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Potential-theoretic methods in the theory of functions of two complex variables

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Potential-theoretic methods in the theory of functions of two complex variables [1])

by

S. Bergman and M. Schiffer

1. The space of functions of two complex variables. Basic geometric assumptions.

The fact that the geometry of the space of two complex variables in contrast to the situation for one variable, differs from that of the Euclidean space of four real variables, has a decisive influence on the structure of the theory. This difference is that many geometrical entities which appear in investigations of functions of two variables are completely unlike those with which we usually operate in the Euclidean space. This results primarily from the fact that a function of one variable vanishes at a point and *the point* is the basic element with the help of which the manifolds of interest in the theory are generated, while an analytic function of two complex variables vanishes only in an analytic surface, and thus the analogous role is played by an analytic surface (and a segment of this surface).

Therefore, in building up manifolds in the geometry of functions of two variables, we have to introduce as elements: analytic hypersurfaces (one-parameter family of analytic surfaces) and segments of such hypersurfaces, domains which are bounded by finitely many segments of analytic hypersurfaces and on the boundary of which lies the distinguished boundary surface (the sum of intersections of the above analytic hypersurfaces), etc. (See [1, 2, 4]) [2]). It is natural to associate with the elements of the space of functions of two complex variables various „measures", i.e., numbers which describe quantitatively certain properties of the manifolds arising in this geometry. See [3]. The present paper is devoted to the derivation of inequalities between certain

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[2]) Numbers in brackets refer to the bibliography.

"measures" which refer to the behavior of analytic functions of two variables.

Although we shall consider a rather special class of domains, the methods and procedures developed in the following can be applied in the case of much more general domains. However, in order to present more clearly the basic ideas of the methods which can be developed in the theory of functions of two complex variables and to avoid the technical difficulties which arise while considering more general domains, bounded by finitely many analytic hypersurfaces, we restrict ourselves to the domains described in the following:

Let $h(\zeta, z_2)$, $\zeta = \xi + i\eta$, $z_2 = x_2 + iy_2$, be a continuously differentiable function of the four real variables x_2, y_2, ξ, η for $|\zeta| \leqq 1$, $|z_2| \leqq 1$. Let for $1 - \varepsilon \leqq |\zeta| \leqq 1$, $\varepsilon > 0$, and $|z_2| \leqq 1$, $h(\zeta, z_2)$ be an analytic function of the two complex variables ζ, z_2. We further assume that for every fixed z_2, $|z_2| \leqq 1$, $z_1 = h(e^{i\lambda}, z_2)$, $0 \leqq \lambda \leqq 2\pi$ is a closed curve which does not intersect itself, i.e., $h(e^{i\lambda_1}, z_2) \neq h(e^{i\lambda_2}, z_2)$, for $\lambda_1 \not\equiv \lambda_2 \pmod{2\pi}$, and which includes the origin, $z_1 = 0$, in its interior.

The union of segments of analytic surfaces [3]),

$$\bar{\mathfrak{h}}^3 = \sum_{\lambda=0}^{2\pi} [\mathfrak{H}^2(e^{i\lambda}) + \mathfrak{h}^1(e^{i\lambda})], \tag{1.1}$$

$$\mathfrak{H}^2(e^{i\lambda}) = [z_1 = h(e^{i\lambda}, z_2),\ |z_2| < 1,\ \lambda \text{ fixed}], \tag{1.1a}$$

$$\mathfrak{h}^1(e^{i\lambda}) = [z_1 = h(e^{i\lambda}, z_2),\ |z_2| = 1,\ \lambda \text{ fixed}], \tag{1.1b}$$

is a segment of an analytic hypersurface. The union of segments of analytic surfaces

$$\bar{\mathfrak{d}}^3 = \sum_{\varphi_2=0}^{2\pi} [\mathfrak{D}^2(e^{i\varphi_2}) + \mathfrak{d}^1(e^{i\varphi_2})], \tag{1.2}$$

$$\mathfrak{D}^2(z_2) = [z_1 = h(\zeta, z_2),\ z_2 = z_2;\ |\zeta| < 1,\ z_2 \text{ fixed}], \tag{1.2a}$$

$$\mathfrak{d}^1(z_2) = [z_1 = h(\zeta, z_2),\ z_2 = z_2;\ |\zeta| = 1,\ z_2 \text{ fixed}], \tag{1.2b}$$

also forms a segment of an analytic hypersurface.

