COMPOSITIO MATHEMATICA

J. DE GROOT
H. DE VRIES

Convex sets in projective space

Compositio Mathematica, tome 13 (1956-1958), p. 113-118
<http://www.numdam.org/item?id=CM_1956-1958__13__113_0>

Convex sets in projective space

by

J. de Groot and H. de Vries

INTRODUCTION. We consider the following properties of sets in n-dimensional real projective space $P_n (n > 1)$: a set is *semiconvex*, if any two points of the set can be joined by a (line)segment which is contained in the set;
a set is *convex* (STEINITZ [1]), if it is semiconvex and does not meet a certain P_{n-1}.

The main object of this note is to characterize the convexity of a set by the following interior and simple property: a set is convex if and only if it is semiconvex and does not contain a whole (projective) line; in other words: a subset of P_n is convex if and only if any two points of the set can be joined *uniquely* by a segment contained in the set. In many cases we can prove more; see e.g. theorem 2.

H. KNESER [2] gives a detailed survey of the different semiconvex sets in P_2 and P_3; see also HAALMEYER [3]. Though, surprisingly enough, he nowhere states the characterization-property just mentioned, this property may easily be concluded from the material contained in his paper. However, his method works only for $n = 2, 3$. Only theorem 3, stated below, serving as a lemma, is due to H. Kneser for all n. Recently we learned that D. DEKKER [4] also discovered the mentioned characterization, but only for *open* sets. Theorem 1, therefore, gives no news in the case of open sets. However, our proof is somewhat different. The authors are indebted to A. HEYTING, who drew their attention to this characterization-problem.

It is possible, of course, to state analogous problems for the n-sphere S_n (or for certain other spaces as well), replacing lines by great circles. The result is, roughly spoken, that the characterization-property holds for open sets in S_n, but breaks down for arbitrary ones. However, since the results for the S_n may be obtained far easier than for the P_n, we do not discuss them here.

We shall denote points by small Latin letters, lines by small Greek letters or two small Latin letters if these denote points of the line, and segments by the end-points of the segment if this is stated explicitly.

Theorem 1. *In order than an open or closed semi-convex set V in P_n $(n > 1)$ is convex, it is necessary and sufficient that V does not contain a whole line. Moreover: if V satisfies this condition, then for every point $q \in P_n \backslash V$, there exists an $(n-1)$-dimensional hyperplane, which contains q, and has no points in common with V.*

Proof. The necessity of the condition is obvious. To prove the sufficiency and the second part of the theorem, we proceed by induction.

Let $n = 2$. Take a point $v \in V$ and let ω be the line qv. Let μ_t be a variable line through q, always $\mu_t \neq \omega$. μ_t separates $P_2 \backslash \omega$ into two disjoint connected parts, I_t and II_t. We see to it that always $I_t \cap I_{t'} \neq 0$, $II_t \cap II_{t'} \neq 0$. We decompose $V_t = \mu_t \cap V$ into two disjoint sets A_t and B_t in the following way: we put $a_t \in V_t$, resp. $b_t \in V_t$, in A_t, resp. B_t, if one of the segments $a_t v$, resp. $b_t v$, of the line $a_t v$, resp. $b_t v$, lies entirely within $V \cap I_t$, resp. within $V \cap II_t$. Since V is semiconvex and does not contain a whole line, we have

$$V_t = A_t \cap B_t, \; A_t \cap B_t = 0.$$

If V is open, then A_t is an open set on the line μ_t, because, if $a_t \in A_t$, there is, by applying the Heine-Borel theorem, an open neighbourhood of a segment $a_t v$ which is contained in V. But then also B_t is open on μ_t. Since V_t is connected, we may conclude that for a definite t either A_t or B_t is empty.

[If V is closed, then A_t is closed on μ_t since the limit of a converging sequence of closed segments $a_t^i v$, $a_t^i \in A_t$, entirely lying within $(V \cap I_t) \cup \omega \cup \mu_t$ is again a closed segment lying within $(V \cap I_t) \cup \omega \cup \mu_t$. Then also B_t is closed on μ_t, and we may conclude that either A_t or B_t is empty.]

Now we project $\bigcup_t A_t$ and $\bigcup_t B_t$ from q upon a line λ through v, $\lambda \neq \omega$; the projections will be L_1 and L_2 respectively. Then obviously $L_1 \cap L_2 = 0$.

If V is open, then L_1 is open on λ: a point $p_1 \in L_1$ is the projection of a point $p \in \bigcup_t A_t$, p has on the line vp a neighbourhood belonging to $\bigcup_t A_t$, so p_1 has a neighbourhood on λ belonging to L_1. Then also L_2 is open, $L_1 \cap L_2 = 0$, so there exists a point $q' \in \lambda$, $q' \notin L_1 \cup L_2 \cup \{v\}$, and the line qq' lies entirely within the complement of V.

