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RAOUF DOSS

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On Riemann Integrability and Almost Periodic Functions

by

Raouf Doss

Let $f(x)$ be a Bohr almost periodic (Bohr a.p.) function. To every $\epsilon > 0$ we can associate a $\delta > 0$ and numbers $\pi_1, \ldots, \pi_m$ such that

(1) $$\sup_t | f(t + \tau_t) - f(t) | < \epsilon,$$

provided

(2) $$| \tau_t | < \delta \pmod{\pi_k} \qquad (k = 1, \ldots, m).$$

Conversely, if to every $\epsilon > 0$ there corresponds a $\delta > 0$ and numbers $\pi_1, \ldots, \pi_m$ such that relations (2) imply (1), then $f(x)$ is a Bohr a.p. function.

This suggests the following definition:

DEFINITION 1. A bounded function $f(x)$ is called *almost periodic in the sense of Riemann-Stepanoff* [1]) (R.S.a.p.) if to every $\epsilon > 0$ there corresponds a $\delta > 0$ and numbers $\pi_1, \ldots, \pi_m$ such that

(3) $$\sup_x \overline{\int_x^{x+1}} | f(t + \tau_t) - f(t) | \, dt < \epsilon,$$

provided

$$| \tau_t | < \delta \pmod{\pi_k} \qquad (k = 1, \ldots, m).$$

Here $\overline{\int_a^b}$ means an upper Lebesgue integral.

To define the R.W.a.p. or the R.B.a.p. classes we just replace (3) by

$$\overline{\lim_{l\to\infty}} \sup_x \frac{1}{l} \overline{\int_x^{x+l}} | f(t + \tau_t) - f(t) | \, dt < \epsilon$$

or

$$\overline{\lim_{l\to\infty}} \frac{1}{2l} \overline{\int_{-l}^{+l}} | f(t + \tau_t) - f(t) | \, dt < \epsilon.$$

respectively.

It will be seen below (theorem 2) that the R.W.a.p. and the R.B.a.p. classes are identical.

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[1]) The approximation theorem below (theorem 2) justifies the name of Riemann.

The Stepanoff, Weyl, and Besicovitch distances between two summable functions $f(x)$, $g(x)$ are defined in the usual manner and will be denoted by $D_S(f, g)$, $D_W(f, g)$ and $D_B(f, g)$ respectively.

We denote by R the additive group of reals. Let E be a measurable set in R and let $c_E(t)$ be its characteristic function. We write

$$S(E) = \sup_x . \int_x^{x+1} c_E(t)dt$$

$$W(E) = \lim_{l\to\infty} \sup_x . \frac{1}{l} \int_x^{x+l} c_E(t)dt$$

$$B(E) = \overline{\lim_{l\to\infty}} \frac{1}{2l} \int_{-l}^{+l} c_E(t)dt.$$

The complementary of E with respect to R will be denoted by $\tilde{E}$.

We have the following theorem:

THEOREM 1. *In order that the bounded function $f(x)$ be R.S.a.p. it is necessary and sufficient that to every $\epsilon > 0$ there corresponds a measurable set E and numbers $\delta > 0$, $\pi_1, \ldots, \pi_m$ such that*

(i) $$S(\tilde{E}) < \epsilon,$$

and such that

$$| f(x) - f(x') | < \epsilon,$$

provided $x \in E$ and

$$| x - x' | < \delta \pmod{\pi_k} \qquad (k = 1, \ldots, m)$$

To have the corresponding theorem for the R.W.a.p. or the R.B.a.p. classes we just replace (i) *by*

$$W(\tilde{E}) < \epsilon$$

or

$$B(\tilde{E}) < \epsilon$$

respectively.

We introduce the following definition:

DEFINITION 2. A function $f(x)$ is called K,S.a.p. [2]) if to every $\epsilon > 0$ there corresponds a measurable set E and numbers $\delta > 0$, $\pi_1, \ldots, \pi_m$ such that

(i) $$S(\tilde{E}) < \epsilon$$

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[2]) Cf. A. S. Kovanko, „Sur la correspondance entre les diverses classes de fonctions presque-périodiques généralisées „Bull. (Izvestiya) Inst. Math. Mech. Univ. Tomsk, 3, 1—38, (1946).

and such that

$$| f(x) - f('x) | < \epsilon$$

provided $x \in E$, $x' \in E$ and

$$| x - x' | < \delta \pmod{\pi_k} \qquad (k = 1, \ldots, m).$$

To have the corresponding definition for the K.W.a.p. or the K.B.a.p. classes we just replace (i) by

$$W(\tilde{E}) < \epsilon$$

or

$$B(\tilde{E}) < \epsilon$$

respectively.

