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Sphere-geometrical Unitary Field Theories

by

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In this paper I will give proofs of the conclusions **1, 2, 3** and **4** [1]) stated below as well as in my previous paper [21] [2]). **1** and **2** correspond respectively to the Kaluza—Klein's unitary field theory [1], [2] and the Einstein—Mayer's [3], **3** and **4** respectively to the Hoffmann's generalization of the Kaluza—Klein's [4] and Einstein—Mayer's [5]. Then I add six further sphere-geometrical unitary field theories **5, 6, 7, 8, 9** and **10** stated below, of which **5, 7** and **9** correspond to the unitary field theories of P. G. Bergmann [19], B. Hoffmann [17] and B. Hoffmann [18] respectively, while **6, 8** and **10** are new. To each of these ten theories there corresponds a new sphere-geometrical connection-geometry, of which the Laguerre connection-geometry finds its origin in the work of Y. Tomonaga [13], [20], [26]. The main purpose of this paper is *to indicate the four-dimensional sphere-geometrical laws for the unitary field theories, so that the assumptions made therein are fulfilled automatically and we are able to avoid the fifth or the sixth dimension in our line of thought, though the fifth or the sixth dimension survives in abstract sense.* The principal importance of this paper seems to lie in the following points: (i) all the figures representing the generalizations of the "Weltpunkte" are realized within the Einstein space V_4, so that the question of four-dimensionality exists no longer; (ii) thus we are lead to the connection geometries of Laguerre's carrier instead of those of the conformal ones, although we have long waited for the latter one. (It is to be noticed that the space of special relativity is the three-dimensional Laguerre space). (iii) My theories have lead us to new sphere-geometrical connection geometries, which have hitherto been considered to be

[1]) The heading numbers **1, 2, . . ., 10** will be retained throughout.

[2]) The numbers in the square brackets refer to the bibliography at the end of this paper.

rather difficult to develop, since the elements of space are other things than points.

In all cases the Einstein space V_4 as basic space is considered to be provided with tangent

N.E. || Euclidean

space in the sense of the

N.E. || limiting

case of the Veblen's projective theory (cfr. [16] and [10]. Also [10], [23]). If we denote the tangential spaces arising in the theories **1**, **2**, ..., and **10** by (1), (2), ..., and (10) respectively, then the tangential spaces, which are realized in the tangent

N.E. || Euclidean

manifold, are situated among them as follows:

(N.E. space) : (Euclidean space) : (N.E. equiform space) : (equiform space)
= (1) : (2) : (7) : (8) = (3) : (4) : (9) : (10) = (5) : (6) : (x) : (y),

where

(1) = dual-conformal (i.e. N.E. Laguerre) space,
(2) = Laguerre space,
(3) = Space of Lie's higher hypersphere geometry,
(4) = "parabolic Lie space" which is new,
(5) = the space which arises from the dual-conformal space by a kind of expansion of each hypersphere and is new,
(6) = the space which arises from the Laguerre space by a kind of expansion of each hypersphere and is new,
(7) = "equiform dual-conformal space" which is new,
(8) = "equiform Laguerre space" which is new,
(9) = "equiform Lie space" which is new,
(10) = "equiform parabolic Lie space" which is new.

The existence of (x) and (y) is thus suggested. The reader will see below what the new ones are.

The corresponding connection geometries (spaces) will be called respectively:

1. dual-conformal connection geometry (space),
2. Laguerre connection geometry (space),
3. Lie connection geometry (space),
4. parabolic Lie connection geometry (space),
5. B-dual-conformal connection geometry (space),
6. B-Laguerre connection geometry (space),

7. equiform dual-conformal connection geometry (space),
8. equiform Laguerre connection geometry (space),
9. equiform Lie connection geometry (space),
10. equiform parabolic Lie connection geometry (space),
x. B-equiform Lie connection geometry (space),
y. B-equiform parabolic Lie connection geometry (space)

when the base manifold is a general Riemannian space X_4, the details of which will be developed in a separate paper. [3])

Since all these geometries are substantially Riemannian geometries, the procedures are comparatively easy as long as one is concerned with normalized vectors. The techniques of the Riemannian geometry will help us very much for sphere-geometrical interpretations.

§ 1. The Unitary Field Theories of Kaluza—Klein and Einstein—Mayer as seen from the View Points of the Sphere-geometries.

In the case of the **unitary field theory of**

1. Kaluza—Klein [1], [2], || **2. Einstein—Mayer** [3],

the fundamental quadratic differential form was

$$d\sigma^2 = \gamma_{\alpha\beta}\, dx^\alpha\, dx^\beta = g_{ij}(x^a)\, dx^i\, dx^j + (\varphi_\alpha(x^a)\, dx^\alpha)^2,$$
$$(a, i, j, \ldots = 1, 2, 3, 4;\ \alpha, \beta, \gamma, \ldots = 1, 2, \ldots, 5),$$

where $\varphi_5 = 1$ and $g_{ij}\, dx^i\, dx^j$ is the fundamental quadratic form of the Einstein space V_4 and

$$\frac{e}{m} = \varphi_\alpha \frac{dx^\alpha}{d\sigma} = \gamma_{\alpha\beta}\, \varphi^\beta\, \gamma^{\alpha\gamma}\, u_\gamma = \delta^\gamma_\beta\, \varphi^\beta\, u_\gamma = \varphi^\alpha\, u_\alpha = u_5$$

$$= \sin\frac{r}{k}\ {}^{4)} \quad \Big\| \quad = r$$

$$= \text{const.},$$

the r being the radius of the hypersphere with the center

$$(\gamma^{\alpha\beta}\varphi_\beta) = (\varphi^\alpha) = (0, 0, 0, 0, 1)$$

in the tangent

N.E. || Euclidean

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[3]) There exist a series of sphere-geometrical connection geometries, based upon a hyperplane manifold. (In preparation).