[If V is closed, then L_1 and L_2 are each closed on $\lambda\backslash v$ as can easily be seen. Since $\lambda\backslash v$ is connected, there exists a point $q' \in \lambda\backslash v$, $q' \notin L_1 \cup L_2$, and the line qq' lies entirely within the complement of V.]

So the theorem is proved for $n = 2$.

Now we assume the theorem to be true for $n = k-1$ and prove the theorem for $n = k$ $(k > 2)$.

We take $q \in P_k\backslash V$ and Ω as a P_{k-1} in P_k not containing q. We project V from q upon Ω, thus obtaining $V' \subset \Omega$. Then with $p_1', p_2' \in V'$, V' contains obviously one of the segments of $p_1'p_2'$, and does not contain both segments: the two-dimensional plane generated by q, p_1', p_2' contains a line through q, which avoids V (the intersection of a semiconvex set not containing a whole line with a hyperplane is a similar set), and so this line intersects $p_1'p_2'$ in a point, not belonging to V'. Using the induction-hypothesis we get that V' avoids a P_{k-2} lying in Ω. The $(k-1)$-dimensional hyperplane through that P_{k-2} and q avoids V, q.e.d.

THEOREM 2. *If V is an open or closed convex set in a P_n $(n > 1)$, and H is a P_{n-k} $(k > 0)$ avoiding V, then there exists a P_{n-1} containing H and avoiding V.*

PROOF. For $k = 1$ the theorem is trivial. We assume further $k > 1$. We proceed by induction with respect to n.

For $n = 2$ the theorem has been proved in theorem 1. Assuming the theorem to be true for $n-1$, we prove the theorem for n.

Choose an $(n-1)$-dimensional hyperplane S containing H. $S \cap V$ is convex. So, according to the induction-hypothesis, there exists a $P_{n-2} \subset S$ containing H and avoiding $S \cap V$, thus also avoiding V. Let Ω be a P_{n-1} avoiding V. If $\Omega \supset H$ we are ready. Assume $\Omega \not\supset H$. Let F_t be a variable $(n-1)$-dimensional hyperplane containing the above-mentioned P_{n-2}, thus also containing H. $F_t\backslash(P_{n-2} \cup \Omega)$ is decomposed by P_{n-2} and $\Omega \cap F_t$ into two disjoint connected parts of which only one may contain points of V. If V is open, we get by varying F_t continuously "a first situation" in which this part contains no points of V. Neither can in this situation the other part contain points of V, since if it would then that part would also contain points of V in "an earlier situation". If V is closed the theorem is proved similarly.

THEOREM 3. (KNESER). *If a semiconvex set V in a P_n contains $n+1$ linearly independent points $p_0, p_1, \ldots, p_n$, then each of these points, for instance p_0, is vertex of an n-dimensional simplex the* *interior points of which belong to V. This includes, that every point of V is accumulation point of interior points of V.*

Proof. The theorem is true for $n = 1$. We assume the theorem to be true for $n = N-1$ and prove it for $n = N$.

The P_{N-1} defined by $p_1, \ldots, p_N$ contains an $(N-1)$-dimensional simplex Σ, the interior of which belongs to V, according to the induction-hypothesis applied to the semiconvex set $V \cap P_{N-1}$. In the N-dimensional projective space P_N, the simplex Σ and the point p_0 define two N-dimensional simplices Λ_1, Λ_2 of which Σ is a face. Let M_1, resp. M_2, be the set of points of Σ which can be joined with p_0 within $V \cap \Lambda_1$, resp. $V \cap \Lambda_2$. M_1 and M_2 are not necessarily disjoint. If one of the sets M_i contains interior points, then it contains an $(N-1)$-dimensional simplex P, and hence there exists an N-dimensional simplex, defined by P and p_0, contained in V; so the theorem is proved.

We prove that necessarily at least one of the M_i contains interior points.

Let L_1, resp. L_2, be the interior of Λ_1, resp. Λ_2. The set $P_N \backslash V$ is semiconvex as can easily be seen. We distinguish the following three cases: the maximal number of linearly independent points of $(P_N \backslash V) \cap (L_1 \cup L_2)$ is 1°. $N+1$, 2°. N, 3°. $<N$. In the second case we have two possibilities: all points of $(P_N \backslash V) \cap (L_1 \cup L_2)$ lie on an $(N-1)$-dimensional hyperplane containing either not p_0 or p_0. In case of 1°. and the first possibility of 2°. there exists an $(N-1)$-dimensional hyperplane Q, such that $p_0 \notin Q$, and $Q \cap (P_N \backslash V) \cap (L_1 \cup L_2)$ contains N linearly independent points. Using the induction-hypothesis, we easily get that $Q \cap (P_N \backslash V) \cap$ $\cap (L_1 \cup L_2)$ contains interior points. This implies $Q \cap (P_N \backslash V) \cap L_1$ or resp. $Q \cap (P_N \backslash V) \cap L_2$ contains interior points, which gives that M_2 or resp. M_1 contains interior points. In case of the second possibility of 2°. and 3°., $(P_N \backslash V) \cap (L_1 \cup L_2)$ is included in an $(N-1)$-dimensional hyperplane containing p_0, and both M_1 and M_2 have interior points.