The union of segments of analytic surfaces

$$\mathfrak{D} = \sum_{|z_2|<1} \mathfrak{D}^2(z_2) \tag{1.3}$$

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[3]) The upper index on a manifold indicates the dimension of the manifold. In the case of four-dimensional manifolds, we omit the upper index 4. The sums (1.1) and (1.2) are to be understood in the point-set sense. We note that every segment $\mathfrak{H}^2(e^{i\lambda}) + \mathfrak{h}^1(e^{i\lambda})$ and $\mathfrak{D}^2(e^{i\varphi_2}) + \mathfrak{d}^1(e^{i\varphi_2})$ lies in a different surface.

form a domain. From the continuity of $h(e^{i\lambda}, z_2)$ considered as a function of λ, x_2, y_2, and the fact that $\mathfrak{h}^1(e^{i\lambda})$ is a simple closed curve, it follows that $\mathfrak{D}$ is simply-connected.

LEMMA 1.1. *The boundary of* $\mathfrak{D}$ *is* $\bar{\mathfrak{d}}^3 + \bar{\mathfrak{h}}^3$.

PROOF. 1. We shall at first show that in the neighborhood of every point, say (z_1^1, z_2^1), of $\bar{\mathfrak{d}}^3 + \bar{\mathfrak{h}}^3$, there are interior points of $\mathfrak{D}$. We shall distinguish three cases: a) $(z_1^1, z_2^1) \in \mathfrak{H}^2(e^{i\lambda})$; b) $(z_1^1, z_2^1) \in \mathfrak{D}^2(e^{i\varphi_2})$; c) $(z_1^1, z_2^1) \in \mathfrak{d}^1(e^{i\varphi_2})$.

a) The curve $[z_1 = h(e^{i\lambda}, z_2^1),\ 0 \leqq \lambda \leqq 2\pi]$, divides the plane $z_2 = z_2^1$, into two parts, $\mathfrak{D}^2(z_2^1)$ and the exterior of $\mathfrak{D}^2(z_2^1)$. Let z_1^0 be an interior point of $\mathfrak{D}^2(z_2^1)$. We wish to show that (z_1^0, z_2^1) is an interior point of $\mathfrak{D}$. Since z_1^0 is an interior point of $\mathfrak{D}^2(z_2^1)$ the distance between z_1^0 and the boundary curve $\mathfrak{d}^1(z_2^1)$ is ϱ, $\varrho > 0$. From the continuity of $h(e^{i\lambda}, z_2)$ it follows that there exists a neighborhood $|z_2 - z_2^1| \leqq \varepsilon$, $1 - |z_2^1| > \varepsilon > 0$, so that the distance of the points z_1 from $\mathfrak{d}^1(z_2^1)$, $|z_2 - z_2^1| \leqq \varepsilon$, is larger than $\varrho/2$. Thus, the bicylinder $[|z_2 - z_2^1| \leqq \varepsilon$, $|z_1 - z_1^0| \leqq \varrho/2]$ will consist of interior points of the domain D and therefore (z_1^0, z_2^1) is an interior point of $\mathfrak{D}$.

b) Exactly in the same manner, it can be shown that if $z_1^1 \in \mathfrak{D}^2(e^{i\varphi_2})$, $z_2^1 = e^{i\varphi_2}$, (z_1^1, z_2^1) is a boundary point of $\mathfrak{D}$.

c) Let, finally, $z_1^1 = h(e^{i\lambda}, e^{i\varphi_2})$, $z_2^2 = e^{i\varphi_2}$.
Let $z_1^0 \in \mathfrak{D}^2(e^{i\varphi_2})$, such that $|z_1^0 - z_1^1| \leqq \varepsilon$, and let us suppose that the distance between z_1^0 and $\mathfrak{d}^1(e^{i\varphi_2})$ is larger than ϱ, $\varrho > 0$. From the continuity of $h(e^{i\lambda}, z_2)$ it follows that for $|z_2 - e^{i\varphi_2}|$ sufficiently small, say $< \gamma$, the distance between z_1^0 and $\mathfrak{d}^1(z_2)$ will be larger than $\varrho/2$, and therefore the point (z_1^0, z_2^0), $z_2^0 = (1 - \gamma)e^{i\varphi_2}$ will be an interior point of $\mathfrak{D}$.