[4]) The k will be expressed in terms of the constant R below in Theorem 2°.

space as well as the geodesic radius of the corresponding generalized (i.e. geodesic) hypersphere with center (φ^α) in V_4. Then the

$$u_\alpha = \gamma_{\alpha\beta}\frac{dx^\beta}{d\sigma},\ \left(\gamma_{\alpha\beta}\frac{dx^\alpha}{d\sigma}\frac{dx^\beta}{d\sigma}=1\right) \Bigg\| \ u_i = \gamma_{i\beta}\frac{dx^\beta}{d\sigma},\ u_5 = \gamma_{\alpha 5}\frac{dx^\alpha}{d\sigma} = -p$$

are the oriented hyperplane coordinates in the four-dimensional

tangential N.E. || tangential Euclidean

space of V_4 as well as the coordinates of the totally geodesic hypersurfaces enveloping the geodesic hypersphere in V_4.

The field equations were

$$(R^{ij} - \tfrac{1}{2}g^{ij}R) - 2(g^{st}\varphi^i_s\varphi^j_t + \tfrac{1}{4}g^{ij}\varphi^t_s\varphi^s_t) = 0,$$

$$\mathfrak{J}^m = \varphi^{ms}_{\ ,s} = 0,$$

$$\Sigma\frac{\partial\varphi_{ij}}{\partial x^k} = 0,$$

where the latter two are Maxwell's equations and

$$\varphi_{jk} = \tfrac{1}{2}\left(\frac{\partial\varphi_j}{\partial x^k} - \frac{\partial\varphi_k}{\partial x^j}\right),\ \varphi^i_k = g^{ij}\varphi_{jk}.$$

The equations of motion were

$$\frac{d^2x^i}{d\sigma^2} + \left\{ {i \atop jk} \right\}\frac{dx^j}{d\sigma}\frac{dx^k}{d\sigma} + \frac{e}{m}\varphi^i_j\frac{dx^j}{d\sigma} = 0.$$

The space with which we are concerned is a four-dimensional

dual-conformal || Laguerre

connection space which is special in the sense that the base manifold is V_4 instead of a general four-dimensional Riemannian manifold. If we interpret the geodesic hyperspheres as points it is nothing but a special five-dimensional Riemannian space which will be realized within the Einstein space V_4 as a special

dual-conformal || Laguerre

connection space by means of a minimal projection [5]) of the points of the tangent five-dimensional

N.E. || Euclidean

space of V_4 as well as by its generalization in the five-dimensional

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[5]) T. Takasu, [24].

Riemannian space. This fact is legitimated by the formula (O. Veblen, [16], p. 46):

$$\frac{\partial \log (\gamma_{\alpha\beta} X^\alpha X^\beta)}{\partial x^\gamma} = 0$$

as wel as by the following theorems (S. Sasaki—K. Yano, [23]):

THEOREM 1⁰. If the group of holonomy of a space with a normal projective connection P_n is a subgroup of the group of all projective transformations in P_n which fix a non-degenerate hyperquadric Q, the P_n is a space with a projective normal connection corresponding to the class of affinely connected spaces with corresponding paths including an Einstein space with non-vanishing constant scalar curvature. In other words the P_n is projective to an Einstein space with non-vanishing scalar curvature. The converse is also true. Correspondingly for the case of vanishing scalar curvature.

THEOREM 2⁰. Let the group of holonomy of the space with a normal projective connection P_n corresponding to Einstein space V_4 with positive definite fundamental tensor and of non-vanishing scalar curvature fix a real or imaginary hyperquadric according as the scalar curvature R is negative or positive. Then the arc length of a geodesic segment PQ is expressed by

$$PQ = \frac{k}{2i} \log (PQ, YZ), \qquad k = \sqrt{3/|R|},$$

where Y and Z are the points of intersections of PQ with the invariant hyperquadric in the tangent N.E. space.

Correspondingly for the case of vanishing scalar curvature.

Thus *the lengths as well as angles are common to the base manifold* V_4 *and the tangent N.E. or Euclidean space, so that the so-called geodesic polar coordinates are also common to them*:

$$\begin{cases} \mathfrak{x}^i = \sin \dfrac{r}{k} C^i, \\ \mathfrak{x}^5 = \cos \dfrac{r}{k}, \end{cases} \begin{pmatrix} C^i = \cos \theta^i, \\ g_{ij} C^i C^j = 1 \end{pmatrix}, \quad (g_{ij}\mathfrak{x}^i\mathfrak{x}^j + \mathfrak{x}^5\mathfrak{x}^5 = 1.) \quad \Bigg\| \quad \begin{cases} \mathfrak{x}^i = rC^i, \\ \mathfrak{x}^5 = 1, \end{cases} \begin{pmatrix} C^i = \cos \theta^i, \\ g_{ij} C^i C^j = 1 \end{pmatrix}, \quad (g_{ij}\mathfrak{x}^i\mathfrak{x}^j = r^2.)$$

