Compositio Mathematica

Mansoor Ahmad

On exceptional values of entire and meromorphic functions

Compositio Mathematica, tome 13 (1956-1958), p. 150-158
<http://www.numdam.org/item?id=CM_1956-1958__13__150_0>

On Exceptional Values of Entire and Meromorphic Functions

by

Mansoor Ahmad

1. Let $f(z)$ be an entire function. A value α is said to be an exceptional value (e.v.) E, if

$$\lim_{r\to\infty} \frac{\log M(r)}{n(r,\ \alpha)\varphi(r)} > 0$$

for some $\varphi \subset E$.

It has been proved by Shah [1] that an entire function $f(z)$ of finite order can not have more than one v.e. E.

The purpose of this paper is to give more precise results of this type for functions of finite order and for a class of functions of infinite order.

2. Let $k(r)$ denote a positive onn-decreasing function which takes an integral value for every value of r. We say that an entire function $f(z)$ belongs to the k-class, if there exists a set of fixed positive numbers α, β, H such that

$$T(r, f) < A + \beta e_{k(\alpha r)-1}(\alpha r)^H$$

for all r, where $T(r, f)$ is Nevanlinna's characteristic function, A is independent of r; and $e_1(x) = e^x$, $e_2(x) = e^{e^x}$ and so on.

On the otherhand, if for every set of fixed positive numbers α, β, H there exists a value r_0 of r, such that

$$T(r, f) > \beta e_{k(\alpha r)-1}(\alpha r)^H$$

for all $r > r_0$, then we say that $f(z)$ does not belong to the k-class.

THEOREM 1. If $f(z)$ is an entire function of order ϱ, then

$$\lim_{r\to\infty} \frac{T(r, f)}{n(r, f-f_1)} \leqq \frac{2}{\varrho}$$

for every entire function $f_1(z)$, with one possible exception, provided that any one of the following conditions is satisfied.

(i) ϱ is finite and non-zero and $f_1(z)$ is of finite order less than ϱ;

(ii) $f(z)$ is of finite k-th order but of infinite $(k-1)$-th lower order; and $f_1(z)$ is of finite $(k-1)$-th order;

(iii) $f(z)$ belongs to the k-class but not to the $(k-1)$-class; and $f_1(z)$ belongs to the $(k-1)$-class, where the k-th order and the k-th lower order of $f(z)$ are defined as

$$\varrho_k = \overline{\lim_{r\to\infty}} \frac{l_k T(r, f)}{\log r}$$

and

$$\lambda_k = \underline{\lim_{r\to\infty}} \frac{l_k T(r)}{\log r},$$

$l_k x$ being the k-th iterate of $\log x$.

3. Lemma. If $\chi(x)$ is a positive function, continuous almost every where in every interval (r_0, r); and if

$$\overline{\lim_{r\to\infty}} \frac{\log \chi(r)}{\log r} = \varrho,$$

then

$$\underline{\lim_{r\to\infty}} \frac{\int_{r_0}^{r} \frac{\chi(x)}{x} dx}{\chi(r)} \leqq \frac{1}{\varrho}.$$

Proof. The lemma is obviously true, if ϱ is zero. Hence we need consider the case when ϱ is either finite and non-zero or infinite. Let

$$\xi(r) = \int_{r_0}^{r} \frac{\chi(x)}{x} dx$$

and suppose that the lemma is false.

We have then $\xi(r) > p\chi(r)$ for all $r \geqq \delta = \delta(p) > r_0$, where $p > 1/\varrho$ is a constant. $\xi'(r)$ exists and is equal to $\chi(r)/r$ almost every where. Hence

$$\frac{\xi'(x)}{\xi(x)} < \frac{\chi(r)}{pr\chi(r)} = \frac{1}{pr}$$

almost every where in every interval (δ, r).
Therefore, we have

$$\log \xi(r) = \log \xi(\delta) + \int_{\delta}^{r} \frac{\xi'(x)}{\xi(x)} dx < \log \xi(\delta) + \frac{1}{p} \log \frac{r}{\delta}$$

for all large r.

Consequently, we have

$$\log p + \log \chi(r) < \log \xi(r) < \log \xi(\delta) + \frac{1}{p} \log \frac{r}{\delta}.$$

Hence

$$\overline{\lim_{r\to\infty}} \frac{\log \chi(r)}{\log r} \leqq \frac{1}{p} < \varrho;$$

which contradicts the hypothesis; and thus the lemma is proved.

PROOF OF THEOREM 1. Let us suppose that the condition (i) is satisfied; and let

$$\underline{\lim_{r\to\infty}} \frac{T(r, f)}{n(r, f-F)} \geqq L > \frac{2}{\varrho}, \tag{1}$$

where F denotes each of the functions f_1 and f_2, these functions being of order less than ϱ.