Theorem 4. *An arbitrary semiconvex set V, not containing a whole line, in n-dimensional projective space P_n $(n > 1)$, avoids an $(n-1)$-dimensional hyperplane. Moreover: if a is an interior point of $P_n \backslash V$, then there exists an $(n-1)$-dimensional hyperplane containing a and avoiding V.*

Proof. We proceed by induction. First we prove the theorem for $n = 2$.

If V is contained in a line, the theorem is trivial. If V is not contained in a line, then the closure $\overline{W}$ of the interior W of V contains V, according to theorem 3. Clearly W does not contain a whole line. We prove, that W is also semiconvex (and thus convex according to theorem 1).

Choose $p, q \in W$, $p \neq q$. Join p and q by a segment S within V. Take a point $r \in pq$, $r \notin V$. Let $O_p \subset W$ and $O_q \subset W$ be two connected neighbourhoods of p resp. q. Draw a variable line λ_t through r meeting O_p and O_q. S is contained in the interior of the sum of the segments in V on the λ_t, joining points of O_p and O_q, that means $S \subset W$, q.e.d.

$\overline{W} = P_2$ implies that $P_2 \backslash W$ is a line: the semiconvex set $P_2 \backslash W$ cannot contain three linearly independent points by theorem 3, but $P_2 \backslash W$ contains at least one whole line by theorem 1, thus $P_2 \backslash W$ is a line. Then $W = V$, since otherwise V would contain a whole line. So the theorem holds in this case.

If $\overline{W} \neq P_2$, there exists an open convex neighbourhood O of a, $O \subset P_2 \backslash \overline{W}$. According to theorem 1, we can draw a line α through a, avoiding W. Let β be an arbitrary line through a, $\alpha \neq \beta$. α and β decompose $P_2 \backslash (\alpha \cup \beta)$ into two connected parts, I and II. If V avoids α, we are ready. α cannot contain accumulation points of $W \cap \beta$. If α only contains accumulation points of $W \cap I$ or of $W \cap II$, we can find a line α' through a avoiding $\overline{W}$, thus avoiding V, by turning α a little around a. On the other hand, if $s \in \alpha$, resp. $t \in \alpha$, is an accumulation-point of $W \cap I$, resp. $W \cap II$ while moreover $s \neq t$, we could find a line joining the points $x \in W \cap I$ and $y \in W \cap II$, x and y near s resp. t, intersecting α in u and β in v while $v \in O$; but in that case x, y and u, v would form separated pairs, x and y in W, u and $v \notin W$, in contradiction with the semiconvexity of W. If $s = t$ we proceed as follows: W is not contained in a line, so we can find c, $d \in W$, $sc \neq sd$. Be III one of the parts of $P_2 \backslash (sc \cup sd)$ into which $P_2 \backslash (sc \cup sd)$ is decomposed by sc and sd, α not lying in III. All points z of III belong to W: the interval $III \cap cd$ of cd lies in W, so we can connect z with a point z' of W sufficiently near to s, such that the points z and z' do not separate the intersection-points z'', z''' of zz' with α resp. cd, thus the segment of $z'z'''$ which contains z lies in W, and therefore $z \in W$. Then certainly a whole line through s minus s lies in W, so $s \notin V$ and $\alpha \cap V = 0$, which completes the proof for $n = 2$.

If the theorem is assumed to be true for $n = k-1$, we can prove it for $n = k$ in exactly the same way as has been done in the proof of theorem 1. We have only to assume that $P_k \backslash V$

contains interior points. If this is not the case, then $P_k \backslash V$ is exactly a P_{k-1} using theorem 3 by a reduction ad absurdum and the theorem holds.

REMARK. If V is an arbitrary convex set in n-dimensional projective space $P_n (n > 1)$, and H is an $(n-2)$-dimensional hyperplane lying in the interior of $P_n \backslash V$, then there exists an $(n-1)$-dimensional hyperplane containing H and avoiding V. This can be proved in a way analogous to that used in the proof of theorem 4.

Mathematisch Instituut
University of Amsterdam.

REFERENCES.

E. STEINITZ

[1] Bedingt konvergente Reihen und konvexe Systeme, J. für die reine und angew. Math. 146, p. 34, (1916).

H. KNESER

[2] Eine Erweiterung des Begriffes "konvexer Körper". Math. Ann. 82, p. 287—296, (1921).

B. P. HAALMEYER

[3] Bijdragen tot de theorie der elementairoppervlakken, Amsterdam, 1917.

D. DEKKER

[4] Convex regions in projective space, The Amer. Math. Monthly, 62, no. 6, p. 430—431, (1955).

(Oblatum 29-8-55).