\phi \wedge \psi$ whenever it holds for ϕ and ψ. Suppose now that 1.6(ii) holds for a given formula ϕ, and consider the formula $\bigvee v_k\phi$ (where k is any natural number). If a sequence $x \in A^{(\omega)}$ satisfies $\bigvee v_k\phi$ in $\mathfrak{S}$, then, by 1.10(ii), we can find an element $a \in A$ such that $x(k/a)$ satisfies ϕ in $\mathfrak{S}$; since $x(k/a) \in A^{(\omega)}$, we conclude from our inductive assumption that $x(k/a)$ satisfies ϕ in $\mathfrak{R}$, and therefore, by 1.1(v), x satisfies $\bigvee v_k\phi$ in $\mathfrak{R}$. By means of a similar (though still simpler) argument, one shows that also, conversely, if x satisfies $\bigvee v_k\phi$ in $\mathfrak{R}$, it satisfies $\bigvee v_k\phi$ in $\mathfrak{S}$ as well. Thus condition 1.6(ii) holds for $\bigvee v_k\phi$ whenever it holds for ϕ. It follows now that 1.6(ii) holds for every formula, and the proof is completed.

Still another characterization of the notion of arithmetical extension, and, indeed, a reduction of this notion to that of arithmetical equivalence is given in the following

THEOREM 1.11. *Given any relational system* $\mathfrak{R} = \langle A, R\rangle$ *and any extension* $\mathfrak{S} = \langle B, S\rangle$ *of* $\mathfrak{R}$, *the following two conditions are equivalent*:

(i) $\mathfrak{S}$ *is an arithmetical extension of* $\mathfrak{R}$;

(ii) *for every finite sequence* $\langle a_0, \ldots, a_{n-1}\rangle$ *of elements of* A, *the systems* $\langle A, R, a_0, \ldots, a_{n-1}\rangle$ *and* $\langle B, S, a_0, \ldots, a_{n-1}\rangle$ *are arithmetically equivalent.*

If, moreover, we let $\mathfrak{R}' = \langle A, R, a_0', \ldots, a_\xi', \ldots\rangle_{\xi<\alpha}$ *and* $\mathfrak{S}' = \langle B, S, a_0', \ldots, a_\xi', \ldots\rangle_{\xi<\alpha}$, *where* $\langle a_0', \ldots, a_\xi', \ldots\rangle_{\xi<\alpha}$ *is any (finite or transfinite) sequence such that the set of all its terms coincides with* A, *then condition* (i) *is equivalent, as well, to each of the following two conditions*:

(iii) $\mathfrak{S}'$ *is an arithmetical extension of* $\mathfrak{R}'$;

(iv) $\mathfrak{R}'$ *and* $\mathfrak{S}'$ *are arithmetically equivalent.*

PROOF: Condition (ii) is, of course, to be understood as referring, not to the original formalized theory, T, but to formalized theories T_n obtained from T by including in its vocabulary n new non-logical constants — the individual constants $c_0, \ldots, c_{n-1}$. Similarly, conditions (iii) and (iv) refer to the theory T' obtained by adding the α-termed sequence of distinct individual constants $\langle c_0, \ldots, c_\xi, \ldots\rangle_{\xi<\alpha}$.

To prove the theorem, we establish four implications: (i)→(iii) → (iv) → (ii) → (i). In the argument we use certain intuitively obvious properties of satisfaction and truth which apply to formalized theories with individual constants or concern connections between two theories that differ from each other only by the presence or absence of certain constants. These properties can be rigorously (and very easily) established by induction on formulas based upon formal definitions of satisfaction and truth for the theories involved.

To obtain (i) → (iii) suppose that (i) holds and consider a formula ϕ of T' and a sequence $x \in A^{(\omega)}$ which satisfies ϕ in $\mathfrak{R}'$. In general ϕ contains individual constants. For simplicity assume that ϕ contains only one such constant, say c_ξ. Let v_k be a variable not occurring in ϕ and let ϕ', be the formula obtained from ϕ by replacing c_ξ by v_k everywhere. By the definition of satisfaction for T', the sequence $x(k/a_\xi')$ satisfies ϕ' in $\mathfrak{R}'$. Since ϕ' contains no individual constants and hence is a formula of T, this sequence also satisfies ϕ' in $\mathfrak{R}$; therefore, by (i) and 1.6(ii), it satisfies ϕ' in $\mathfrak{S}$ and, hence, also in $\mathfrak{S}'$; consequently, x satisfies ϕ in $\mathfrak{S}'$. In case ϕ contains more than one individual constant, the proof is essentially the same. Thus, by 1.6, $\mathfrak{S}'$ is an arithmetical extension of $\mathfrak{R}'$ and (iii) holds.

The implication (iii) $\rightarrow$ (iv) follows directly from Corollary 1.7 (applied, not to the original theory T, but to the theory T'). The implication (iv) $\rightarrow$ (ii) is obvious.

Finally, to establish (ii) $\rightarrow$ (i), suppose that (ii) holds. Consider a formula ϕ (of T) and a sequence $x \in A^{(\omega)}$ which satisfies ϕ in $\mathfrak{R}$. Replace all the distinct variables $v_{k_0}, \ldots, v_{k_{n-1}}$ which occur free in ϕ by the constants $c_0, \ldots, c_{n-1}$, respectively; the resulting expression ϕ' is a sentence of T_n. Clearly, ϕ' is true in the system $\langle A, R, x_{k_0}, \ldots, x_{k_{n-1}}\rangle$. Hence, by (ii) and 1.3, ϕ is also true in $\langle B, S, x_{k_0}, \ldots, x_{k_{n-1}}\rangle$, and therefore x satisfies ϕ in $\mathfrak{S}$. Consequently, $\mathfrak{S}$ is an arithmetical extension of $\mathfrak{R}$ and (i) holds. This completes the proof.[2])

Theorems 1.10 and 1.11 have presented criteria for one relational system to be an arithmetical extension of another. In Theorem 1.12 we give a criterion for one system to be isomorphic to an arithmetical extension of another.