2. We now wish to show that the boundary of $\mathfrak{D}$ consists *only* of the points of $\bar{\mathfrak{d}}^3 + \bar{\mathfrak{h}}^3$. Suppose (z_1^0, z_2^0) is an interior, and (z_1^1, z_2^1) an exterior point of $\mathfrak{D}$. Let further $\mathfrak{c}^1$ be an oriented curve which begins at (z_1^0, z_2^0) and ends at (z_1^1, z_2^1), and let (z_1^2, z_2^2) be the first point of $\mathfrak{c}^1$ which does not belong to $\mathfrak{D}$ and which we meet moving along $\mathfrak{c}^1$ from (z_1^0, z_2^0). Since the interior of $\mathfrak{D}$ is given by the points $z_1 = h(\varrho e^{i\lambda}, \tau e^{i\varphi_2})$, $z_2 = \tau e^{i\varphi_2}$, $\varrho < 1$, $\tau < 1$, and the curves $h(e^{i\lambda}, e^{i\varphi_2})$, $z_2 = e^{i\varphi_2}$ vary continuously, one of the following three possibilities must hold for (z_1^2, z_2^2): a) $\varrho = 1$, $\tau < 1$; b) $\varrho < 1$, $\tau = 1$; or c) $\varrho = 1$, $\tau = 1$. In all three cases (z_1^2, z_2^2) belongs to $\bar{\mathfrak{h}}^3 + \bar{\mathfrak{d}}^3$, which proves our assertion.

The intersection of $\bar{\mathfrak{h}}^3$ and $\bar{\mathfrak{d}}^3$

$$(1.4)\quad \mathfrak{S}^2 = [z_1 = h(e^{i\lambda}, e^{i\varphi_2}),\ z_2 = e^{i\varphi_2},\ 0 \leqq \lambda \leqq 2\pi,\ 0 \leqq \varphi_2 \leqq 2\pi],$$

forms a closed surface which lies on the boundary of $\mathfrak{D}$ and is called the *distinguished boundary surface of* $\mathfrak{D}$. It follows from (1.1b) and (1.2b) that

$$\mathfrak{S}^2 = \sum_{\lambda=0}^{2\pi} \mathfrak{h}^1(e^{i\lambda}) = \sum_{\varphi_2=0}^{2\pi} \mathfrak{d}^1(e^{i\varphi_2}). \tag{1.5}$$

By the transformation

$$z_1 = h(\zeta, z_2), \; z_2 = z_2 \tag{1.6}$$

the bicylinder $|\zeta| \leqq 1$, $|z_2| \leqq 1$ is mapped into the closed domain $\overline{\mathfrak{D}}$.

Here the segment $[\zeta = e^{i\lambda}, |z_2| \leqq 1]$ goes into the segment $\overline{\mathfrak{H}}^2(e^{i\lambda}) = \mathfrak{H}^2(e^{i\lambda}) + \mathfrak{h}^1(e^{i\lambda})$ of $\mathfrak{D}$, and $[|\zeta| < 1, z_2 = e^{i\varphi_2}]$ goes into $\mathfrak{T}^2(e^{i\varphi_2})$.

Lemma 1.2. *The decomposition of a simply-connected segment of an analytic hypersurface in a one parameter family of segments of analytic surfaces is essentially unique. That is*

$$\mathfrak{h}^3 = \sum_{\lambda=0}^{\lambda_1} \mathfrak{H}^2(e^{i\lambda}) = \sum_{\mu=0}^{\mu_1} \mathfrak{T}^2(e^{i\mu}),$$

where $\mathfrak{H}^2(e^{i\lambda})$ *and* $\mathfrak{T}^2(e^{i\mu}) = [z_1 = t(e^{i\mu}, z_2)]$ *are segments of analytic surfaces, and* $t(\zeta, z_2)$ *satisfies the same differentiability conditions as* $h(\zeta, z_2)$, *see p. 214, then* $\lambda = \lambda(\mu)$ *is a continuous, monotonically-increasing (or decreasing) function of* μ, *and* $\mathfrak{H}^2(e^{i\lambda(\mu)}) = \mathfrak{T}^2(e^{i\mu})$.