We have the following approximation theorems:

THEOREM 2. *In order that the function* $f(x)$ *be R.S.a.p. it is necessary and sufficient that to every* $\epsilon > 0$ *there corresponds two trigonometric polynomials* $p(x)$, $q(x)$ *such that*

(i) $$p(x) \ll f(x) \ll q(x),$$

(here $a \ll b$ *means* $\operatorname{Re} a \leqq \operatorname{Re} b$ *and* $Ia \leqq Ib$*),*

(ii) $$D_S(p, q) < \epsilon.$$

To have the corresponding theorem for the R.W.a.p. or the R.B.a.p. classes we just replace (ii) *by*

$$D_W(p, q) < \epsilon$$

or

$$D_B(p, q) < \epsilon.$$

respectively.

Since for polynomials (or Bohr a.p. functions) $p(x)$, $q(x)$ we have

$$D_W(p, q) = D_B(p, q),$$

we see that *the two classes R.W.a.p. and R.B.a.p. are identical.*

THEOREM 3. *In order that the function* $f(x)$ *be K.S.a.p. it is necessary and sufficient that to every* $\epsilon > 0$ *we can associate a trigonometric polynomial* $q(x)$ *and a measurable set* E *such that*

(i) $$S(\tilde{E}) < \epsilon$$

and

(ii) $$| f(x) - q(x) | \leqq \epsilon, \quad \textit{for } x \in E.$$

To have the corresponding theorem for the K.W.a.p. or the K.B.a.p. classes we just replace (1) *by*

$$W(\tilde{E}) < \epsilon$$

or

$$B(\tilde{E}) < \epsilon$$

respectively [3]).

Let $f(x)$ be a R.B.a.p. function. By means of theorem 2 we can easily extend to $f(x)$ a classical property due to H. Weyl [4]) of R-integrable, purely periodic functions: we can find two numbers ξ and M with the property:

To every $\epsilon > 0$ there corresponds an integer n such that

$$\left| \frac{1}{n} \sum_{l=0}^{n-1} f(x + l\xi) - M \right| < \epsilon$$

whatever be x.

Combining this property with almost periodicity we obtain

Theorem 4. *Let $f(x)$ be a R.B.a.p. function; then we can find two numbers ξ and M possessing the following property:*

To every $\epsilon > 0$ there corresponds an integer n and numbers $\delta > 0$, $\pi_1, \ldots, \pi_m$ such that

$$\left| \frac{1}{n} \sum_{l=0}^{n-1} f(x_l + l\xi) - M \right| < \epsilon$$

provided

$$| x_i - x_j | < \delta \pmod{\pi_k} \qquad \begin{pmatrix} i, j = 0, \ldots, n-1 \\ k = 1, \ldots, m \end{pmatrix}$$

Conversely, if there are two numbers ξ and M with the above property, then $f(x)$ is a R.B.a.p. function [5]).

There is no corresponding theorem for the R.S.a.p. functions.

Proof of theorem 1

Necessity. Let $f(x)$ be a R.S.a.p. function. Let $\epsilon > 0$ be given, and let $\delta > 0$, $\pi_1, \ldots, \pi_m$ be such that

$$\text{(1)} \qquad \sup_x \overline{\int_x^{x+1}} | f(t + \tau_t) - f(t) | \, dt < \frac{\epsilon^2}{4},$$

provided

$$| \tau_t | < \delta \pmod{\pi_k} \qquad (k = 1, \ldots, m).$$

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[3]) This theorem is close to the main theorem in E. Følner, „On the structure of generalized almost periodic functions „Danske Vid. Selsk. Math. Phys. Medd. 21, no 11, 30 p. (1945).

[4]) H. Weyl, „Über die Gleichverteilung von Zahlen mod. Eins." Math, Ann. 77, 313—352, (1916).

[5]) This theorem has been stated without proof in Raouf Doss" Sur une nouvelle classe de fonctions presque-périodiques" C. R. Acad. Sci. Paris, 238, 317—318, (1954).