It is well known that the Einstein space V_4 is totally umbilical and the oriented hypersphere (as well as the oriented geodesic hypersphere in V_4) with center (φ^α) is given by the equation:

$$r = \text{const.},$$

The N.E. || The

Laguerre coordinates of the oriented hypersphere (as well as of the oriented geodesic hypersphere in V_4) $(\mathfrak{x}^\alpha, \varrho)$ are

$\begin{cases} \xi^i = \mathfrak{x}^i / \cos \frac{\varrho}{k}, \\ \xi^5 = i(g_{ij}\mathfrak{x}^i\mathfrak{x}^j + \mathfrak{x}^5\mathfrak{x}^5)^{\frac{1}{2}} \tan \frac{\varrho}{k}, \\ \xi^6 = \mathfrak{x}^5/\cos \frac{\varrho}{k}, \end{cases}$ $(g_{ij}\xi^i\xi^j + \xi^5\xi^5 + \xi^6\xi^6 = 1).$ The N.E.	$\begin{cases} \xi^i = \mathfrak{x}^i, \\ \xi^5 = i\varrho, \\ \mathfrak{x}^6 = 1, \end{cases}$ $(\xi^6 = \mathfrak{x}^6 = 1).$ The

Laguerre coordinates of the oriented hyperplane (as well as of the oriented totally geodesic hypersurface in V_4) (u_α) are

$\begin{cases} \sigma . U_\alpha = u_\alpha, \\ \sigma . U_6 = i, \\ (\gamma^{\alpha\beta} U_\alpha U_\beta + U_6 U_6 = 0), \end{cases} \quad (\sigma \neq 0)$	$\begin{cases} \sigma . U_i = u_i, \\ \sigma . U_5 = u_5 = i, \\ \sigma . U_6 = -p, \end{cases} \quad (\sigma \neq 0)$ $(g^{ij} U_i U_j + U_5 U_5 = 0),$

so that the equation to the oriented hypersphere (as well as to the oriented geodesic hypersphere in V_4) $(\xi^A(x^a))$ is

$$\xi^A U_A = 0, \quad (A = 1, 2, \ldots, 6),$$

which forms a hypercomplex (a system of ∞^4 oriented geodesic hyperspheres).

The space of

Kaluza—Klein	*Einstein—Mayer*

may thus be considered to have arisen from the Einsteinean V_4 by expansion of each point of V_4 to an oriented geodesic sphere of constant radius r such that e/m

$= \sin \frac{r}{k}.$	$= r.$

From this I have concluded as follows:

1. *The Kaluza—Klein's*	**2.** *The Einstein—Mayer's*

space is equivalent to the Einstein's space V_4 (special-)

dual-conformal	*Laguerre*

connection geometrically so that the points in V_4 correspond to the geodesic hyperspheres of equal geodesic radii, whose developments in the

N.E.	*Euclidean*

tangential spaces are hyperspheres of equal radii.

[N.B.] According to the new view points in differential geometry in the large of Shing-Shen Chern [27], the fundamental orthogonal differential form for $d\sigma^2$ is, except for quadratic transformations, expressible in the form

$$d\sigma^2 = \omega_1^2 + \omega_2^2 + \ldots + \omega_5^2,$$

where ω_i are Pfaffians. In our case we have

$$\omega_5 = \varphi_\alpha \, dx^\alpha$$

$$= \sin \frac{r}{k} \, d\sigma \qquad \| \qquad = r d\sigma$$

and the part

$$-ds^2 = \omega_1^2 + \omega_2^2 + \omega_3^2 + \omega_4^2$$

corresponds to the Einsteinean V_4.

§ 2. The Hoffmann's Generalization of the Unitary Field Theories of Kaluza—Klein and Einstein—Mayer as seen from the View Points of the Sphere-geometries.

In the Hoffmann's generalization of the

3. Kaluza—Klein [4] ‖ **4. Einstein—Mayer** [5]

space, the fundamental quadratic differential form was

$$dS^2 = \gamma_{AB} \, dx^A dx^B = g_{ij}(x^a) \, dx^i dx^j + (\varphi_A(x^a) \, dx^A)^2 + (\psi_A(x^a) dx^A)^2,$$
$$(a, b, \ldots, i, j, \ldots = 1, 2, 3, 4; \alpha, \beta, \ldots = 1, 2, \ldots, 5;$$
$$A, B, \ldots = 1, 2, \ldots, 6),$$

where $g_{ij} dx^i dx^j$ is the fundamental quadratic differential form of the Einstein space V_4, and

$\varphi_5 = 1, \ \varphi_6 = 0,$	$\psi_5 = 0, \ \psi_6 = 1,$
$e/m = \varphi_A \dfrac{dx^A}{dS}$	$\mu/m = \psi_A \dfrac{dx^A}{dS}$
$= \sin \dfrac{r}{k} \ \| \ = r$	$= \sin \dfrac{r'}{k} \ \| \ = r'$

$$= \text{const.}\ ^{6}),$$

the r ‖ the r'

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[6]) The subsequent interpretations of (φ^A, r) and (ψ^A, r') will be made in two ways exposed on the both sides of |.