Proof: Let $\mathfrak{h}_\varkappa^3$, $\varkappa = 1, 2$, denote open manifolds

$$\mathfrak{h}_1{}^3 = \sum_{\lambda=0}^{\lambda_0} \mathfrak{H}^2(e^{i\lambda}), \; \mathfrak{h}_2{}^3 = \sum_{\lambda=\lambda_0}^{\lambda_1} \mathfrak{H}^2(e^{i\lambda}), \; 0 < \lambda_0 < \lambda_1.$$

Then $\mathfrak{h}^3$ is divided by $\mathfrak{H}^2(e^{i\lambda_0})$ into two disconnected parts, $\mathfrak{h}_1{}^3$ and $\mathfrak{h}_2{}^3$. For every value of λ_3, where $|\lambda_3 - \lambda_0| \leqq \varepsilon$, ε sufficiently small, $\lambda_3 < \lambda_0$, there exists a „sphere"[4]) $\mathfrak{S}^3(P)$ with a center at $P \in \mathfrak{H}^2(e^{i\lambda_3})$ which lies completely in $\mathfrak{h}^3$ and which includes in its interior points of $\mathfrak{h}_2{}^3$. The manifold which separates the points of $\mathfrak{h}_1{}^3$ from those of $\mathfrak{h}_2{}^3$ must be at least a *two*-dimensional manifold and since it belongs to $\mathfrak{H}^2(e^{i\lambda_0})$, $\mathfrak{H}^2(e^{i\lambda_0})$ is a two-dimensional manifold.

Suppose now that $\mathfrak{H}^2(e^{i\lambda_0})$ does not coincide with any $\mathfrak{T}^2(e^{i\mu})$, $0 \leqq \mu \leqq \mu_1$. Then the intersection with every $\mathfrak{T}^2(e^{i\mu})$, consists

[4]) I.e. the set of points of $\mathfrak{h}^3$, whose distance from P is smaller than ϱ, ϱ being a sufficiently small positive constant.

of finitely many points. Let for $\mu = \mu_0$, the intersection points be $(z_1^{(0,\nu)}, z_2^{(0,\nu)}$, $\nu = 1, 2, \ldots m(\mu_0)$. Therefore according to our hypotheses

$$(1.7) \qquad [h_0(e^{i\lambda_0}) - t_0(e^{i\mu})] + [h_1(e^{i\lambda_0}) - t_1(e^{i\mu})]z_2 + [h_2(e^{i\lambda_0}) - t_2(e^{i\mu})]z_2^2 + \ldots = 0,$$

$$h_n(e^{i\lambda_0}) = \left.\frac{\partial^n h(e^{i\lambda_0}, z_2)}{\partial z_2^n}\right|_{z_2=0}, \quad t_n(e^{i\mu}) = \left.\frac{\partial^n t(e^{i\mu}, z_2)}{\partial z_2^n}\right|_{z_2=0}, \quad n = 0, 1, 2, \ldots,$$

has for $\mu = \mu_0$ finitely many solutions, say $z_2 = z_2^{(0,\nu)}$, $\nu = 1, 2, \ldots m(\mu_0)$, for which $|z_2^{(0,\nu)}| < 1$. Let $z_2^{(\nu)}(\mu)$ denote the solution of the equation (1.7) in the neighborhood of $\mu = \mu_0$. Obviously there must exist a smallest n, $n > 0$, for which $h_n(e^{i\lambda_0}) \neq t_n(e^{i\mu_0})$. Therefore there exists a neighborhood, say $[\mu_0 - \varepsilon \leqq \mu \leqq \mu_0 + \varepsilon]$, where $h_n(e^{i\lambda_0}) \neq t_n(e^{i\mu})$. Since

$$(\partial^n[h(e^{i\lambda_0}, z_2) - t(e^{i\mu}, z_2)]/\partial z_2^n)_{z_2=0} \neq 0$$

and t is a continuously differentiable function of μ, by the theory of implicit functions, the intersection curve $\mathfrak{H}^2(e^{i\lambda_0})$ with $\mathfrak{T}^2(e^{i\mu})$, $\mu_0 - \varepsilon \leqq \mu \leqq \mu_0 + \varepsilon$, consists of at most n branches