Put

$$\varphi(t) = \sup_{\tau} . \mid f(t+\tau) - f(t) \mid,$$

where τ is subject to the condition

$$\mid \tau \mid < \delta \pmod{\pi_k} \qquad (k = 1, \ldots, m).$$

Then, by (1) and the definition of an upper Lebesgue integral

$$\text{(1, } S\text{)} \qquad \sup_{x} . \overline{\int_x^{x+1}} \varphi(t)dt < \frac{\epsilon^2}{4}.$$

Let n be a fixed positive or negative integer and call D_n the set of points t of the interval $(n, n+1)$ at which $\varphi(t) \geqq \epsilon$. D_n is not necessarily measurable, but there exists a partition of $(n, n+1)$ into a finite number of disjoint measurable sets $E_1, \ldots E_s$ such that

$$\text{(2)} \qquad \Sigma_{i=1}^{s} M_i \mu(E_i) \leqq \overline{\int_n^{n+1}} \varphi(t)dt + \frac{\epsilon^2}{4},$$

where $\mu(E_i)$ is the measure of E_i and M_i is the sup. of $\varphi(t)$ for t on E_i. The set D_n above meets a number of E_i, say $E_1, \ldots, E_r$ $(r \leqq s)$, so that

$$\epsilon \leqq M_i \qquad \text{for} \quad i = 1, \ldots, r.$$

By (2)

$$\Sigma_{i=1}^{r} \epsilon\, \mu(E_i) \leqq \overline{\int_n^{n+1}} \varphi(t)dt + \frac{\epsilon^2}{4};$$

The set

$$C_n = \bigcup_{i=1}^{r} E_i$$

possesses therefore the property that

$$\text{(3)} \qquad \mu(C_n) \leqq \frac{1}{\epsilon} \overline{\int_n^{n+1}} \varphi(t)dt + \frac{\epsilon}{4},$$

and

$$\varphi(t) < \epsilon \qquad \text{for } t \in (n, n+1),\ t \,\bar{\epsilon}\, C_n.$$

Let

$$C = \bigcup_{n=-\infty}^{\infty} C_n,$$

and let $E = \tilde{C}$ be the complementary of C. Then, clearly

$$\text{(4)} \qquad \varphi(t) < \epsilon \quad \text{for} \quad t \in E.$$

Also, by (3) and (1, S)

$$(5)\qquad S(\tilde{E}) = S(C) \leqq \frac{2}{\epsilon}\ \sup_{n}.\ \overline{\int_{n}^{n+1}} \varphi(t)dt + 2\frac{\epsilon}{4} < \epsilon.$$

Since $\epsilon > 0$ is arbitrary, relations (4) and (5) show that $f(x)$ satisfies the condition of the theorem.

If we start with a R.W.a.p. or a R.B.a.p. function, relation (1, S) should be replaced by

$$(1,\ W)\qquad \overline{\lim_{l\to\infty}}\ \sup_{x} \frac{1}{l} \overline{\int_{x}^{x+l}} \varphi(t)dt < \frac{\epsilon^2}{4}$$

or

$$(1,\ B)\qquad \overline{\lim_{l\to\infty}} \frac{1}{2l} \overline{\int_{-l}^{l}} \varphi(t)dt < \frac{\epsilon^2}{4},$$

respectively. Relation (4) is still true, but (3) would then give easily

$$W(C) < \epsilon$$

or

$$B(C) < \epsilon$$

respectively.

The necessity is now proved.

Sufficiency. The sufficiency of the condition of the theorem is immediate if we take into account the boundedness of $f(x)$.

Lemma. *Let $f(x)$ be a real function and $E \subset E'$ be two subsets of R. Let*

$$|\, f(x)\, | \leqq M \quad \textit{for} \quad x \in E'.$$

Let $\epsilon > 0$, $\delta > 0$, $\pi_1, \ldots, \pi_m$ be numbers such that

$$|\, f(x) - f(x')\, | < \epsilon$$

provided $x \in E$, $x' \in E'$ and

$$|\, x - x'\, | < \delta \ (\mathrm{mod}\ \pi_k) \qquad (k = 1, \ldots, m).$$

Then there exists a Bohr a.p. function $q(x)$ such that

$$\text{(i)}\qquad f(x) \leqq q(x) \leqq M \quad \textit{for} \quad x \in E'$$

and

$$\text{(ii)}\qquad |\, f(x) - q(x)\, | \leqq \epsilon \quad \textit{for} \quad x \in E.$$