THEOREM 1.12. *Let T' be the formalized theory obtained by adjoining the sequence of distinct individual constants $\langle c_0, \ldots, c_\xi, \ldots\rangle_{\xi<\alpha}$ to the theory T, where α is the smallest ordinal with the power $\mathfrak{a}$. Let $\mathfrak{R} = \langle A, R\rangle$ be a relational system of the power $\mathfrak{a}$. Then, in order that a system $\mathfrak{S} = \langle B, S\rangle$ be an arithmetical extension of a system isomorphic to $\mathfrak{R}$, the following condition is necessary and sufficient:*

(i) *Given any set Σ of sentences of the theory T', if there exists a sequence $a \in A^\alpha$ such that every sentence of Σ is true in the system $\langle A, R, a_0, \ldots, a_\xi, \ldots\rangle_{\xi<\alpha}$, then there exists a sequence $b \in B^\alpha$ such that every sentence of Σ is true in the system $\langle B, S, b_0, \ldots, b_\xi, \ldots\rangle_{\xi<\alpha}$.*

PROOF. The necessity of (i) follows immediately from 1.11(i), (iv) and 1.5. To establish the sufficiency of (i), we suppose that (i) holds and let $\langle a_0, \ldots, a_\xi, \ldots\rangle_{\xi<\alpha}$ be a transfinite sequence, the set of whose terms coincides with A. Let Σ be the set of all sentences of T' true in the system $\mathfrak{R}' = \langle A, R, a_0, \ldots, a_\xi, \ldots\rangle_{\xi<\alpha}$. By (i), there exists a (finite or transfinite) sequence $b \in B^\alpha$ such that all sentences of Σ are true in the system $\mathfrak{S}' = \langle B, S, b_0, \ldots, b_\xi, \ldots\rangle_{\xi<\alpha}$. Let C be the set of all terms of the transfinite sequence b, and let

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[2]) It was noticed by Vaught that, using either 1.10 or 1.11, one can easily obtain the following result which concerns the notion of Γ-completeness, recently introduced by Henkin in [10]:

Every model of a complete and Γ-complete theory is an arithmetical extension of the subsystem formed by its Γ-designated elements.

$\mathfrak{T} = \langle C, S \cap C^3 \rangle$ and $\mathfrak{T}' = \langle C, S \cap C^3, b_0, \ldots, b_\xi, \ldots \rangle_{\xi < \alpha}$. All the sentences of the forms $c_\xi \equiv c_\eta$ or $P(c_\xi, c_\eta, c_\zeta)$ $(\xi, \eta, \zeta < \alpha)$ which are true in $\mathfrak{R}'$ are also true in $\mathfrak{S}'$, and hence are true in the subsystem $\mathfrak{T}'$ of $\mathfrak{S}'$; the same applies to the negations of these sentences. Consequently, the systems $\mathfrak{R}'$ and $\mathfrak{T}'$ are isomorphic. Therefore by **1.5**, all sentences true in $\mathfrak{T}'$ are true in $\mathfrak{R}'$ and, hence, in $\mathfrak{S}'$. By the remarks after **1.6**, it follows that $\mathfrak{S}'$ and $\mathfrak{T}'$ are arithmetically equivalent. Therefore, by **1.11**, $\mathfrak{S}$ is an arithmetical extension of $\mathfrak{T}$, which is isomorphic to $\mathfrak{R}$. This completes the proof.

It may be remarked that condition (i) of **1.12** may be considered as stating that certain propositions involving infinite conjunctions must hold in $\mathfrak{S}$ if they hold in $\mathfrak{R}$. As a result, (i), unlike some similar conditions we have considered earlier, does not imply its own converse.

We close this section on elementary properties of arithmetical extensions with some examples showing that various possible partial converses of **1.5** and **1.7** as well as an antisymmetry law for the relation "isomorphic to an arithmetical extension of" fail to hold. As we shall show, it can happen that (1) *a system* $\mathfrak{S}$ *is an isomorphic extension but not an arithmetical extension of a system* $\mathfrak{R}$; (2) *a system* $\mathfrak{S}'$ *is an arithmetically equivalent extension of a system* $\mathfrak{R}'$ *but is not an arithmetical extension of any system isomorphic to* $\mathfrak{R}'$; *and* (3) *each of two systems* $\mathfrak{R}''$ *and* $\mathfrak{S}''$ *is an arithmetical extension of a system isomorphic to the other, though* $\mathfrak{R}''$ *and* $\mathfrak{S}''$ *are not isomorphic.* In the examples we give, all systems involved are denumerable. The problems whether (2) could occur at all and whether (3) could occur for denumerable systems were proposed to us by Roland Fraïssé, who had earlier found non-denumerable examples of (3).