$$[z_1 = t(e^{i\mu}, Z_2^{(0,\nu,\varkappa)}(\mu)), \; z_2 = Z_2^{(0,\nu,\varkappa)}(\mu)], \quad \nu = 1, 2, \ldots m(\mu_0), \quad \varkappa = 1, 2, \ldots n_1(\nu, \mu_0),$$

$$Z_2^{(0,\nu,\varkappa)}(\mu_0) = z_2^{(0,\nu)},$$

where $Z_2^{(0,\nu,\varkappa)}(\mu)$, for $\mu_0 - \varepsilon \leqq \mu \leqq \mu_0 + \varepsilon$, are continuously differentiable functions of μ.

Using the Heine-Borel theorem it follows that the interval $(0, \mu_1)$ can be divided into finitely many sub-intervals, $[\mu_\varrho - \varepsilon \leqq \mu \leqq \mu_\varrho + \varepsilon]$ such that in every sub-interval the intersection of $\mathfrak{H}^2(e^{i\lambda_0})$ and $\sum_{\mu_\varrho-\varepsilon}^{\mu_\varrho+\varepsilon} \mathfrak{T}^2(e^{i\mu})$ can be represented in the above described manner. Such a set $\mathfrak{j}^1$ of curves *cannot* fill out a two-dimensional neighborhood. Since points which lie in $\mathfrak{h}_1^3 + \mathfrak{h}_2^3$ cannot belong to the division manifold of $\mathfrak{h}_1^3$ with $\mathfrak{h}_2^3$, it follows that a *one*-dimensional set $\mathfrak{j}^1$ separates $\mathfrak{h}_1^3$ from $\mathfrak{h}_2^3$, which is a contradiction, since we have shown that this manifold includes a two-dimensional set.

Thus, $\mathfrak{H}^2(e^{i\lambda_0})$ must coincide with some $\mathfrak{T}^2(e^{i\mu})$, say $\mathfrak{T}^2(e^{i\mu_0})$.

Consider now three segments $\mathfrak{H}^2(e^{i\lambda_\varkappa})$, $\varkappa = 2, 3, 4$, $0 < \lambda_2 < \lambda_3 < \lambda_4 < \lambda_1$, (or $\lambda_2 > \lambda_3 > \lambda_4$), and the corresponding images $\mathfrak{T}^2(e^{i\mu_\varkappa})$, $\varkappa = 2, 3, 4$. The segment $\mathfrak{T}^2(e^{i\mu_3})$ divides the domain

$\sum_{\lambda=\lambda_2}^{\lambda_4} \mathfrak{H}^2(e^{i\lambda})$ into two parts. Since $\sum_{\lambda=\lambda_2}^{\lambda_4} \mathfrak{H}^2(e^{i\lambda})$ is simply connected, and $\mathfrak{T}^2(e^{i\mu_3}) = \mathfrak{H}^2(e^{i\lambda_3})$, we have, necessarily, $\mu_2 < \mu_3 < \mu_4$, i.e., $\mu(\lambda)$ must be a monotonically-increasing (or decreasing) function of λ, and vice-versa.

In every $\sum_{\mu_\varrho-\varepsilon}^{\mu_\varrho+\varepsilon} \mathfrak{T}^2(e^{i\mu})$, we consider the curve

$$[z_1 = t(e^{i\mu}, z_2^{(0,1)}), z_2 = z_2^{(0,1)}, \mu_\varrho - \varepsilon \leqq \mu \leqq \mu_\varrho + \varepsilon].$$

Let $s(\mu)$ be the length of this curve counted from the point $[t(e^{i\mu_\varrho}, z_2^{(0,1)}), z_2^{(0,1)}]$. Obviously $s(\mu)$ is a bi-continuous function of μ. The same holds for $s(\lambda)$. Therefore λ is a bi-continuous function of μ. Q. e. d.