being the radius (generalized in the sense of common tangential segment) of the oriented linear hypercomplex (of oriented hyperspheres with equal radii

r')	r)

with its "center" (oriented hypersphere with radius

r' and center $(\gamma^{AB}\psi_B)=(\psi^A)=(0,0,0,0,0,1))$: (φ^A)	r and center $(\gamma^{AB}\varphi_B)=(\varphi^A)=(0,0,0,0,1,0))$: (ψ^A)

in the tangential four-dimensional

N.E.	Euclidean

space as well as the geodesic radius (generalized in the sense of common geodesic tangential segment) of the corresponding oriented linear hypercomplex (of oriented geodesic hyperspheres with equal geodesic radii

r')	r)

with its "center" (oriented geodesic hypersphere with geodesic radius

r' and center $(\psi^A))$: (φ^A)	r and center $(\varphi^A))$: (ψ^A)

realized in the Einstein space V_4. Thereby the

$=\gamma_{AB}\frac{dx^B}{dS}$,	$u_A=\gamma_{AB}\frac{dx^B}{dS}$,	$u_\alpha=\gamma_{\alpha B}\frac{dx^B}{dS}$,	$u_{\alpha'}=\gamma_{\alpha' B}\frac{dx^B}{dS}$,
${}_B\frac{dx^A}{dS}\frac{dx^B}{dS}=1)$	$(\gamma_{AB}\frac{dx^A}{dS}\frac{dx^B}{dS}=1)$	$u_6=\gamma_{B6}\frac{dx^B}{dS}=-P$,	$u_5=\gamma_{B5}\frac{dx^B}{dS}=-P$
		$(\alpha=1, 2, \ldots, 5)$	$(\alpha'=1, 2, \ldots, 4,$

are the coordinates of the oriented linear hypercomplex [7]) of the oriented hyperspheres touching properly two oppositely oriented hyperplanes in the tangential four-dimensional

N.E.	Euclidean

space of V_4 as well as the coordinates of the corresponding oriented generalized totally geodesic linear hypercomplexes (belonging to the generalized linear hypercomplex of the oriented hyperspheres under consideration realized in V_4).

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[7]) A four-dimensional generalization of the system of oriented spheres touching properly a pair of oppositely oriented planes.

The field equations were

$$(R^{mn}-\tfrac{1}{2}g^{mn}R)+2(g^{st}\varphi_s^m\varphi_t^n+\tfrac{1}{4}g^{mn}\varphi_t^s\varphi_s^t)-2(g^{st}\psi_s^m\psi_t^n+\tfrac{1}{4}g^{mn}\psi_t^s\psi_s^t)=0,$$

$$\mathfrak{J}^m=\varphi^{ms}_{\ \ ,s}=0,$$

$$\psi^{ms}_{\ \ ,s}=0,$$

$$\Sigma\frac{\partial\varphi_{ij}}{\partial x^k}=0,$$

$$\Sigma\frac{\partial\psi_{ij}}{\partial x^k}=0,$$

where

$\varphi_{jk}=\frac{1}{2}\left(\frac{\partial\varphi_j}{\partial x^k}-\frac{\partial\varphi_k}{dx^j}\right),$	$\psi_{jk}=\frac{1}{2}\left(\frac{\partial\psi_i}{\partial x^k}-\frac{\partial\psi_k}{\partial x^j}\right),$
$\varphi_k^i=g^{ij}\varphi_{jk}.$	$\psi_k^i=g^{ij}\psi_{jk}.$

The equations of motion were

$$\frac{d^2x^i}{dS^2}+\left\{ {i \atop jk} \right\}\frac{dx^j}{dS}\frac{dx^k}{dS}+\frac{e}{m}\varphi_j^i\frac{dx^j}{dS}+\frac{\mu}{m}\frac{dx^i}{dS}=0.$$

(i) The space in consideration is a four-dimensional

Lie	parabolic Lie

connection space which is special in the sense that the basic manifold is V_4 instead of a general four-dimensional Riemannian space. If we interpret the linear hypercomplexes of geodesic hyperspheres as points it is nothing but a special six-dimensional Riemannian space which will be realized in the Einstein space V_4 as a special

Lie	parabolic Lie

connection space by means of two successive minimal projections[8]) of the points of the tangent six-dimensional

N.E.	Euclidean

space of V_4 from two different mutually orthogonal directions upon the four-dimensional

N.E.	Euclidean

tangent space of V_4 as well as by their generalization in the six-dimensional Riemannian space.

[8]) T. Takasu, [24].

(ii) The space under consideration which is a special four-dimensional

Lie	parabolic Lie

connection space is nothing but a special five-dimensional

dual-conformal	Laguerre

connection space if the oriented linear hypercomplexes of oriented geodesic hyperspheres are interpreted as oriented hyperspheres and this

dual-conformal	Laguerre

connection space will be realized in the Einstein space V_4 by means of one minimal projection of the points of the tangent five-dimensional

dual-conformal	Laguerre

space upon the four-dimensional

N.E.	Euclidean

tangent space of V_4 as well as its generalization in the five-dimensional

dual-conformal	Laguerre

connection space.