Proof. Denote by T_k the additive group of reals modulo π_k and let $\varphi_k(x)$ be the canonical homomorphism of R on T_k. T_k

is metrized and the distance between two elements ξ, $\bar{\xi}$ will be denoted by $\varrho_k(\xi, \bar{\xi})$. We introduce in R a new distance $\varrho(x, \bar{x})$ defined as follows

$$\varrho(x, \bar{x}) = \Sigma_{k=1}^{m} \varrho_k(\varphi_k(x), \varphi_k(\bar{x})).$$

It is clear that to every $\alpha > 0$ there corresponds a $\beta > 0$ such that

$$\varrho(x, \bar{x}) < \beta \tag{1}$$

implies

$$| x - \bar{x} | < \alpha \ (\text{mod } \pi_k) \qquad (k = 1, \ldots, m). \tag{2}$$

Conversely, to every $\beta > 0$ corresponds an $\alpha > 0$ such that relations (2) imply relation (1).

We now put for a positive integer n

$$f_n(x) = \sup_{x' \in E'} [f(x') - n\varrho(x, x')].$$

We shall show that $f_n(x)$ is a Bohr a.p. function. In fact

$$f_n(\bar{x}) = \sup_{x' \in E'} [f(x') - n\varrho(\bar{x}, x')].$$

Hence, for $x' \in E'$

$$f_n(\bar{x}) \geqq f(x') - n\varrho(\bar{x}, x').$$

$$- f_n(\bar{x}) \leqq - f(x') + n\varrho(\bar{x}, x') \leqq - f(x') + n\varrho(\bar{x}, x) + n\varrho(x, x')$$

$$f(x') - n\varrho(x, x') \leqq f_n(\bar{x}) + n\varrho(\bar{x}, x).$$

This relation holding for any $x' \in E'$, we conclude

$$f_n(x) \leqq f_n(\bar{x}) + n\varrho(x, \bar{x}).$$

In the same way we prove

$$f_n(\bar{x}) \leqq f_n(x) + n\varrho(\bar{x}, x),$$

so that

$$| f_n(x) - f_n(\bar{x}) | \leqq n\varrho(x, \bar{x}).$$

Let $\eta > 0$ be given; take $\beta = \eta/n$ and let $\alpha > 0$ be the number associated to β in such a way that relations (2) imply relation (1) Relations (2) imply therefore

$$| f_n(x) - f_n(\bar{x}) | \leqq n(\eta/n) = \eta,$$

and this proves that $f_n(x)$ is a Bohr a.p. function.

It is clear that (whatever be n)

(i′) $$f(x) \leqq f_n(x) \leqq J \quad \text{for} \quad x \in E'.$$

To complete the proof of the lemma we shall show that for some n we have

(ii′) $$| f(x) - f_n(x) | \leqq \epsilon \quad \text{for} \quad x \in E.$$

In fact, by hypothesis

(3) $$f(x') \leqq f(x) + \epsilon$$

provided $x \in E$, $x' \in E'$ and

(4) $$| x - x' | < \delta \pmod{\pi_k} \qquad (k = 1, \ldots, m).$$

Let $\delta' > 0$ be the number associated to δ in such a way that

(5) $$\varrho(x, x') < \delta'$$

implies relations (4). Thus relation (5), combined with $x \in E$, $x' \in E'$ implies (3).

Take n such that $n\delta' > 2M$; then, for a fixed $x \in E \subset E'$ we have

$$\sup_{x' \in E',\, \varrho(x, x') \geqq \delta'} [f(x') - n\varrho(x, x')] < -M \leqq f_n(x).$$

We conclude

$$f_n(x) = \sup_{x' \in E',\, \varrho(x, x') < \delta'} . [f(x') - n\varrho(x, x')],$$

$$f_n(x) \leqq \sup_{x' \in E',\, \varrho(x, x') < \delta'} . [f(x')],$$

so that, by (3)

$$f_n(x) \leqq f(x) + \epsilon \quad (\text{for } x \in E).$$

This, combined with (i′) gives the required relation (ii′).

The lemma is now proved.