A function $f(z_1, z_2)$ of two complex variables assumes a constant value, say v, in an analytic surface, say $\mathfrak{F}_v^2 = [f(z_1, z_2) = v]$, and it is of interest to associate certain "measures" with the intersections of $\mathfrak{F}_v^2$ with segments of analytic surfaces and hypersurfaces. In the following in (1.8)—(1.9) and (1.10)—(1.11) we shall introduce several quantities of this type:

1) The intersection, $\mathfrak{F}_v^2 \cap \mathfrak{D}^2(t_2)$, forms the set $\boldsymbol{\alpha}(t_2)$ of points

$$z_1 = \alpha_\nu(t_2), \quad z_2 = t_2, \quad \nu = 1, 2, \ldots m(t_2).$$

We shall consider the following "measures" of the set $\boldsymbol{\alpha}(t_2)$:

$$\boldsymbol{g}(v, t_1; t_2) = \sum_{\nu=1}^{m(t_2)} \gamma[\Omega(t_1, t_2), \Omega(\alpha_\nu(t_2), t_2)], \tag{1.8}$$

$$\boldsymbol{l}(v, t_1; t_2) = \sum_{\nu=1}^{m(t_2)} \log | \Omega(t_1, t_2) - \Omega(\alpha_\nu(t_2), t_2) |, \tag{1.9}$$

where $\Omega(t_1, t_2)$, $\Omega(0, t_2) = 0$, $\Omega'(0, t_2) > 0$, is the function mapping $\mathfrak{D}^2(t_2)$ onto the unit circle, and γ the Green's function of the unit circle.

(1.8) is the potential defined in $\mathfrak{D}^2(t_2)$ at t_1 of charges at the points $\alpha_\nu(t_2)$ which vanishes on the boundary $\mathfrak{b}^1(t_2)$.

(1.9) is the sum of the logarithms of the Euclidean distances from $\Omega(t_1, t_2)$ of the images of the points $\alpha_\nu(t_2)$ of the set $\boldsymbol{\alpha}(t_2)$.

2) According to Lemma 1.2, a segment, say $\mathfrak{h}_\varrho^3$, of analytic hypersurfaces can be represented in a unique manner as a sum of segments of analytic surfaces,

$$\mathfrak{h}_\varrho^3 = \sum_{\lambda=0}^{2\pi} \mathfrak{H}^2(\varrho e^{i\lambda}), \ \mathfrak{H}^2(\varrho e^{i\lambda}) = [z_1 = h(\varrho e^{i\lambda}, z_2), \ | z_2 | < 1],$$
$$0 \leqq \lambda \leqq 2\pi.$$

The intersection of $\mathfrak{F}_v^2$ with the segment $\mathfrak{h}_\varrho^3 = [z_1 = h(\varrho e^{i\lambda}, z_2),$ $0 \leqq \lambda \leqq 2\pi,\ |z_2| \leqq 1]$, ϱ-constant, is the line

$$\mathfrak{b}_\varrho^1 = [z_1 = h(\varrho e^{i\lambda},\ a_\nu(\varrho e^{i\lambda})),\ z_2 = a_\nu(\varrho e^{i\lambda}),\ \nu = 1, \ldots M(\varrho e^{i\lambda}),\ 0 \leqq \lambda \leqq 2\pi]$$

which consists of a number of branches $\mathfrak{b}_\nu^1$.

We introduce now the "measures":

$$G(\mathfrak{F}_v^2 \cap \bar{\mathfrak{h}}_\varrho^3,\ t_2,\ A) \equiv \overline{G(v, t_2, \varrho, A)} = \frac{1}{2\pi} \int_{\lambda=0}^{2\pi} \sum_{\nu=1}^{M(\exp(i\lambda))} \log \left| \frac{1 - t_2 a_\nu(\varrho e^{i\lambda})}{t_2 - \quad a_\nu(\varrho e^{i\lambda})} \right| d\,A\,(t_1, t_2, \lambda) \tag{1.10}$$

and

$$L(\mathfrak{F}_v^2 \cap \bar{\mathfrak{h}}^3, t_2, A) \equiv L(v, t_2, A) = = \frac{1}{2\pi} \int_{\lambda=0}^{2\pi} \sum_{\nu=1}^{M(\exp(i\lambda))} \log |\, t_2 - a_\nu(e^{i\lambda}) \,|\, dA(t_1, t_2, \lambda) \tag{1.12}$$

$G(v, t_2, \varrho, A)$ represents the average (with respect to the weight A, see (3.18)) of the potentials (defined in the unit circle) at t_2 of charges at the points $a_\nu(\varrho e^{i\lambda})$ which vanishes on the boundary of the unit circle.