Consequently, we have

$$T(r, f) > Ln(r, f-f_1)$$

and

$$T(r, f) > Ln(r, f-f_2) \tag{2}$$

for all $r > r_0$.

By the second fundamental theorem of Nevanlinna [2, § 34], we have

$$T(r, \varphi) = T(r) < N(r, 0) + N(r, 1) + N(r, \infty) \\ + 8 \log T(cr) + O(\log r) \tag{3}$$

for all sufficiently large r. where c is a fixed number greater than 1.

Putting $\varphi(z) = \dfrac{f(z)-f_1(z)}{f(z)-f_2(z)}$ in (3), we have

$$T(r, f) = T(r) < N(r, f-f_1) + N(r, f-f_2) + 8 \log T(cr) \\ + aT(r, f_1) + bT(r, f_2) + O(\log r) \tag{4}$$

for all $r > r_1$, where $N(r, f-f_1)$ and $N(r, f-f_2)$ refer to the functions $n(r, f-f_1)$ and $n(r, f-f_2)$ respectively; and a, b are certain positive constants.

Let $\varrho(r)$ be a proximate order of $f(z)$. Then

$$\lim_{r\to\infty} \varrho(r) = \varrho, \ \lim_{r\to\infty} r\varrho'(r) \log r = 0$$

$$T(r) \leqq r^{\varrho(r)} \text{ for all } r > r_2$$

$$T(r) = r^{\varrho(r)} \text{ for an infinity of } r.$$

Now, if $r_3 = \max(r_0, r_1, r_2)$, (by (2), we have

$$\overline{\lim_{r\to\infty}}\frac{N(r, f-F)}{r^{\varrho(r)}} \leqq \overline{\lim_{r\to\infty}}\frac{\int_{r_3}^{r}\frac{T(x, f)}{x}dx}{Lr^{\varrho(r)}}$$

$$\leqq \overline{\lim_{r\to\infty}}\frac{\int_{r_3}^{r} x^{\varrho(x)-1}\,dx}{Lr^{\varrho(r)}}$$

$$\leqq \overline{\lim_{r\to\infty}}\frac{1}{L(r\varrho'(r)\log r+\varrho(r))}$$

$$=\frac{1}{L\varrho}. \tag{5}$$

Therefore, combining (4) and (5), we have

$$1\leqq\frac{2}{L\varrho}.$$

This contradicts (1), and so the theorem is proved.

If the condition (ii) or (iii), is satisfied, it can be easily seen that

$$\lim_{r\to\infty}\frac{\log T(cr)}{T(r)}=0$$

and

$$\lim_{r\to\infty}\frac{T(r, F)}{T(r)}=0,$$

where F denotes each of the functions f_1 and f_2.

Also, if r_2 is sufficiently large, by (2), and (4), we have

$$1<\frac{N(r, f-f_1)+N(r, f-f_2)}{T(r)}+\frac{\log T(cr)}{T(r)}+\frac{aT(r, f_1)}{T(r)}+\frac{bT(r, f_2)}{T(r)}+\frac{O(\log r)}{T(r)}$$

$$<\frac{2}{L}\frac{\int_{r_2}^{r}\frac{T(x)}{x}dx}{T(r)}+o(1)$$

Consequently, we have

$$\underline{\lim_{r\to\infty}}\frac{\int_{r_2}^{r}\frac{T(x)}{x}dx}{T(r)}\geqq\frac{L}{2}>\frac{1}{\varrho}.$$

This contradicts the lemma; and so the theorem is proved.

THEOREM 2 (i) If $f(z)$ is an entire function of order ϱ for which the deficiency sum (excluding $\alpha=\infty$) $\Sigma\delta(\alpha)=\sigma>0$; and if $n'(r, \alpha)$ denotes the number of simple zeros of the function $f(z)-\alpha$ in the region $|z|\leqq r$, then

$$\varliminf_{r\to\infty} \frac{T(r, f)}{n'(r, \alpha)} \leqq \frac{2}{\varrho \cdot \sigma}$$

for every finite value of α, with one possible exception, provided that $f(z)$ satisfies any-one of the conditions of Theorem 1.

PROOF. If $N'(r, \alpha)$ and $N'(r, \beta)$ refer to $n'(r, \alpha)$ and $n'(r, \beta)$ respectively, we have

$$N'(r, \alpha)+N'(r, \beta)+2N_1(r)+O(\log r)>N(r, \alpha)+N(r, \beta).$$

Also, by the theorem of Nevanlinna (loc. cit.), we have

$$\begin{aligned} T(r)&<N(r, \alpha)+N(r, \beta)-N_1(r)+8\log T(cr)+O(\log r)\\ &<N'(r, \alpha)+N'(r, \beta)+N_1(r)+8\log T(cr)+O(\log r), \end{aligned}$$

where $N_1(r)$ has the same meaning as in [3, § 33, (16)].