These facts are also legitimated by remembering that the analytical apparatus is common to the tangential space of V_4 and the space realized in V_4 itself, which is stated in § 1, the meaning of k being the same. The four-dimensional

Lie	parabolic Lie

coordinates of the oriented hypersphere (as well as of the oriented geodesic hypersphere in V_4) may be constructed as follows:

$\begin{cases} \sigma \cdot \zeta^A = \xi^A, & (\sigma \neq 0) \\ \sigma \cdot \zeta^7 = i(g_{ij}\xi^i\xi^j + \xi^5\xi^5 + \xi^6\xi^6) \end{cases}$ $= i,$ $(\gamma_{AB}\zeta^A\zeta^B + \zeta^7\zeta^7 = 0).$	$\begin{cases} \zeta^\alpha = \xi^\alpha, \\ \zeta^6 = \xi^6 = 1, \\ \zeta^7 = i\varrho. \end{cases}$
The Lie	The parabolic Lie

coordinates

$(U_L) = (\gamma_{LM} U^M)$, $(\gamma_{LM} U^L U^M = 1)$, $(L, M = 1, 2, \ldots, 7)$	$\begin{cases} U_\alpha = \gamma_{\alpha\beta} U^\beta, \\ U_6 = -P, \\ U_7 = U^7 = i, \ (U^\alpha U_\alpha + U^7 U_7 = 0) \end{cases}$

of the oriented linear hypercomplex of the oriented hyperspheres (as well as of oriented geodesic hyperspheres) (ζ^L) are such that [9])

$$U_L \zeta^L = 0, \quad (L = 1, 2, \ldots, 7).$$

The space of the Hoffmann's generalization of the

Kaluza—Klein	*Einstein—Mayer*

space may thus be considered to have arisen from the

Kaluza—Klein's	*Einstein—Mayer's*

by expansion of each oriented geodesic hyperspheres (φ^α, r) *of* V_4 *to an oriented linear hypercomplex of the constant geodesic generalized (in the sense of common tangential geodesic segment) radius* r' *such that*

$$\frac{\mu}{m}$$

$= \sin \frac{r'}{k}.$	$= r'.$

From this I have concluded as follows:

The Hoffmann's generalization of the

Kaluza—Klein's	*Einstein—Mayer's*

space is equivalent to the Einstein's space V_4 *special*

Lie	*parabolic Lie*

connection geometrically so that the points in the Einstein space V_4 *correspond to the special linear hypercomplexes of generalized (geodesic) hyperspheres, whose developments in the tangent*

N.E.	*Euclidean*

space are hyperspheres of equal radii.

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[9]) This equation for the righthand side shows, when it is interpreted in a space of six dimensions, that the point (ξ^α, ξ^7) lies on the minimal hyperplane (with coordinates $(U_\alpha, U_7, -P, (U^\alpha U_\alpha + U^7 U_7 = 0))$ expressed in its Hesse's normal form.

§ 3. The Unitary Field Theory of Jordan—Bergmann as seen from the View Point of a Sphere-geometry and a New Allied Theory.

In the case of the

5. Jordan—Bergmann's space [19],	6. B-Laguerre connection space introduced in the following lines,

the fundamental quadratic differential form is

$$d\sigma^2 = \gamma_{\alpha\beta}\, dx^\alpha\, dx^\beta = g_{ij}(x^a)\, dx^i\, dx^j + C^2(x^a)(\varphi_\alpha(x^a)\, dx^\alpha)^2,$$
$$(i, j, \ldots = 1, 2, 3, 4;\ \alpha, \beta, \ldots = 1, 2, \ldots, 5)$$

leading to the variable

$$e/m(x^a) = C(x^a)\varphi_\alpha \frac{dx^\alpha}{d\sigma}$$

$= \sin\dfrac{r}{k}$,	$= r$,

where $\varphi_5 = 1$ and $g_{ij}dx^i dx^j$ is the fundamental quadratic form of the Einstein space V_4, the r being the radius of the hypersphere with center $(\varphi^\alpha) = (\gamma^{\alpha\beta}\varphi_\beta) = (0, 0, 0, 0, 1)$ in the tangential four-dimensional

N.E.	Euclidean

hyperspace as well as the geodesic radius of the corresponding geodesic hypersphere with center (φ^A) in V_4. Therefore the

$u_\alpha = \gamma_{\alpha\beta}\dfrac{dx^\beta}{d\sigma}$, $\left(\gamma_{\alpha\beta}\dfrac{dx^\alpha}{d\sigma}\dfrac{dx^\beta}{d\sigma} = 1\right)$	$u_i = \gamma_{i\beta}\dfrac{dx^\beta}{d\sigma}$, $u_5 = \gamma_{\alpha 5}\dfrac{dx^\alpha}{d\sigma} = -p$

are the oriented hyperplane coordinates in the tangential four-dimensional

N.E.	Euclidean

space of V_4 as well as the coordinates of the totally geodesic hypersurface enveloping the geodesic hypersphere under consideration in V_4.

Putting

$$A_\alpha = C\varphi_\alpha,\ \psi = \log C(x^s),\ \varphi_{mn} = \frac{1}{C}\gamma^\mu_m\gamma^\nu_n(A_{\mu,\nu} - A_{\nu,\mu}) = \varphi_{m,n} - \varphi_{n,m},$$

the following results were obtained:

$$\gamma^{\varkappa\lambda} R_{\varkappa\lambda} = g^{kl} R_{kl} + \tfrac{1}{4} C^2 \varphi^{rs} \varphi_{rs} + \frac{2}{C} g^{rs} C_{\ rs},$$

$$\varphi^{rs}_{,s} = -2\varphi^{rs} \psi_{,s} \text{ (Maxwell's equations)},$$

$$A^{\varrho}_{,\varrho} = 0,$$

$$g^{rs}\psi_{,r}\psi_{,s} + \tfrac{1}{8}\varphi^{rs}\varphi_{rs} = 0 \text{ (the fifteenth equation)}$$

accompanied by the identities:

$$A_{\mu,\nu} + A_{\nu,\mu} + \psi_{,\mu} A_{\nu} + \psi_{,\nu} A_{\mu} = 0,$$

$$\psi_{,\varrho} A^{\varrho} = 0,$$

$$(B^{\varrho} = A^{\varrho}_{,\sigma} A^{\sigma}, \ \ B_{\varrho} = -\psi_{,\varrho}).$$

For arguments for and against the theory we refer to the original paper of Bergmann [19].