In all the examples, we deal with systems $\langle A, R \rangle$, where R is a binary relation which simply orders A. In particular, we shall be partly concerned with discretely ordered systems having a first element and no last element, i.e., with simply ordered systems having a first element, and in which every element has an immediate successor, and every element except the first has an immediate predecessor. We shall make use of some consequences of the decision method for the first-order theory of such systems, which was originally found by Langford. One such consequence is that two ordered systems of the respective order types ω and $\omega + \omega^* + \omega$ are arithmetically equivalent (cf. [15], p. 301). Another consequence is that, roughly speaking, any formula, of the first-order theory of such systems, involving n elements (i.e., having n free variables) can be reduced to a canonical form in which it is represented as a disjunction of conjunctions of formulas, each 'stating' that a pair of the elements are equal, or that one precedes another, or that there are at least (or exactly) a certain natural number of elements between two of the elements (or after one of the elements, or before one of the elements).[3])

To obtain an example of (1), let $\mathfrak{S}$ be the natural numbers together with their usual ordering, and $\mathfrak{R}$ be the positive integers with the same ordering. Then $\mathfrak{S}$ is an isomorphic extension of $\mathfrak{R}$, but not an arithmetical extension of $\mathfrak{R}$, for the first element of $\mathfrak{R}$ is the second element of $\mathfrak{S}$.

For (2), let $\mathfrak{S}'$ be a simply ordered system of the order type $\omega \cdot \eta$ (where η is the order type of the rational numbers together with their usual ordering). Since $\mathfrak{S}'$ has a subsystem of the type η, it has, by the well known theorem of Cantor, a subsystem of any given denumerable order type, and in particular one of type $(\omega+\omega^*+\omega) \cdot \eta$, which we take for $\mathfrak{R}'$. As we saw, any two systems of the respective types ω and $\omega+\omega^*+\omega$ are arithmetically equivalent; by a result of FEFERMAN [3], it follows that the ordinal products of these systems by a system of type η are again arithmetically equivalent. Hence $\mathfrak{R}'$ and $\mathfrak{S}'$ are arithmetically equivalent. Now, it is clear that $\mathfrak{R}'$ has a sequence $\langle x_0, \ldots, x_n \ldots \rangle$ of members such that, for each k, x_{k+1} is the immediate predecessor of x_k; on the other hand, $\mathfrak{S}'$ has no such sequence of members. From these facts and 1.12 it follows easily that $\mathfrak{S}'$ is not an arithmetical extension of any system isomorphic to $\mathfrak{R}'$.

Finally, to give an example of (3), we let $\mathfrak{R}''$ be a simply ordered system of the type $\omega+(\omega^*+\omega) \cdot \eta$ and $\mathfrak{S}''$ be one of the type $\omega+(\omega^*+\omega) \cdot \eta+\omega^*+\omega$. It is easily seen that each system is isomorphic to an initial segment of the other. Moreover, each of the two systems is an arithmetical extension of any of its initial segments isomorphic to the other; this is easily seen from 1.6, by noting that both systems are discretely ordered with a first element and no last element, and by referring to the canonical form of formulas (in the theory of such systems), which was previously described. Thus, each of $\mathfrak{R}''$ and $\mathfrak{S}''$ is an arithmetical extension of a system isomorphic to the other. On the other hand, it is clear that $\mathfrak{R}''$ and $\mathfrak{S}''$ are not isomorphic to each other.

[3]) Essentially this result is stated and proved in [8], pp. 234—263. The proof is by the method of eliminating quantifiers.

§ 2. Applications to the Löwenheim-Skolom Theorem.

We turn now to problems related to the Löwenheim-Skolem theorem. One of the known generalizations of this theorem can be formulated as follows:

If $\mathfrak{R} = \langle A, R \rangle$ is a relational system of an infinite power $\mathfrak{a}$ and $\mathfrak{b}$ is any infinite cardinal, then there is a relational system $\mathfrak{S} = \langle B, S \rangle$ of power $\mathfrak{b}$ which is arithmetically equivalent to $\mathfrak{R}$.[4])

The question naturally arises whether a system $\mathfrak{S}$ with the desired properties can be found among subsystems of $\mathfrak{R}$ in case $\mathfrak{b} \leqq \mathfrak{a}$, and among extensions of $\mathfrak{R}$ in case $\mathfrak{a} \leqq \mathfrak{b}$. The answer is known to be affirmative (but the proof for the case $\mathfrak{b} \leqq \mathfrak{a}$ is not available in the literature).[5]) We want to improve this result here by showing that in case $\mathfrak{b} \leqq \mathfrak{a}$ we can find a system $\mathfrak{S}$ (of power $\mathfrak{b}$) of which $\mathfrak{R}$ is an arithmetical extension, while in case $\mathfrak{a} \leqq \mathfrak{b}$ we can find a system $\mathfrak{S}$ (different from $\mathfrak{R}$) which is an arithmetical extension of $\mathfrak{R}$.

In case $\mathfrak{b} \leqq \mathfrak{a}$, our result is an immediate consequence of the

Theorem 2.1. *Let $\mathfrak{R} = \langle A, R \rangle$ be a relational system of an infinite power $\mathfrak{a}$, let C be a subset of A of power $\mathfrak{c}$, and let $\mathfrak{b}$ be an infinite cardinal for which $\mathfrak{c} \leqq \mathfrak{b} \leqq \mathfrak{a}$. Then there exists a system $\mathfrak{S} = \langle B, S \rangle$ of power $\mathfrak{b}$ such that $C \subseteq B$ and $\mathfrak{R}$ is an arithmetical extension of $\mathfrak{S}$.*

Proof: We begin by choosing a fixed well-ordering of A, to which we shall be referring when we speak of the first element in A having a certain property.