Further, by the same theorem, we have

$$\Sigma\,\delta(\alpha)+\varlimsup_{r\to\infty}\frac{N_1(r)}{T(r)} \leqq 1+\varlimsup_{r\to\infty}\frac{\log T(cr)}{T(r)},$$

But, under the conditions of Theorem 1, we have

$$\varlimsup_{r\to\infty}\frac{\log T(cr)}{T(r)}=0.$$

Therefore

$$\varlimsup_{r\to\infty}\frac{N_1(r)}{T(r)} \leqq 1-\Sigma\,\delta(\alpha)=1-\sigma.$$

The rest of the proof, now, follows the same lines as in the second part of the proof of Theorem 1.

THEOREM 2(ii). If $f(z)$ is an entire function of finite and non-zero order ϱ, for which the deficiency sum (excluding $\alpha=\infty$) $\Sigma\delta(\alpha)=\sigma>1-\dfrac{2^{-\varrho}}{3}$, then

$$\varliminf_{r\to\infty}\frac{\log M(r)}{n'(r, \alpha)} \leqq \frac{6}{\varrho(3\sigma+2^{-\varrho}-3)}$$

for every finite value of α, with one possible exception.

The proof follows the same lines as in Theorem 2(i) and in the first part of the proof of Theorem 1.

THEOREM 3(i). If $f(z)$ is an entire function of finite and non-zero order ϱ; and if $(r_m)m=1, 2, 3, \ldots$ is any sequence of positive numbers, tending to infinity, then

$$\varliminf_{m\to\infty}\frac{T(r_m, f)}{N(r_m, f-f_1)} \leqq 2$$

for every entire function $f_1(z)$ of order less than ϱ, with one possible exception.

(ii) if $f(z)$ is an entire function of finite and non-zero k-th order ϱ_k; and if $(r_m)m = 1, 2, 3, \ldots$ is a sequence of positive numbers tending to infinity, such that

$$\lim_{m\to\infty} \frac{l_k T(r_m, f)}{\log r_m} = \varrho_k,$$

then

$$\underline{\lim}_{m\to\infty} \frac{T(r_m, f)}{N(r_m, f-f_1)} \leqq 2$$

for every entire function $f_1(z)$ of k-th order less than ϱ_k, with one possible exception.

(iii) If $f(z)$ is an entire function of finite k-th order but of infinite $(k-1)$-th order; and if $(r_m)m = 1, 2, 3, \ldots$ is a sequence of positive numbers, tending to infinity, such that

$$\lim_{m\to\infty} \frac{l_{k-1} T(r_m, f)}{\log r_m} = \infty,$$

then

$$\underline{\lim}_{m\to\infty} \frac{T(r_m, f)}{N(r_m, f-f_1)} \leqq 2$$

for every entire function $f_1(z)$ of finite $(k-1)$-th order, with one possible exception.

Theorem 4(i). If $f(z)$ is an entire function of zero order (not a polynomial), then

$$\underline{\lim}_{r\to\infty} \frac{\log M(r)}{N(r, f-f_1)} \leqq 6$$

for every polynomial $f_1(z)$, with one possible exception.

Proof. Let $\log M(r) = r^{u(r)}$ and suppose that the theorem does not hold. Then, we have

$$N(r, f-f_1) < \frac{r^{u(r)}}{L}$$

and

$$N(r, f-f_2) < \frac{r^{u(r)}}{L}$$

for all sufficiently large r, where f_1 and f_2 are polynomials; and $L > 6$.

Consequently, by (4), we have

$$T(r) < \frac{2r^{u(r)}}{L} + O(\log r).$$

Therefore, we have

$$r^{u(r)} \leqq 3T(2r) < \frac{6}{L} \cdot 2^{u(2r)} \cdot r^{u(2r)-u(r)} \cdot r^{u(r)} + O(\log 1)$$

or

$$1 \leqq \frac{6}{L} \cdot \varliminf_{r\to\infty} r^{u(2r)-u(r)}. \tag{6}$$

Now, we prove that

$$\varliminf_{r\to\infty} r^{u(2r)-u(r)} \leqq 1.$$

If not, then we have

$$u(2r)-u(r) > \frac{\varepsilon}{\log r}$$

for all $r \geqq r_0 > 1$.

By this inequality, we have

$$u(2^n r_0)-u(r_0) > \varepsilon \sum_{k=0}^{n} \frac{1}{\log r_0 + k \log 2}.$$

If we make n tend to infinity in this inequality, we reach a contradiction.

Hence we have

$$\varliminf_{r\to\infty} r^{u(2r)-u(r)} \leqq 1.$$

Proving thereby that (6) does not hold, and thus the theorem follows.