We will refer to the connection geometry corresponding to

$$d\sigma^2 = \gamma_{\alpha\beta}\, dx^{\alpha}\, dx^{\beta} = g_{ij}(x^a)\, dx^i\, dx^j + C^2(x^a)(\varphi_{\alpha}(x^a)\, dx^{\alpha})^2$$

for the general Riemannian quadratic form $g_{ij}dx^i dx^j$ as

B-dual-conformal || *B-Laguerre*

geometry and conclude as follows:

The Jordan-Bergmann's || *The B-Laguerre's*

space is equivalent to the Einstein space V_4 special

B-dual-conformal || *B-Laguerre*

connection geometrically, so that the points in V_4 corresponds to the geodesic hyperspheres in the

Jordan—Bergmann's space, || *B-Laguerre space,*

whose developments in the four-dimensional tangential

N.E. || *Euclidean*

space are hyperspheres of variable radii r such that

$$\frac{e}{m(x^a)} = C(x^a)\left(\varphi_{\alpha}\frac{dx^{\alpha}}{d\sigma}\right)$$

$$= \sin\frac{r}{k}. \quad \Big\| \quad = r.$$

[10]) For the reason, cfr. the conclusion of § 1.

§ 4. The Hoffmann's Field Theory Unifying the Gravitation and the Vector Meson Fields as seen from the View Points of a Sphere-Geometry and a New Allied Theory.

In the case of the

7. Hoffmann [17] space	**8. equiform Laguerre space** as will be introduced in the following lines

for vector meson and gravitation fields, the fundamental quadratic differential form is

$$d\sigma^2 = G_{\alpha\beta}dx^\alpha dx^\beta = G_{55}(g_{ij}(x^a)dx^i dx^j + (\varphi_\alpha(x^a)dx^\alpha)^2 = G_{55}\gamma_{\alpha\beta}dx^\alpha dx^\beta,$$
$$(a, b, \ldots, i, j, \ldots = 1, 2, 3, 4;\ \alpha, \beta, \ldots = 1, 2, \ldots, 5),$$

where $\varphi_5 = 1$ and $g_{ij}dx^i dx^j$ is the fundamental quadratic form of the Einstein space V_4 and $G_{55} = \Phi^2(x^a, x^0) = e^{2N}x^0 f(x^a)$ is a scalar of index N and is

$$G_{55}\,\varphi_\alpha \frac{dx^\alpha}{d\sigma}$$

$= \sin \dfrac{r}{k}$,	$= r$,

r being the radius of the hypersphere with center $(\varphi^\alpha) = (\gamma^{\alpha\beta}\varphi_\beta) = (0, 0, 0, 0, 1)$ in the tangential four-dimensional

N.E.	Euclidean

space as well as the geodesic radius of the corresponding geodesic hypersphere with center (φ^α) in V_4. Therefore

$u_\alpha = G_{\alpha\beta}\dfrac{dx^\beta}{d\sigma}$, $\left(G_{\alpha\beta}\dfrac{dx^\alpha}{d\sigma}\dfrac{dx^\beta}{d\sigma} = 1\right)$	$u_i = G_{i\beta}\dfrac{dx^\beta}{d\sigma}$, $u_5 = G_{\alpha 5}\dfrac{dx^\alpha}{d\sigma} = -p$

are the oriented hyperplane coordinates in the tangential four-dimensional

N.E.	Euclidean

space of V_4 as well as the coordinates of the totally geodesic hypersurface enveloping the geodesic hypersphere under consideration in V_4.

The field equations are

$$(R^{ab}-\tfrac{1}{2}g^{ab}R)-6N^2g^{ab}+\tfrac{1}{2}(g^{cd}\theta^a_c\theta^b_d+\tfrac{1}{4}g^{ab}\theta^c_d\theta^d_c)-12N^2(\theta^a\theta^b-\tfrac{1}{2}g^{ab}\theta^c\theta_c)=0,$$

(a) $$\theta^{ab}_{,b}+12N^2\theta^a=0,$$

(b) $$\theta^a_{,a}=0,$$

where

$$\Phi_\alpha = N^{-1}\frac{\partial \log \Phi}{\partial x^\alpha}, \quad (\Phi_0 = 1),$$

$$\theta_\alpha = \varphi_\alpha - \Phi_\alpha,$$

$$\theta_{ab} = 2\varphi_{ab} = \frac{\partial\varphi_a}{\partial x_b} - \frac{\partial\varphi_b}{\partial x_a} = \frac{\partial\theta_a}{\partial x^b} - \frac{\partial\theta_b}{\partial x^a},$$

$$\varphi^{ab} = g^{ac}g^{bd}\varphi_{cd}, \quad \varphi^b_a = g^{bc}\varphi_{ca}.$$

Here (b) is a direct consequence of (a).