We now define recursively an increasing sequence $\langle D_0, \ldots, D_n, \ldots \rangle$ of subsets of A. For D_0 we take any subset of A which includes C and is of power $\mathfrak{b}$. D_{n+1} is defined as the set of all elements b of A such that, for some sequence $x \in D_n^{(\omega)}$, some natural number k, and some formula ϕ, b is the first element in A for which $x(k/b)$ satisfies ϕ in $\mathfrak{R}$. Using the formula $v_0 \equiv v_1$ we easily see that $D_n \subseteq D_{n+1}$ for every n. Let now B be the union of all the sets

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[4]) For relational systems of finite or denumerable orders this result was obtained by Tarski. For systems of non-denumerable orders the result requires an additional assumption (see the concluding remarks of this section), and with this assumption it was established by Henkin, and independently, by A. Robinson. More detailed historical information can be found in [20], footnote 3, pp. 467—8, as well as [6], footnote 7, p. 413, and footnote 34, p. 427.

[5]) For the case $\mathfrak{b} \leqq \mathfrak{a}$ (and for systems of at most denumerable orders) this result was obtained by Tarski and stated in [16], p. 712, Theorem 23. Regarding the case $\mathfrak{a} \leqq \mathfrak{b}$, see the proof of Theorem 2.2 below.

D_n ($n = 0, 1, 2, \ldots$), and let $\mathfrak{S} = \langle B, S \rangle$ be the corresponding subsystem of $\mathfrak{R}$ ($S = R \cap B^3$). Obviously, $C \subseteq B$, and it is a simple matter to show that B is of the power $\mathfrak{b}$. We want now to show that condition 1.10(ii) holds. Thus, suppose that the sequence $x \in B^{(\omega)}$ satisfies the formula $\bigvee v_k \phi$ in $\mathfrak{R}$. Since x has only finitely many different terms, there must be a natural number n such that $x \in D_n^{(\omega)}$. Furthermore, by 1.1(v) there is an element $b \in A$ for which $x(k/b)$ satisfies ϕ in $\mathfrak{R}$. It we take b as the first element of A with this property, then, by definition, $b \in D_{n+1}$, and therefore $b \in B$. Thus 1.10(ii) is seen to hold. Consequently, by 1.10, $\mathfrak{R}$ is an arithmetical extension of $\mathfrak{S}$, which completes the proof.

2.1 implies as a direct consequence the original Löwenheim-Skolem theorem, by which every infinite relational system is arithmetically equivalent to a denumerable system. Hence the method used in establishing 2.1 automatically provides a simple proof for the original Löwenheim-Skolem theorem. [6]) [7])

The second result in this section, Theorem 2.2, is an analogue of Theorem 2.1 for the case $\mathfrak{a} \leqq \mathfrak{b}$ (or, more precisely, it is an analogue of the particular case which is obtained by taking the empty set for C). While the proof of 2.1 is rather simple in itself, we do not know of any method which would permit us to derive this theorem directly from results whose proofs are available in the literature. We shall see that, on the contrary, such a method can be successfully applied to obtain a simple proof of 2.2.

THEOREM 2.2. *Let $\mathfrak{R} = \langle A, R \rangle$ be an infinite relational system of power $\mathfrak{a}$ and let $\mathfrak{b}$ be any cardinal such that $\mathfrak{a} \leqq \mathfrak{b}$. Then there exists a system $\mathfrak{S} = \langle B, S \rangle$ of power $\mathfrak{b}$ which is a proper arithmetical extension of $\mathfrak{R}$.*

[6]) In Skolem's papers [12] and [13] we find a result which is stronger than what we have just formulated as the original Löwenheim-Skolem theorem and to which 2.1 is more closely related; namely, the result that every infinite relational system has an arithmetically equivalent denumerable subsystem. The methods applied in [12] and [13] could be used to obtain new proofs of 2.1. They seem to us, however, to be less simple and natural than the method we have actually applied. Unlike our method, they are based upon a reduction of formulas to normal forms, and the resulting proofs, when presented in a completely precise and detailed manner, are rather involved.

[7]) It may be remarked that Theorem 2.1, as applied to a model of any set of axioms for set theory (e.g., either the Zermelo-Fraenkel or von Neumann-Bernays axiom system) yields the so-called Skolem paradox (cf. [13]) in the stronger form: *any model $\mathfrak{R}$ of a set of axioms for set theory has a denumerable subsystem $\mathfrak{S}$ such that $\mathfrak{R}$ is an arithmetical extension of $\mathfrak{S}$, i.e., such that every arithmetical notion is $\mathfrak{R}$—$\mathfrak{S}$ absolute* (in the sense of Gödel [5], p. 42).

PROOF: By applying the well-ordering principle we arrange the elements of A in a transfinite sequence $\langle a_0, \ldots, a_\xi, \ldots \rangle_{\xi<\alpha}$ without repeating terms, and we put

$$\mathfrak{R}' = \langle A, R, a_0, \ldots, a_\xi, \ldots \rangle_{\xi<\alpha}.$$

It is known that every relational system $\mathfrak{R}$ of an infinite power $\mathfrak{a}$ has a proper arithmetically equivalent extension of any given power $\mathfrak{b} \geqq \mathfrak{a}$. This result was established independently by Henkin ([6], p. 417, Theorem 2) and A. Robinson ([10], p. 74, Theorem 6.6.5). It applies to every system $\mathfrak{R}$ of an arbitrary order γ, provided that the power $\mathfrak{c}$ of γ at most equals $\mathfrak{b}$. Thus we can apply this result to the system $\mathfrak{R}'$ (since in this case $\gamma = \alpha$, and $\mathfrak{c} = \mathfrak{a} \leqq \mathfrak{b}$, and we obtain a system $\mathfrak{S}'$ of the form

$$\mathfrak{S}' = \langle B, S, a_0, \ldots, a_\xi, \ldots \rangle_{\xi<\alpha}$$

which has the power $\mathfrak{b}$ and is an arithmetically equivalent extension of $\mathfrak{R}'$. Hence, by 1.11, the system $\mathfrak{S} = \langle B, S \rangle$, which is of power $\mathfrak{b}$, is an arithmetical extension of the system $\mathfrak{R}$, and the theorem is proved.