THEOREM 4 (ii) If $f(z)$ is an entire function of zero order, for which the deficiency sum (excluding $\alpha = \infty$) $\Sigma\, \delta(\alpha) = \sigma > \frac{2}{3}$, then

$$\varliminf_{r\to\infty} \frac{\log M(r)}{N'(r, \alpha)} \leqq \frac{6}{3\sigma-2}$$

for every finite value of α, with one possible exception.

The proof follows the same lines as in the preceding theorem and in Theorem 2 (i)

4. THEOREM 5. If $f(z)$ is a meromorphic function of order ϱ, then

$$\varliminf_{r\to\infty} \frac{T(r, f)}{n(r, f-f_1)} \leqq \frac{3}{\varrho}$$

for every meromorphic function $f_1(z)$, with two possible exceptions, provided that $f(z)$ and $f_1(z)$ satisfy any-one of the conditions of Theorem 1.

THEOREM 6. If $f(z)$ is a meromorphic function of order ϱ, for which the deficiency sum (including $\alpha = \infty)\Sigma\delta(\alpha) = \sigma > 1$, then

$$\lim_{r\to\infty} \frac{T(r, f)}{n'(r, \alpha)} \leqq \frac{3}{\varrho(\sigma-1)}$$

for every value of α, with two possible exceptions, provided that $f(z)$ satisfies any-one of the conditions of Theorem 1.

PROOF OF THEOREM 5. If f_1, f_2 and f_3 are meromorphic functions then putting

$$\varphi(z) = \frac{f(z)-f_1(z)}{f(z)-f_2(z)} \cdot \frac{f_3(z)-f_2(z)}{f_3(z)-f_1(z)}$$

in (3), we have

$$T(r, f) = T(r) < N(r, f-f_1)+N(r, f-f_2)+N(r, f-f_3)+$$
$$+8 \log T(cr)+\alpha T(r, f_1)+\beta T(r, f_2)+\gamma T(r, f_3)+O(\log r).$$

for all sufficiently large r, where α, β and γ are certain positive constants. The rest of the proof, now, follows the same lines as that of Theorem 1. The proof of Theorem 6 is similar to that of Theorem 2(i).

5. THEOREM 7(i). If $f(z)$ is an entire function of order ϱ, for which the deficiency sum (excluding $\alpha = \infty)\Sigma\delta(\alpha) = \sigma$, then

$$\lim_{r\to\infty} \frac{T(r)}{n'(r, \alpha)} \leqq \frac{3}{\varrho(1+\sigma)}$$

for every finite value of α, with two possible exceptions, provided that $f(z)$ satisfies any-one of the conditions of Theorem 1.

(ii) If $f(z)$ is an entire function of zero order, then

$$\lim_{r\to\infty} \frac{T(r)}{N'(r, \alpha)} \leqq \frac{3}{1+\sigma}$$

for every finite value of α, with two possible exceptions, where σ has the same meaning as before.

THEOREM 8(i) If $f(z)$ is an entire function of order ϱ, then

$$\lim_{r\to\infty} \frac{T(r)}{n''(r, \alpha)} \leqq \frac{4}{\varrho(1+\sigma)}$$

for every finite value of α, with one possible exception, where $n''(r, \alpha)$ denotes the number of simple and double zeros of $f(z)$

in the region $|z| \leqq r$, provided that $f(z)$ satisfies any-one of the conditions of Theorem 1.

(ii) If $f(z)$ is an entire function of zero order, then

$$\lim_{r\to\infty} \frac{T(r)}{N''(r, \alpha)} \leqq \frac{4}{1+\sigma}$$

fore every finite value of α, with one possible exception, where $N''(r, \alpha)$ refers to the function $n''(r, \alpha)$; and σ has the same meaning as before.

The proof of Theorem 7 depends on the theorem of Nevanlinna (loc. cit.), with $q = 4$, and follows the same lines as that of Theorem 2 (i). For the proof of Theorem 8, it should be observed that

$$N(r, \alpha)+N(r, \beta)<N''(r, \alpha)+N''(r, \beta)+\tfrac{3}{2}N_1(r)+O\ (\log r).$$

Remarks (i) Theorem 3 generalises a theorem [4]; and it shows that an e.f.B. (exceptional function in the sense of Borel) is also an exceptional function in the sense of this theorem for the particular class of functions of infinite order.

(ii) Theorem 2 generalises Theorem 2 of Shah (loc. cit.) in a number of ways.

REFERENCES.

S. M. Shah

[1] On exceptional values of entire functions, Compositio Mathematica 9 (1951), 227—238.

R. Nevanlinna

[2], [3], Fonctions Meromorphes (Paris, 1929).

Mansoor Ahmad

[4] On exceptional values of entire functions of infinite order, Indian Math. Soc. Journal, Vol. XVIII (1954), 19—21.

Muslim University.
Aligarh (India)

(Oblatum 17-10-55).