The reciprocal of the index N has the significance that except for a numerical factor, it is the range of the meson force.

x° is the gauge variable:

$$x'^0 = x^0 + \{\log f(x^a)\}^{1/N}$$

When the quadratic differential form $g_{ij}dx^i dx^j$ is a general one we will call the connection space corresponding to

$$d\sigma^2 = G_{55}(\gamma_{\alpha\beta}dx^\alpha dx^\beta) = G_{55}(a^a, x^0)\,[g_{ij}(x^a)dx^i dx^j + \{\varphi_\alpha(x^a)dx^\alpha\}^2]$$

the *equiform*

dual-conformal || *Laguerre*

connection space and the corresponding geometry the equiform

dual-conformal || *Laguerre*

connection geometry.

The space under consideration is nothing but a special five-dimensional Weyl (i.e. equiform connection) space if the geodesic hyperspheres are interpreted as points and this Weyl space will be realized within the Einstein space V_4 as a special equiform

dual-conformal || Laguerre

connection space by means of a minimal projection (accompanied by a kind of tangential dilatation) of the points of the tangential five-dimensional

equiform N.E. || equiform

space upon the four-dimensional

N.E.	Euclidean

tangential space of V_4 as well as by its generalization in the five-dimensional Weyl space.

From this [11]) I conclude as follows:

The special equiform dual-conformal connection space of Hoffmann	*The special equiform Laguerre connection space*

is equivalent to the Einstein space V_4 special equiform

dual-conformal	*Laguerre*

connection geometrically so that the points in V_4 correspond to the geodesic hyperspheres whose developments in the

N.E.	*Euclidean*

tangential spaces of V_4 are hyperspheres of the radii r such that

$$G_{55}\,\varphi_\alpha \frac{dx^\alpha}{d\sigma}$$

$= \sin \frac{r}{k}.$	$= r.$

§ 5. The Hoffmann's Field Theory unifying the Gravitation, the Electromagnetism and the Vector Meson as seen from the View Points of a Sphere-geometry and a New Allied Theory.

In the case of the

9. Hoffmann's space [18]	**10. equiform parabolic Lie space** as will be introduced in the following lines

for vector meson, gravitation and electromagnetic fields, the fundamental quadratic differential form is

$$dS^2 = S_{66}[\gamma_{\alpha\beta}(x^a)dx^\alpha dx^\beta + \{\psi_A(x^a)dx^A\}^2] = S_{AB}dx^A dx^B = S_{66}s_{AB}dx^A dx^B$$
$$= e^{2NX^0} f(x^a)[g_{ij}(x^a)dx^i dx^j + \{\varphi_\alpha(x^a)dx^\alpha\}^2 + \{\psi_A(x^a)dx^A\}^2], \quad (s_{66}=1),$$
$$(a, b, \ldots, i, j, \ldots = 1, 2, \ldots, 5; \; B, A, \ldots = 1, 2, \ldots, 6),$$

[11]) Cfr. also the conclusion of § 1.

where $g_{ij}dx^i dx^j$ is the fundamental quadratic differential form of the Einstein space V_4 and

$\varphi_5 = 1,\ \varphi_6 = 0,$	$\psi_5 = 0,\ \psi_6 = 1,$

$$S_{66} = \Psi^2(x^a, x^0) = e^{2NX^0} f(x^a),$$

$S_{66}\,\varphi_\alpha \dfrac{dx^\alpha}{dS}$	$e/m = S_{66}\,\psi_A \dfrac{dx^A}{dS}$
$= \sin \dfrac{r'}{k}, \quad \Big\vert \quad = r',$	$= \sin \dfrac{r}{k}, \quad \Big\vert \quad = r,$

k being the same as in § 1 and

r'	r

being the radius (generalized in the sense of common tangential segment) of the oriented hypercomplex (of oriented hyperspheres with radii

r)	r')

with its "center" (oriented hypersphere with radius

r and center $(s^{AB}\psi_B) = (\psi^A)$ $= (0, 0, 0, 0, 0, 1))$: (φ^A)	r' and center $(s^{AB}\varphi_B) = (\varphi^A)$ $= (0, 0, 0, 0, 1, 0))$: (ψ^A)

in the tangential four-dimensional

N.E.	Euclidean

space as well as the geodesic radii (generalized in the sense of common geodesic tangential segment) of the corresponding oriented linear hypercomplex (of oriented geodesic hyperspheres with geodesic radii

r)	r')

with its "center")oriented geodesic hyperspheres with geodesic radius

r and center $(\psi^A))$: (φ^A)	r' and center $(\varphi^A))$: (ψ^A)

realized in the Einstein space V_4. Therefore

$u_A = S_{AB}\dfrac{dx^B}{dS},$	$u_A = S_{AB}\dfrac{dx^B}{dS},$	$u_\alpha = S_{\alpha B}\dfrac{dx^B}{dS},$	$u_{\alpha'} = S_{\alpha' B}\dfrac{dx^B}{dS},$
$(S_{AB}\dfrac{dx^A}{dS}\dfrac{dx^B}{dS} = 1)$	$(S_{AB}\dfrac{dx^A}{dS}\dfrac{dx^B}{dS} = 1)$	$u_6 = S_{B6}\dfrac{dx^B}{dS} = -P,$	$u_5 = S_{B5}\dfrac{dx^B}{dS} = -P',$
		$(\alpha = 1, 2, \ldots, 5)$	$(\alpha' = 1, 2, \ldots, 4, 6)$

are the coordinates of the oriented linear hypercomplex of the oriented hyperspheres touching properly two oppositely oriented hyperplanes in the tangential four-dimensional

N.E. || Euclidean

space of V_4 as well as the coordinates of the corresponding oriented generalized totally geodesic linear hypercomplexes [12]) (belonging to the generalized linear hypercomplex of the oriented geodesic hyperspheres under consideration realized in V_4).