By analyzing the proof just outlined we notice that this proof, even when applied exclusively to relational systems of at most denumerable order, essentially depends upon the discussion of formalized theories with non-denumerably many symbols. This may be regarded as a defect reducing the esthetic value of the proof, and the problem naturally arises of finding a simple and elegant proof of 2.2 which would be free from this defect. It would suffice to find such a proof for the weaker statement that every infinite system has at least one proper arithmetical extension; for this partial result combined with 1.9 can be used as a base for transfinite induction yielding arithmetical extensions of arbitrarily large powers.[8])

In proving both 2.1 and 2.2 we have made full use of the axiom of choice. (At the beginning of the proof of 2.2, and in the statement and proof of 1.11, the application of the well-ordering principle can be avoided if we agree to consider relational systems in which

[8]) A proof, along these lines, of Theorem 2.2 is known, but is by no means simple. It is essentially the same proof which was originally found by Tarski, in 1928, for the generalized Löwenheim-Skolem theorem (see footnote 5), although in the original argument the notion of arithmetical extension was only implicitly involved. The method used in the first part of the proof (i.e., in showing that every infinite system has at least one proper arithmetical extension) is related to one developed in [12]. Altogether the proof is complicated and will not be reproduced here.

relations and distinguished elements are indexed, not necessarily by ordinals, but by elements of arbitrary sets. However, the axiom of choice will still be involved in the proof of the Henkin-Robinson result from which 2.2 was derived.) The use of the axiom of choice in these proofs is essential. Indeed, Vaught has shown that *each of the theorems* 2.1 *and* 2.2 (*as well as certain other generalizations of the Löwenheim-Skolem theorem*) *implies the axiom of choice in its general form.* (Cf. [22].)

Theorems 2.1 and 2.2 are the only formally stated results of this paper whose proofs do not automatically extend to relational systems of arbitrary (possibly non-denumerable) order δ. To make such an extension possible the hypotheses of each of the two theorems must be provided with the additional assumption that the power $\mathfrak{d}$ of δ at most equals $\mathfrak{b}$. It is easily seen that in the case of 2.1 this assumption is essential. On the other hand, Robinson in [10] (pp. 76f.) outlines a method which would permit us to prove 2.2 with no additional assumption. Unfortunately, his argument seems to contain an error, and it actually enables us to establish 2.2 only if the condition $\mathfrak{a} \leqq \mathfrak{b}$ is replaced in the hypothesis by the stronger condition $2^{\mathfrak{a}} \leqq \mathfrak{b}$. Thus it still remains dubious whether 2.2 applies to relational systems of arbitrary order in case $\mathfrak{a} = \mathfrak{b}$, and also (unless we assume the generalized continuum hypothesis) in case $\mathfrak{a} < \mathfrak{b} < 2^{\mathfrak{a}}$.

§ 3. Applications to special algebraic systems.

We begin by obtaining a condition of a purely algebraic character which is sufficient for a system $\mathfrak{R}$ to be an arithmetical extension of a system $\mathfrak{S}$.

THEOREM 3.1. *The following two conditions are (jointly) sufficient for a system* $\mathfrak{S} = \langle B, S \rangle$ *to be an arithmetical extension of a system* $\mathfrak{R} = \langle A, R \rangle$:

(i) $\mathfrak{S}$ *is an extension of* $\mathfrak{R}$.

(ii) *for any finite subset* A' *of* A *and any element* b *of* B, *there exists an automorphism* f *of* $\mathfrak{S}$ *such that* $f(a') = a'$ *for every* $a' \in A'$, *and* $f(b) \in A$.

PROOF: Assuming conditions (i) and (ii) above, we shall show that condition (ii) of Theorem 1.10 holds. Suppose that ϕ is any formula, k is a natural number, x is a sequence belonging to $A^{(\omega)}$, and x satisfies $\bigvee v_k \phi$ in $\mathfrak{S}$. By 1.1(v), there is a member b of B such that $x(k/b)$ satisfies ϕ in $\mathfrak{S}$. Since x belongs to $A^{(\omega)}$, the set

A', consisting of all the terms x_i for which $i \neq k$, is finite. By (ii), there is an automorphism f of $\mathfrak{S}$ such that $f(x_i) = x_i$ for every $i \neq k$, and $f(b) \in A$. Now

$$x(k/f(b)) = \langle f(x_0), \ldots, f(x_{k-1}), f(b), f(x_{k+1}), \ldots \rangle.$$

Therefore $x(k/f(b))$ satisfies ϕ in $\mathfrak{S}$, by 1.2, while $f(b) \in A$. Thus 1.10(ii) is established, and the theorem follows, by 1.10.[9])

It should be emphasized that 3.1 holds as well for relational systems having arbitrarily many relations.