Proof of theorem 2

Necessity. Let $f(x)$ be a R.S.a.p. function. It will suffice to prove that to every $\epsilon > 0$ we can associate two Bohr a.p. functions $p(x)$ and $q(x)$ satisfying conditions (i) and (ii) of the theorem. Moreover, we can suppose that $f(x)$ is real.

Let

$$| f(x) | < M \qquad \text{for} \quad x \in R.$$

Let $\epsilon > 0$ be given. By theorem 1 we can find a measurable set E and numbers $\delta > 0$, $\pi_1, \ldots, \pi_m$ such that

(1) $$S(\tilde{E}) < \frac{\varepsilon}{6M}$$

and such that

$$| f(x) - f(x + \tau) | < \frac{\epsilon}{3}$$

provided $x \epsilon E$ and

$$| \tau | < \delta \pmod{\pi_k} \qquad (k = 1, \ldots, m).$$

By the lemma, taking $E' = R$ we can find a Bohr a.p. function $q(x)$ such that

$$f(x) \leqq q(x) \leqq M \qquad \text{for} \quad x \epsilon R$$

and

$$| q(x) - f(x) | \leqq \frac{\epsilon}{3} \quad \text{for} \quad x \epsilon E.$$

In the same way we can find a Bohr a.p. function $p(x)$ such that

$$- M \leqq p(x) \leqq f(x) \quad \text{for} \quad x \epsilon R$$

and

$$| f(x) - p(x) | \leqq \frac{\epsilon}{3} \quad \text{for} \quad x \epsilon E.$$

Then, by (1)

$$\sup_x \overline{\int_x^{x+1}} | q(t) - p(t) | \, dt < \frac{2\varepsilon}{3} + 2M \frac{\varepsilon}{6M} = \varepsilon.$$

For R.W.a.p. or R.B.a.p. functions the proof is quite similar.

Sufficiency. Let $f(x)$ satisfy the condition of the theorem. Let $\epsilon > 0$ be given and let $p(x)$, $q(x)$ be two Bohr a.p. functions such that

$$p(x) \ll f(x) \ll q(x)$$

and

(1) $$\sup_x \overline{\int_x^{x+1}} | q(t) - p(t) | \, dt < \frac{\varepsilon}{5}.$$

Choose $\delta > 0$, $\pi_1, \ldots, \pi_m$ such that

$$| p(t + \tau_t) - p(t) | < \frac{\varepsilon}{5}$$

$$| q(t + \tau_t) - q(t) | < \frac{\varepsilon}{5}$$

provided

$$|\tau_t| < \delta \pmod{\pi_k} \qquad (k = 1, \ldots, m). \tag{2}$$

Then

$$|f(t+\tau_t) - f(t)| \leqq |f(t+\tau_t) - p(t+\tau_t)| + |p(t+\tau_t) - p(t)| + |f(t) - p(t)|$$
$$\leqq |q(t+\tau_t) - p(t+\tau_t)| + |p(t+\tau_t) - p(t)| + |q(t) - p(t)|$$
$$\leqq |q(t+\tau_t) - q(t)| + |q(t) - p(t)| + |p(t) - p(t+\tau_t)| + |p(t+\tau_t) - p(t)| + |q(t) - p(t)|.$$

Thus, relations (2) imply

$$|f(t+\tau_t) - f(t)| \leqq \frac{3\varepsilon}{5} + 2\,|q(t) - p(t)|.$$

The same relations (2), therefore, imply by (1)

$$\overline{\sup_x}\int_x^{x+1} |f(t+\tau_t) - f(t)|\,dt \leqq \frac{3\epsilon}{5} + 2\frac{\epsilon}{5} = \epsilon.$$

This proves, since ϵ is arbitrary, that $f(x)$ is a R.S.a.p. function.

For the R.W.a.p. or the R.B.a.p. classes the proof is quite similar.

Proof of theorem 3

Necessity. Let $f(x)$ be a K.S.a.p. function and let $\epsilon > 0$ be given. We can find a measurable set E and numbers $\delta > 0$, $\pi_1, \ldots, \pi_m$ for which $S(\tilde{E}) < \epsilon$, and

$$|f(x) - f(x')| < \epsilon$$

provided $x \epsilon E$, $x' \epsilon E$ and

$$|x - x'| < \delta \pmod{\pi_k} \quad (k = 1, \ldots, m).$$

If we show that there is a constant M such that

$$|f(x)| \leqq M \quad \text{for} \quad x \epsilon E,$$

then, by the lemma, taking $E' = E$ we can find a Bohr a.p. function $q(x)$ such that

$$|f(x) - q(x)| \leqq \epsilon \quad \text{for} \quad x \epsilon E,$$

and the condition of the theorem will be proved.