The transformations are of the type:

$$\begin{cases} \bar{x}^a = \bar{x}^a(x^b), \\ \bar{x}^0 = x^0 + \dfrac{1}{N} \log k, \quad k = e^{N(\bar{x}^0 - x^0)} \\ \bar{x}^6 = \dfrac{1}{k^2} x^6, \end{cases}$$

so that

$$S_{AB} = e^{2N\bar{x}^0} \bar{S}_{AB}.$$

The field equations are

$$(R^{ab} - \tfrac{1}{2} g^{ab} R) + 2(g^{cd} \psi^a_c \psi^b_d + \tfrac{1}{4} g^{ab} \psi^c_d \psi^d_c) + \tfrac{1}{2} (g^{cd} \theta^a_c \theta^b_d + \tfrac{1}{4} g^{ab} \theta^c_d \theta^d_c) - 20N^2(\theta^a \theta^b - \tfrac{1}{2} g^{ab} \theta^c \theta_c) = 0,$$

Maxwell's equations: $\psi^{ab}_{,b} = 0,$

Vector meson field equations: $\theta^{ab}_{,b} + 20N^2 \theta^a = 0,$

where

$$\Psi_A = \frac{1}{N} \frac{\partial \log \Psi}{\partial x^A}, \quad \Psi_5 = 0,$$

$$\psi_{\alpha\beta} = \tfrac{1}{2} \left(\frac{\partial \psi_\alpha}{\partial x^\beta} - \frac{\partial \psi_\beta}{\partial x^\alpha} \right), \quad \psi^\alpha_\beta = \gamma^{\alpha\gamma} \psi_{\gamma\beta},$$

$$\theta_\alpha = \varphi_\alpha - \Psi_\alpha,$$

$$\theta_{ab} = 2\varphi_{ab} = \frac{\partial \varphi_a}{\partial x^b} - \frac{\partial \varphi_b}{\partial x^a} = \frac{\partial \theta_a}{\partial x^b} - \frac{\partial \theta_b}{\partial x^a},$$

$$\varphi^a_b = g^{ac} \varphi_{cb}, \qquad \varphi^{ab} = g^{ac} g^{bd} \varphi_{cd},$$

so that

$$\Psi^A \Psi_B = \Psi^\alpha \Psi_\beta = 1 + \theta^a \theta_b,$$

where

$$\Psi^A = S^{AB} \Psi_B, \qquad S^{AB} S_{AC} = \delta^B_C.$$

[12]) See [7]).

When the quadratic form $g_{ij}dx^i dx^j$ is a general one we will call the connection space corresponding to

$$dS^2 = S_{00}(x^a, x^0)[g_{ij}(x^a)dx^i dx^j + (\varphi_\alpha(x^a)dx^\alpha)^2 + \{\psi_A(x^a)dx^A\}^2]$$

the *equiform*

Lie || *parabolic Lie*

connection space and the corresponding geometry the *equiform*

Lie || *parabolic Lie*

connection geometry.

(A) The space under consideration is nothing but a special five-dimensional equiform

dual-conformal || Laguerre

connection space if the linear hypercomplexes of the geodesic hyperspheres are interpreted as geodesic hyperspheres and this space will be realized within the Einstein space V_4 as a special equiform

Lie || parabolic Lie

connection space by means of a minimal projection (accompanied by a kind of tangential dilatation) of the tangential five-dimensional equiform

dual-conformal || Laguerre

connection space upon the four-dimensional

N.E. || Euclidean

tangent space of V_4 as well as by its generalization in the five-dimensional equiform

dual-conformal || Laguerre

connection space.

(B) The space under consideration is also nothing but a special six-dimensional Weyl (i.e. equiform connection) space if the geodesic hyperspheres are interpreted as points and this Weyl space will be realized within the Einstein space V_4 as a special equiform

Lie || parabolic Lie

connection space by means of a succession of two minimal projections from two mutually orthogonal direction (accompanied by a kind of tangential dilatation) of the points of the tangential six-dimensional

N.E. equiform	equiform

space upon the four-dimensional

N.E.	Euclidean

tangent space of V_4 as well as by its generalization in the six-dimensional Weyl space.

From this I conclude as follows:

The special equiform Lie connection space of Hoffmann	*The special equiform parabolic Lie connection space*

is equivalent to the Einstein space V_4 *special equiform*

Lie	*parabolic Lie*

connection geometrically so that the points in V_4 *correspond to the linear hypercomplexes of the geodesic hyperspheres whose developments in the equiform*

N.E.	*Euclidean*

tangent spaces are linear hypercomplexes of the generalized radii r' *such that*

$$S_{00}\,\varphi_\alpha \frac{dx^\alpha}{dS}$$

$= \sin \dfrac{r'}{k}$	$= -r'$

consisting of hyperspheres of the radii r *such that*

$$S_{00}\,\psi_A \frac{dx^A}{dS}$$

$= \sin \dfrac{r}{k}.$	$= r.$

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