In certain applications of 3.1, we shall be concerned with relational systems $\mathfrak{A} = \langle A, O \rangle$ in which the ternary relation O may be construed as a binary operation, i.e., such that for any $x, y \in A$ there exists at most one z for which $\langle x, y, z \rangle \in O$ (if such a z exists, it is denoted by $O(x, y)$). Such systems will be called *generalized algebras.* If, moreover, $\mathfrak{A}$ is closed under the operation O, i.e., $O(x, y)$ exists for all $x, y \in A$, then $\mathfrak{A}$ is called an *algebra.*

By a *generalized subalgebra* of a *generalized algebra* $\mathfrak{A} = \langle A, O \rangle$ we mean a subsystem $\mathfrak{A}' = \langle A', O \cap A'^3 \rangle$ of $\mathfrak{A}$ such that, wherever $x, y \in A'$ and $O(x, y)$ exists, then $O(x, y) \in A'$. The *generalized subalgebra* of $\mathfrak{A}$ *generated by a subset* X *of* A is the subalgebra $\langle A', O \cap A'^3 \rangle$ of $\mathfrak{A}$, where A' is the intersection of all A'' such that $A'' \supseteq X$ and $\langle A'', O \cap A''^3 \rangle$ is a generalized subalgebra of $\mathfrak{A}$; under the same circumstances, we say that an arbitrary element of A' is *generated* by X. Clearly, every element of A' is generated by some finite subset of X.

The following theorem was suggested to us by Bjarni Jónsson as a means of generalizing three subsequent theorems, 3.3, 3.4, and 3.5, and unifying their proofs [10]):

Theorem 3.2. *Suppose the generalized algebra $\mathfrak{B} = \langle B, O \rangle$ is generated by an infinite set Y in such a way that every permutation of Y can be extended to an automorphism of $\mathfrak{B}$. Let X be any infinite subset of Y and $\mathfrak{A} = \langle A, O' \rangle$ be the generalized subalgebra of $\mathfrak{B}$ generated by X. Then $\mathfrak{B}$ is an arithmetical extension of $\mathfrak{A}$.*

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[9]) Fraïssé in [4], p. 177, has defined, in a purely mathematical way, relations called *n-subparenté bornée* ($n = 1, 2, 3, \ldots$) between systems $\mathfrak{R}$ and $\mathfrak{S}$ (in symbols, $\mathfrak{R} \underset{n}{\sim} \mathfrak{S}$), for which he has proved that a necessary and sufficient condition for $\mathfrak{R}$ to be arithmetically equivalent to $\mathfrak{S}$ is that, for every n, $\mathfrak{R} \underset{n}{\sim} \mathfrak{S}$. Using condition (ii) of Theorem 1.11, we may construct an alternative proof of Theorem 3.1 by showing by induction on n that for any n and p and any members $b_0, \ldots, b_{p-1}$ of B,

$$\langle A, R, b_0, \ldots, b_{p-1} \rangle \underset{n}{\sim} \langle B, S, b_0, \ldots, b_{p-1} \rangle.$$

[10]) In our original version of § 3, we derived 3.3 and 3.5 separately, and directly from 3.1.

PROOF: We shall show that condition 3.1(ii) holds. Suppose that A' is any finite subset of A and b is any member of B. All the elements of A' are generated by some finite subset X_1 of X, and b is generated by some finite subset Y' of Y. Let $Y' = X_2 \cup Y''$, where $X_2 \subseteqq X$ and $Y'' \subseteqq Y-X$. Noting that $X-(X_1 \cup X_2)$ is infinite, one sees easily that there exists a permutation g of Y such that $g(x) = x$ for every $x \in X_1 \cup X_2$, while $g(y) \in X$ for every $y \in Y''$. By hypothesis, g may be extended to an automorphism f of $\mathfrak{B}$. From the facts that g is an automorphism, and that $g(x) = x$ for every $x \in X_1$, and that every member of A' is generated by X_1, it clearly follows that $f(a') = a'$ for every $a' \in A'$. Likewise, from the facts that $f(y') \in A$ for every member y' of Y', and that b is generated by Y', and that f is an automorphism, we see that $f(b) \in A$. Thus 3.1(ii) holds, and the theorem follows by 3.1.

For the next theorem we shall need to rely upon some well known results concerning free algebras. A class $\boldsymbol{K}$ of algebras is called *equational* if, roughly speaking, $\boldsymbol{K}$ consists of all and only those algebras in which each of a certain set Σ of sentences is true, the sentences of Σ all being equations (preceded by universal quantifiers). (For a precise definition, cf. [17], III, p. 57.) Given an equational class $\boldsymbol{K}$, an algebra $\mathfrak{A} = \langle A, O \rangle$ and a set X, we say that *X generates $\mathfrak{A}$ freely over $\boldsymbol{K}$* if (i) $\mathfrak{A} \in \boldsymbol{K}$, (ii) X generates $\mathfrak{A}$, and (iii) any function on X into an algebra $\mathfrak{A}'$ of $\boldsymbol{K}$ can be extended to a homomorphism of $\mathfrak{A}$ into $\mathfrak{A}'$. Under the same circumstances, we say that $\mathfrak{A}$ is a *$\boldsymbol{K}$-free algebra with $\mathfrak{c}$ generators,* where $\mathfrak{c}$ is the power of X. The following facts are well known (cf., e.g., [1], pp. vii–ix): *If $\boldsymbol{K}$ is an equational class of algebras, then* (1) *for every cardinal number $\mathfrak{c} \neq 0$, there exists a $\boldsymbol{K}$-free algebra with $\mathfrak{c}$ generators, and any two such algebras are isomorphic*; (2) *if X generates $\mathfrak{A}$ freely over $\boldsymbol{K}$, then every permutation of X may be extended to an automorphism of $\mathfrak{A}$*; (3) *if Y generates $\mathfrak{B}$ freely over $\boldsymbol{K}$, X is a non-empty subset of Y, and $\mathfrak{A}$ is the subalgebra of $\mathfrak{B}$ generated by X, then X generates $\mathfrak{A}$ freely over $\boldsymbol{K}$.*

THEOREM 3.3. *Let $\boldsymbol{K}$ be an equational class of algebras. Then*

(i) *if Y generates $\mathfrak{B} = \langle B, O \rangle$ freely over $\boldsymbol{K}$, X is an infinite subset of Y, and $\mathfrak{A}$ is the subalgebra of $\mathfrak{B}$ generated by X, then $\mathfrak{B}$ is an arithmetical extension of $\mathfrak{A}$;*

(ii) *of any two $\boldsymbol{K}$-free algebras with infinitely many generators one is isomorphic to an arithmetical extension of the other, and hence the two algebras are arithmetically equivalent.*

PROOF: In view of property (2) of free algebras stated above,

(i) follows at once from 3.2. (ii) follows from (i), in view of properties (1) and (3) above.