So suppose there is a sequence x_n of points of E for which

$$\lim_{n\to\infty} |f(x_n)| = \infty. \tag{1}$$

We can extract from x_n a subsequence $\bar{x}_n$ such that whatever be p, q

$$|\bar{x}_p - \bar{x}_q| < \delta \pmod{\pi_k} \qquad (k = 1, \ldots, m).$$

Then

$$|f(\bar{x}_p) - f(\bar{x}_q)| < \epsilon;$$

but this is incompatible with (1).

For the K.W.a.p. or the K.B.a.p. functions the proof is quite similar.

Sufficiency. Let $f(x)$ satisfy the condition of the theorem. Let $\epsilon > 0$ be given. Let the polynomial $q(x)$ and the measurable set E be such that

$$S(\tilde{E}) < \frac{\epsilon}{3}$$

and

$$(1) \qquad |f(x) - q(x)| < \frac{\epsilon}{3} \qquad \text{for} \quad x \in E.$$

We can find a $\delta > 0$ and numbers $\pi_1, \ldots, \pi_m$ such that

$$(2) \qquad |q(x) - q(x')| < \frac{\epsilon}{3}$$

provided

$$(3) \qquad |x - x'| < \delta \pmod{\pi_k} \qquad (k = 1, \ldots, m).$$

If $x \in E$, $x' \in E$ and if relations (3) hold, then, by (1) and (2)

$$|f(x) - f(x')| \leqq |f(x) - q(x)| + |q(x) - q(x')| + |q(x') - f(x')|$$

$$\leqq \frac{\epsilon}{3} + \frac{\epsilon}{3} + \frac{\epsilon}{3} = \epsilon.$$

$f(x)$ is thus a K.S.a.p. function.

For the K.W.a.p. or the K.B.a.p. classes the proof is quite similar.

Proof of theorem 4

Necessity. Suppose that $f(x)$ is a R.B.a.p. function. Let ϵ_l be a sequence of positive numbers tending to 0 and let $p_l(x)$, $q_l(x)$ be the polynomials associated by theorem 2 to ϵ_l. Let λ_ν be a sequence containing all the non-vanishing exponents of each of the polynomials $p_l(x)$, $q_l(x)$, $l = 1, 2, \ldots$ Choose ξ such that $\xi\lambda_\nu/2\pi$ is never an integer. We may suppose, by considering separately the real and imaginary parts, that $f(x)$ is real and that $p_l(x)$, $q_l(x)$ are real Bohr a.p. functions.

Let

$$M = \overline{\text{bound}}_{l}\, \mathfrak{M}\{p_l(x)\} = \underline{\text{bound}}_{l}\, \mathfrak{M}\{q_l(x)\}.$$

$\epsilon > 0$ being given, let $p(x)$, $q(x)$ be two real Bohr a.p. functions chosen among the $p_l(x)$, $q_l(x)$ such that

(1) $$p(x) \leqq f(x) \leqq q(x)$$

(2) $$\mathfrak{M}\{q(x) - p(x)\} < \frac{\epsilon}{3}.$$

We put

$$p_0 = \mathfrak{M}\{p(x)\}, \qquad q_0 = \mathfrak{M}\{q(x)\}.$$

Then, by (1) and (2)

(3) $$q_0 - \frac{\epsilon}{3} < M < p_0 + \frac{\epsilon}{3}.$$

We see easily, in view of our choice of ξ [6]), that there exists a number n_0 such that for $n \geqq n_0$ and every x_0

(4) $$p_0 - \frac{\epsilon}{3} \leqq \frac{1}{n} \Sigma_{l=0}^{n-1} p(x_0 + l\xi)$$

(5) $$\frac{1}{n} \Sigma_{l=0}^{n-1} q(x_0 + l\xi) \leqq q_0 + \frac{\epsilon}{3}.$$

