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On Certain Periodic Characteristic Functions

by

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1. Introduction.

The purpose of this note is to derive certain properties of periodic characteristic functions and to determine those distributions whose characteristic functions are entire periodic functions.

Let $F(x)$ be a probability distribution, that is, a never-decreasing, right continuous function such that $\lim_{x \to -\infty} F(x) = 0$ and $\lim_{x \to \infty} F(x) = 1$. The Fourier transform of $F(x)$, that is the function

$$(1) \qquad f(z) = \int_{-\infty}^{\infty} e^{izx} dF(x)$$

is called the characteristic function of $F(x)$.

A distribution is called a *lattice distribution* if it is purely discontinuous and if its discontinuity points form a (proper or improper) subset of a sequence of equidistant points.

A characteristic function is said to be an *analytic characteristic function* if it coincides with an analytic function in some neighborhood of the origin.

We prove the following theorems:

THEOREM 1. *An analytic characteristic function which is single valued and periodic has either a real or a purely imaginary period. The period is real if, and only if, the characteristic function belongs to a lattice distribution.*

THEOREM 2. *A characteristic function is an entire periodic function (not $\equiv 1$) if, and only if, it is the characteristic function of a lattice distribution.*

2. The lemmas.

For the proof of these theorems we need two lemmas which we derive in this section.

LEMMA 1. *A characteristic function $f(t)$ assumes the value 1 for some real $t_0 \neq 0$ if, and only if, it is the characteristic function of a lattice distribution.*

PROOF: Let us first assume that for some real $t_0 \neq 0$ the value $f(t_0) = f(0) = 1$. We see then from (1) that

$$\int_{-\infty}^{\infty} (1 - e^{it_0 x}) dF(x) = 0$$

and therefore also

$$\int_{-\infty}^{\infty} (1 - \cos t_0 x) dF(x) = 0$$

Since the function $1 - \cos t_0 x$ is continuous and nonnegative this relation can hold only if $F(x)$ is a purely discontinuous distribution such that its discontinuity points are contained in the point set $2\pi s / t_0$ ($s = 0, \pm 1, \pm 2, \ldots$) where $1 - \cos t_0 x$ vanishes. The distribution function $F(x)$ has therefore necessarily the form

(2) $$F(x) = \sum_{s=-\infty}^{\infty} p_s \varepsilon(x - 2\pi s / t_0)$$

where

$$p_s \geq 0, \quad \sum_{s=-\infty}^{\infty} p_s = 1$$

and where

(3) $$\varepsilon(x) = \begin{cases} 0 & \text{if } x < 0 \\ 1 & \text{if } x \geq 0 \end{cases}$$

is the degenerate distribution. On the other hand, if $F(x)$ has the form (2) then

(4) $$f(t) = \sum_{s=-\infty}^{\infty} p_s \exp(2\pi its / t_0)$$

so that $f(t_0) = 1$.

Lemma 1 is a particular case of a somewhat more general result of A. Wintner ([2], p. 48). Wintner proves that $|f(\xi)| = 1$ for some real $\xi \neq 0$ if, and only if, $F(x)$ is a lattice distribution.

LEMMA 2. *Let $f(z)$ be an analytic characteristic function with the strip of convergence* [1]) *$-\alpha < Im(z) < \beta$ and assume that there are three real numbers n_1, n_2, n_3 such that*

(i) $-\alpha < n_1 < n_2 < n_3 < \beta,$
(ii) $n_1 + n_3 = 2n_2,$
(iii) $f(in_1) = f(in_2) = f(in_3),$

Then $f(z) \equiv 1$.

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[1]) For the properties of analytic characteristic functions see [1].

PROOF: We rewrite (iii) as

(5)
$$\begin{cases} f(in_2)-f(in_1) = 0 \\ f(in_3)-f(in_2) = 0 \end{cases}$$

We conclude from (i) that the three points in_j $(j = 1, 2, 3)$ are in the interior of the strip of convergence of $f(z)$ so that the integral representation (1) is valid at these points. Then

$$f(in_j) = \int_{-\infty}^{\infty} e^{-xn_j}\,dF(x) \text{ for } j = 1, 2, 3$$

and

$$f(in_{j+1})-f(in_j) = \int_{-\infty}^{\infty} [e^{-xn_{j+1}}-e^{-xn_j}]\,dF(x) \text{ for } j = 1, 2.$$

If we put

(6)
$$\begin{cases} g_j(x) = e^{-xn_{j+1}}-e^{-xn_j} & (j = 1, 2) \\ I_j = \int_{-\infty}^{\infty} g_j(x)\,dF(x) & (j = 1, 2) \end{cases}$$

We can write (5) as

(7)
$$I_1 = I_2 = 0$$

We carry the proof of lemma 2 indirectly and assume therefore tentatively that $f(z) \not\equiv 1$ and show that this leads to a contradiction with (7). If $f(z) \not\equiv 1$, then also $f(t) \not\equiv 1$ for real t and the corresponding distribution function is nondegenerate and therefore there exists a real $h>0$ such that at least one of the two relations

(8a)
$$\int_{+h}^{+\infty} dF(x) = 1-F(h)>0 \qquad ,$$

(8b)
$$\int_{-\infty}^{-h} dF(x) = F(-h)>0$$

is satisfied.