The question arises whether this result may be strengthened to apply in some way to $\boldsymbol{K}$-free algebras with finitely many generators. As remarked in the introduction, Tarski has conjectured that any two free groups with at least two generators are arithmetically equivalent. The analogous statement for $\boldsymbol{K}$-free algebras in general is not true, as may be seen, for example, by considering free Boolean algebras, since the free Boolean algebra with n generators is the Boolean algebra with 2^{2^n} elements. One may also ask whether a sentence true in all $\boldsymbol{K}$-free algebras with finitely many generators is necessarily true in the $\boldsymbol{K}$-free algebra with $\aleph_0$ generators. Dana Scott showed us that the answer here is also negative. Again, Boolean algebras provide a simple example, for all the free Boolean algebras with finitely many generators are atomistic, while the free Boolean algebra with $\aleph_0$ generators is atomless.

THEOREM 3.4. *Let* $\mathfrak{C} = \langle C, +, -, \cdot \rangle$ *be a commutative ring and assume that all the elements of* C *have been arranged in a sequence* $\langle c_0, \ldots, c_\xi, \ldots \rangle_{\xi<\alpha}$ *without repeating terms. Then*

(i) *if* $\mathfrak{B} = \langle B, +', -', \cdot', c_0, \ldots, c_\xi, \ldots \rangle_{\xi<\alpha}$ *is the polynomial ring over* $\mathfrak{C}$ *generated by an infinite set* Y *of unknowns, and* $\mathfrak{A}$ *is the subalgebra of* $\mathfrak{B}$ *generated by an infinite subset* X *of* Y, *then* $\mathfrak{B}$ *is an arithmetical extension of* $\mathfrak{A}$;

(ii) *any two polynomial rings over* $\mathfrak{C}$ *with infinitely many unknowns are arithmetically equivalent.*

PROOF. Let $\boldsymbol{K}$ be the class of all systems $\langle D, +'', -'', \cdot'', d_0, \ldots, d_\xi, \ldots \rangle_{\xi<\alpha}$ in which the axioms for commutative rings are true and in which, moreover, $-''d_\xi = d_\eta$, $d_\xi +'' d_\eta = d_\zeta$, or $d_\xi \cdot'' d_\eta = d_\zeta$ whenever $-c_\xi = c_\eta$, $c_\xi + c_\eta = c_\zeta$, or $c_\xi \cdot c_\eta = c_\zeta$, respectively ($\xi, \eta, \zeta < \alpha$). Then $\boldsymbol{K}$ is an equational class of algebras (constants being regarded as 0-termed operations). From the hypothesis of (i), it is easily seen that Y generates $\mathfrak{B}$ freely over $\boldsymbol{K}$. Hence (i) follows from 3.3 (applied to algebras of arbitrary order); (ii) is an immediate consequence of (i).[11])

THEOREM 3.5. *Let* $\mathfrak{C} = \langle C, +, -, \cdot, \div \rangle$ *be a field and assume that all the elements of* C *have been arranged in a sequence* $\langle c_0, \ldots, c_\xi, \ldots \rangle_{\xi<\alpha}$ *without repeating terms. Then*

[11]) We may also prove 3.4 (i) more simply by deriving it from 3.2, which (as was mentioned above) gives a common basis for 3.3, 3.4, and 3.5. However, it is perhaps of interest that 3.4 can be derived directly from 3.3, in the way just indicated.

(i) *if* $\mathfrak{B} = \langle B, +', -', \cdot' \div', c_0, \ldots, c_\xi, \ldots \rangle_{\xi<\alpha}$ *is a pure transcendental extension of* $\mathfrak{C}$ *generated by the infinite set* Y *of elements algebraically independent with respect to* $\mathfrak{C}$, *and* $\mathfrak{A}$ *is the generalized subalgebra of* $\mathfrak{B}$ *generated by an infinite subset* X *of* Y, *then* $\mathfrak{B}$ *is an arithmetical extension of* $\mathfrak{A}$;

(ii) *any two pure transcendental extensions of* $\mathfrak{C}$ *of infinite degree are arithmetically equivalent.*

PROOF: (i) follows from 3.2, in view of the well known fact that any permutation of Y can be extended to an automorphism of B; (ii) follows immediately from (i). It should be noted that we are considering fields as systems having a division operation, and thus not as algebras, but as generalized algebras.