Also we can find a $\delta > 0$ and numbers $\pi_1, \ldots, \pi_m$ such that

$$| p(x_0) - p(x_l) | < \frac{\epsilon}{3} \text{ and } | q(x_0) - q(x_l) | < \frac{\epsilon}{3},$$

provided

$$| x_0 - x_l | < \delta \pmod{\pi_k} \qquad (k = 1, \ldots, m).$$

We conclude, by (3), (4) and (5) that

$$M - \epsilon < p_0 - \frac{2\epsilon}{3} \leqq \frac{1}{n} \Sigma_{l=0}^{n-1} p(x_l + l\xi)$$

$$\frac{1}{n} \Sigma_{l=0}^{n-1} q(x_l + l\xi) \leqq q_0 + \frac{2\epsilon}{3} < M + \epsilon$$

provided

$$| x_i - x_j | < \delta \pmod{\pi_k} \quad \begin{pmatrix} i, j = 0, \ldots, n-1 \\ k = 1, \ldots, m \end{pmatrix}$$

Thus these last relations imply

$$\left| \frac{1}{n} \Sigma_{l=0}^{n-1} f(x_l + l\xi) - M \right| < \epsilon.$$

Sufficiency. Suppose that $f(x)$ satisfies the condition of the theorem. We may suppose again that $f(x)$ is real.

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[6]) Cf. A. S. Besicovitch, „Almost periodic Functions", Cambridge, 1932, p. 44.

Let $\epsilon > 0$ be given, and let n, $\delta > 0$, $\pi_1, \ldots, \pi_m$ be such that

$$\left| \frac{1}{n} \sum_{l=0}^{n-1} f(x_l + l\xi) - M \right| < \frac{\epsilon}{2}$$

provided

$$| x_i - x_j | < 2\delta \pmod{\pi_k} \quad \begin{pmatrix} i, j = 0, \ldots, n-1 \\ k = 1, \ldots, m. \end{pmatrix}$$

Let τ_x be such that

$$(1) \qquad | \tau_x | < \delta \pmod{\pi_k} \qquad (k = 1, \ldots, m).$$

Then, if θ_x, θ'_x are two functions which take only the values 0 and 1, we have, whatever be x

$$\left| \frac{1}{n} \sum_{l=0}^{n-1} f(x + l\xi + \theta_{x+l\xi} \tau_{x+l\xi}) - M \right| < \frac{\epsilon}{2}$$

$$\left| \frac{1}{n} \sum_{l=0}^{n-1} f(x + l\xi + \theta'_{x+l\xi} \tau_{x+l\xi}) - M \right| < \frac{\epsilon}{2}.$$

Hence

$$\left| \frac{1}{n} \sum_{l=0}^{n-1} \left[f(x+l\xi+\theta_{x+l\xi}\tau_{x+l\xi}) - f(x+l\xi+\theta'_{x+l\xi}\tau_{x+l\xi}) \right] \right| < \epsilon.$$

By an appropriate choice of θ_x and θ'_x we see that

$$\frac{1}{n} \sum_{l=0}^{n-1} \left| f(x + l\xi + \tau_{x+l\xi}) - f(x + l\xi) \right| < \epsilon.$$

Let a be arbitrary and let $L = n\xi$. Then

$$\frac{1}{L} \overline{\int_a^{a+L}} | f(x + \tau_x) - f(x) | \, dx$$

$$= \frac{1}{n\xi} \sum_{l=0}^{n-1} \overline{\int_{a+l\xi}^{a+(l+1)\xi}} | f(x + \tau_x) - f(x) | \, dx$$

$$= \frac{1}{n\xi} \sum_{l=0}^{n-1} \overline{\int_a^{a+\xi}} | f(x + l\xi + \tau_{x+l\xi}) - f(x + l\xi) | \, dx$$

$$= \frac{1}{\xi} \overline{\int_a^{a+\xi}} \frac{1}{n} \sum_{l=0}^{n-1} | f(x + l\xi + \tau_{x+l\xi}) - f(x + l\xi) | \, dx \leqq \epsilon.$$

Relations (1) therefore imply

$$\overline{\lim_{T \to \infty}} \frac{1}{2T} \overline{\int_{-T}^{T}} | f(x + \tau_x) - f(x) | \, dx \leqq \epsilon,$$

so that $f(x)$ is a R.B.a.p. function

Remark. The proof shows that $f(x)$ is a R.W.a.p. function. Thus we see again that the R.W.a.p. and the R.B.a.p. classes are identical.

(Oblatum 1-10-'54).