L is the average (with the weighting function A), of the sum of logarithms of euclidean distances of the points $a_\nu(e^{i\lambda})$ from t_2.

By analogy with the interior normal of a curve in the case of functions of one complex variable so in the space of functions of two complex variables we can distinguish two "normals" with every point P of the distinguished boundary surface. The respective intersection of $\mathfrak{S}^2$ with the lamina $\mathfrak{D}^2(e^{i\varphi_2})$ and $\mathfrak{H}^2(e^{i\lambda})$ are the curves $\mathfrak{d}^1(e^{i\varphi_2})$, and $\mathfrak{h}^1(e^{i\lambda})$. Both curves pass through the point $P = [z_1 = h(e^{i\lambda},\ e^{i\varphi_2}),\ z_2 = e^{i\varphi_2}]$ and we define at the point P *two "normal directions"* n_1, n_2, the first lying in $\mathfrak{D}^2(e^{i\varphi_2})$ and perpendicular to $\mathfrak{d}^1(e^{i\varphi_2})$ at P, while n_2 lies in $\mathfrak{H}^2(e^{i\lambda})$, perpendicular to $\mathfrak{h}^1(e^{i\lambda})$ at P. If the normal derivatives are directed into the interior of the respective curves, we shall speak of interior normals.

Let $f(z_1, z_2)$ be a function which is defined on and in the neighborhood of $\mathfrak{S}^2$. In order to simplify the formulas in the present paper it is convenient to introduce the following symbols:

$$f^\dagger(\zeta_1, z_2) = f(h(\zeta_1, z_2), z_2) \tag{1.13}$$

$$D_1[f^\dagger(\varrho e^{i\lambda},\ e^{i\varphi_2})] = \left[\frac{\partial f^\dagger(\varrho e^{i\lambda},\ e^{i\varphi_2})}{\partial \varrho}\right]_{\varrho=1} \tag{1.14}$$

$$(1.15) \qquad T_1[f^\dagger(\varrho e^{i\lambda}, \tau e^{i\varphi_2})] = \left[\frac{\partial f^\dagger(\varrho e^{i\lambda}, \tau e^{i\varphi_2})}{\partial \tau}\right]_{\tau=1}$$

If $f(z_1, z_2)$ is a function for which

$$-D_1[f^\dagger(\varrho e^{i\lambda}, e^{i\varphi_2})] \text{ and } -T_1[f^\dagger(e^{i\lambda}, \tau e^{i\varphi_2})]$$

exist, we shall call them *the derivatives in the first and second normal directions*, respectively.

Our aim is to derive inequalities of the Nevanlinna type relating the „measures" of segments of the surface $f = v = const.$ and the quantities (1.13), (1.14), *see p.* [238] *and* [239]. *See also* [3].

Further in § 3 *we generalize the relations between the kernel-function and Green's functions obtained in* [5] *to the case of two complex variables.*

§ 2. Extended class of functions.

As is well known the real or imaginary part of a function of two complex variables z_1, z_2 satisfies the system of differential equations

$$(2.1) \qquad \frac{\partial^2 u}{\partial x_1^2} + \frac{\partial^2 u}{\partial y_1^2} = 0, \quad \frac{\partial^2 u}{\partial x_2^2} + \frac{\partial^2 u}{\partial y_2^2} = 0$$

$$(2.2) \qquad \frac{\partial^2 u}{\partial x_1 \partial x_2} + \frac{\partial^2 u}{\partial y_1 \partial y_2} = 0, \quad \frac{\partial^2 u}{\partial x_1 \partial y_2} - \frac{\partial^2 u}{\partial x_2 \partial y_2} = 0.$$

The functions satisfying the above system will be called B-harmonic functions [5]).