From (ii) we see that $n_3-n_2 = n_2-n_1$ and obtain therefore from (6)

$$g_1(x)-g_2(x) = -e^{-xn_1}[e^{-x(n_2-n_1)}-1]^2.$$

Therefore

(9)
$$\begin{cases} g_1(x) < g_2(x) \text{ if } x \neq 0 \\ g_1(0) = g_2(0) = 0 \end{cases}$$

Let us first assume that (8a) is satisfied and choose h accordingly. We see then from (9) that

(10a)
$$\int_0^h g_1(x)\,dF(x) \leqq \int_0^h g_2(x)\,dF(x) \leqq 0$$

and

(11) $$0 \leqq \int_{-\infty}^{0} g_1(x)\,dF(x) \leqq \int_{-\infty}^{0} g_2(x)\,dF(x)$$

From (8a) we conclude that there exists a finite $A>h$ such that $\int_{h}^{A} dF(x)>0$. We see from (9) that $g_2(x)-g_1(x)>0$ for $0<h\leqq x\leqq A$ and there exists a real number $\mu>0$ such that $g_2(x)-g_1(x)\geqq\mu>0$ for $h\leqq x\leqq A$. Since $g_2(x)-g_1(x)$ is positive for $x\geqq h$, we see that

$$\int_{h}^{\infty} [g_2(x)-g_1(x)]dF(x) \geqq \int_{h}^{A} [g_2(x)-g_1(x)]dF(x) \geqq \mu\int_{h}^{A} dF(x) > 0$$

and therefore

(10b) $$\int_{h}^{\infty} g_1(x)\,dF(x) < \int_{h}^{\infty} g_2(x)\,dF(x)$$

Adding (10a), (10b) and (11) we obtain

$$\int_{-\infty}^{+\infty} g_1(x)\,dF(x) < \int_{-\infty}^{+\infty} g_2(x)\,dF(x)$$

or

(12) $$I_1<I_2.$$

In case (8b) is valid we obtain by a similar reasoning again (12). Thus if at least one of the conditions (8a) of (8b) is satisfied, relation (12) must be valid in contradiction to (7). Therefore no h exists such that at least one of the relations (8a) or (8b) is satisfied. But this means that $1-F(h) = F(-h) = 0$ for any $h>0$, in other words, $\int_{-h}^{+h}dF(x) = 1$ for any $h>0$, i.e. the distribution $F(x)$ is necessarily equal to the degenerate distribution $\varepsilon(x)$.

In this proof condition (iii) is apparently not used fully. In the argument only the relation $f(in_2)-f(in_1) = f(in_3)-f(in_2) = M$ was used. Nevertheless (iii) can not be weakened since for $M \neq 0$ the function $f(t)$ could not be a characteristic function.

3. Proof of the theorems.

We next prove theorem 1. Let us therefore assume that $f(z)$ is a single valued analytic characteristic function which has the period

(13) $$\omega = \xi+i\eta \qquad (\xi,\ \eta \text{ real}).$$

It is known (see for instance [1]) that the relation

(14) $$f(-\bar{z}) = \overline{f(z)}$$

is satisfied for every analytic characteristic function in its do-

main of regularity. On account of the periodicity of $f(z)$ we see that
$$f(-\xi-i\eta) = f(0) = 1,$$
we deduce then from (14) that
$$f(\xi-i\eta) = 1.$$
Adding to the argument the period $\omega = \xi+i\eta$, we obtain
$$(15) \hspace{4cm} f(2\xi) = 1.$$
We consider first the case where $\xi \neq 0$. In this case we see from (15) and from lemma 1 that $f(z)$ is the characteristic function of a lattice distribution. We see from (2) and (4) that its characteristic function is given by
$$(16) \hspace{2cm} f(t) = \sum_{s=-\infty}^{\infty} p_s \exp(it\pi s/\xi) \hspace{1cm} (p_s \geqq 0, \sum_{-\infty}^{+\infty} p_s = 1)$$
Therefore $f(t)$ is a simply periodic function with the real period 2ξ while $\eta = 0$. If on the other hand $\xi = 0$, then (15) is satisfied and we see from (13) that $\omega = i\eta$, i.e. that the period is purely imaginary. This establishes theorem 1.

The case of a characteristic function which has a purely imaginary period can actually occur. As an example we mention the well known characteristic function $f(t) = \dfrac{1}{\cosh t}$.

From the proof of theorem 1 we obtain the following corollary:

COROLLARY TO THEOREM 1. *A characteristic function which does not reduce to a constant can not be doubly periodic.*

We proceed to prove theorem 2. Let therefore $f(t)$ be an entire characteristic function which does not reduce to a constant and assume that it is periodic. From theorem 1 we see that $f(t)$ must be simply periodic with either a real or a purely imaginary period. From lemma 2 we see that $f(t)$ can not have a purely imaginary period. The theorem follows then from lemma 1.

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[2]   On a class of Fourier Transforms, American Journal of Mathematics 58, 45—90, (1936).