In a recent paper [11], Sikorski has defined a general notion called the ***K**-free product* of an indexed system of algebras, each belonging to a given class $\boldsymbol{K}$ of algebras. (This product does not always exist.) He shows that a $\boldsymbol{K}$-free algebra with $\mathfrak{a}$ generators is the $\boldsymbol{K}$-free product of $\mathfrak{a}$ replicas of the $\boldsymbol{K}$-free algebra with one generator, when $\boldsymbol{K}$ is equational, and that the $\boldsymbol{K}$-free product coincides with the usual free product of groups, when $\boldsymbol{K}$ is the class of all groups.[12]) Using this notion, 3.3 may be generalized as follows:

THEOREM 3.6. *Let* $\langle \mathfrak{A}_i \rangle_{i\in I}$ *be an indexed system of algebras of the class* $\boldsymbol{K}$, *and supposes* J *is a subset of* I *such that, for every* $i \in I$, *either there are only finitely many* $i' \in I$ *for which* $\mathfrak{A}_i$ *and* $\mathfrak{A}_{i'}$ *are isomorphic, and all such* i' *belong to* J, *or there are infinitely many such* i' *and infinitely many of them belong to* J. *Suppose the algebras* $\langle \mathfrak{A}_i \rangle_{i\in I}$ *have a **K**-free product* $\mathfrak{B}$, *and* $\mathfrak{C}$ *in the **K**-free product of the algebras* $\langle \mathfrak{A}_i \rangle_{i\in J}$. *Then* $\mathfrak{B}$ *is (isomorphic to) an arithmetical extension of* $\mathfrak{C}$.

PROOF: The theorem may be derived from 3.1, in a manner analogous to the proof of 3.2.

As other examples where 3.1 may be applied, we may mention weak direct powers and cardinal multiples. Given a set I and a relational system $\mathfrak{R} = \langle A, R \rangle$, having an element e such that $\langle e, e, e \rangle \in R$, the *weak direct power* $\mathfrak{R}^I$ (relative to e) is defined to be the system $\langle B, S \rangle$ where B is the set of all functions f on I into A such that $f(i) = e$ for all but a finite number of $i \in I$, and S is the set of all triples $\langle f, g, h \rangle$ such that $f, g, h \in B$ and, for

[12]) Both here and in the last paragraph of this paper a group must be considered as a system with either one binary operation, left hand (or right hand) division, or else one binary operation, multiplication, and one unary operation, inversion.

each $i \in I$, $\langle f(i), g(i), h(i)\rangle \in R$. Given a relational system $A = \langle A, R\rangle$ and a set I, the *cardinal multiple* $I \cdot \mathfrak{R}$ is defined to be the system $\langle B', S'\rangle$, where B' is the set of all couples $\langle i, a\rangle$ such that $i \in I$, and $a \in A$, and S' is the set of all triples $\langle\langle i, a_0\rangle, \langle i, a_1\rangle, \langle i, a_2\rangle\rangle$ such that $i \in I$ and $\langle a_0, a_1, a_2\rangle \in S$.

By applying 3.1, we see easily that *the weak direct power* $\mathfrak{R}^I$ *of a relational system* $\mathfrak{R}$ *(relative to an element e) is isomorphic to an arithmetical extension of the weak direct power* $\mathfrak{R}^J$, *provided that J is an infinite subset of I.* Similarly, we may show that *the cardinal multiple* $I \cdot \mathfrak{R}$ *is an arithmetical extension of the cardinal multiple* $J \cdot \mathfrak{R}$, *provided that J is an infinite subset of I.* These results may be generalized to apply to infinite weak direct products and infinite cardinal sums in the same way that 3.3 was generalized in 3.6. It should be remarked that generally much stronger results concerning sentences holding in weak direct powers and cardinal multiples have been obtained by other methods by Mostowski and Feferman, respectively (cf. [9], [2], and [3]).

Our various applications of Theorem 3.1 all establish the possibility of reducing the discussion of the arithmetical properties of systems of a given class to those of the denumerable or, roughly speaking, 'denumerable formed', systems of the class. In this way, they resemble the Löwenheim-Skolem theorem. Generally, it would seem, nonetheless, that there is no possibility of obtaining these results directly from the Löwenheim-Skolem theorem. It may be remarked, however, that Theorem 3.3(ii), for the special case of free groups, can be obtained by an application of the Löwenheim-Skolem theorem in the stronger form given in 2.1. Indeed, suppose $\mathfrak{a}$ and $\mathfrak{b}$ are cardinal numbers, with $\mathfrak{a} \geqq \mathfrak{b} \geqq \aleph_0$, and $\mathfrak{A}$ is a group freely generated by a set X of power $\mathfrak{a}$. For the set C of 2.1, we take a subset of X of the power $\mathfrak{b}$. Applying 2.1, we obtain a subgroup $\mathfrak{B}$ of power $\mathfrak{b}$, including C, such that $\mathfrak{A}$ is an arithmetical extension of $\mathfrak{B}$. It is well known that $\mathfrak{B}$ is, as a subgroup of a free group, a free group itself; $\mathfrak{B}$ clearly has at most $\mathfrak{b}$ generators; and, since $\mathfrak{B}$ includes the set C, $\mathfrak{B}$ must have at least $\mathfrak{b}$ generators. Thus, a free group with $\mathfrak{a}$ generators is shown to be isomorphic to an arithmetical extension of a free group with $\mathfrak{b}$ generators.

Examples of arithmetical extensionality, as found in this section by the method of Theorem 3.1, must, of course, be of a rather special nature. It may be interesting to mention, in closing, another source of such examples. Indeed, Tarski has remarded that, in many situations where the so-called "method of eliminating quantifiers" has been successfully applied, the strong results thus obtained (which necessarily apply to formulas as well as sentences) yield interesting and important cases of arithmetical extensionality. He has noted, in particular, that, from the results in his monograph *A decision method for elementary algebra and geometry* (prepared for publication by J. C. C. McKinsey, Berkeley, 1951), it follows at once that

every real closed field is an arithmetical extension of each of its real closed subfields,

and, similarly,

every algebraically closed field is an arithmetical extension of each of its algebraically closed subfields.

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(Oblatum 16-2-56).