As already mentioned in the introduction, the distinguished boundary surface plays in many respects a role similar to that of the boundary curve in the case of functions of one complex variable (see [1, 3, 4]). In one important respect, however, the analogy fails: If a real, sufficiently smooth function is given on the distinguished boundary surface, then in general, there need not be a B-harmonic function (defined in the domain) which assumes the prescribed values on the distinguished boundary surface. In order that the boundary value problem shall always have a solution, it is useful to extend the class of B-harmonic functions. In the case of a bicylinder, this extension can be carried out in a natural way, by enlarging the class of B-harmonic functions to the class of *doubly-harmonic* functions, i.e., of func-

[5]) We use this notation rather than biharmonic functions, since the functions $\psi(x, y)$ which satisfy the equation $\triangle\triangle\psi = 0$ are already called biharmonic functions.

tions which satisfy the system (2.1) but not necessarily the system (2.2). Then the boundary value problem with the boundary values prescribed on the distinguished boundary surface $\mathfrak{S}^2 = [z_1 = h(e^{i\lambda}), e^{i\varphi_2}), z_2 = e^{i\varphi_2}, 0 \leqq \lambda \leqq 2\pi, 0 \leqq \varphi_2 \leqq 2\pi]$ always has a solution. On the other hand the class of doubly-harmonic functions is not invariant with respect to pseudo-conformal transformations [6]), and we proceed to describe functions of an extended class which are invariant with respect to pseudo-conformal transformation for the domains described in § 1.

We shall consider two possible extensions of this type and shall denote the corresponding function classes by $E = E(\mathfrak{D})$, (see also [1, 3, 4]) and $\overset{\frown}{E} = \overset{\frown}{E}(\mathfrak{D})$, respectively. Each of them will be appropriate for a special type of boundary value problem.

Definition 2.1. Let $F(z_1, z_2)$ be a real-valued function defined in $\overline{\mathfrak{D}}$. Let $\mathfrak{L}$ be a subset of $0 \leqq \lambda \leqq \pi$ and let the set of values of λ, $0 \leqq \lambda \leqq 2\pi$, which do not belong to $\mathfrak{L}$ be of measure 0. If $F^{\dagger}(e^{i\lambda}, z_2)$ is a harmonic function of x_2, y_2 in $|z_2| < 1$ for $\lambda \in \mathfrak{L}$, and $F(z_1, z_2)$ is a harmonic function of x_1, y_1 in $|z_1| \leqq 1$, which assumes the value $F(z_1, z_2) = F^{\dagger}(e^{i\lambda}, z_2)$, for every $\lambda \in \mathfrak{L}$ then we say F belongs to the class $E = E(\mathfrak{D})$.

We now construct a function $F(z_1, z_2)$ of the class E which on $\mathfrak{S}^2$ (see (1.4)) assumes the values $f(e^{i\lambda}, e^{i\varphi_2})$, in the case that $f(e^{i\lambda}, e^{i\varphi_2})$ is continuous in both variables λ, φ_2. This will be done by: first defining $F(z_1, z_2)$ on the boundary $\bar{\mathfrak{b}}^3 + \bar{\mathfrak{h}}^3$ of $\mathfrak{D}$ and then extending this definition to the interior of $\mathfrak{D}$.

By (1.2), $\bar{\mathfrak{b}}^3$ is a union of segments $\overline{\mathfrak{D}}^2(e^{i\varphi_2})$, each of which is bounded by $\mathfrak{b}^1(e^{i\varphi_2})$. For every fixed φ_2, we determine that function which is harmonic in x_1, y_1 in $\mathfrak{D}^2(e^{i\varphi_2})$ and assumes on $\mathfrak{b}^1(e^{i\varphi_2})$ the prescribed values, i.e., we write

$$F(t_1, e^{i\varphi_2}) = \frac{1}{2\pi} \int\limits_{\mathfrak{b}^1(e^{i\varphi_2})} f(e^{i\lambda}, e^{i\varphi_2}) \frac{\partial \gamma(t_1, h(e^{i\lambda}, e^{i\varphi_2}); e^{i\varphi_2})}{\partial n_1} ds_1. \tag{2.3}$$

where $\gamma(t_1, z_1; e^{i\varphi_2})$ is Green's function of $\mathfrak{D}^2(e^{i\varphi_2})$, and n_1 and ds_1 are the direction of the interior normal and the line element of $\mathfrak{b}^1(e^{i\varphi_2})$, respectively. By (2.3), $F(z_1, z_2)$ is defined in $\bar{\mathfrak{b}}^3$.

According to (1.5), $\mathfrak{S}^2$ is the union of curves $\mathfrak{h}^1(e^{i\lambda})$. For every λ, we determine in $\mathfrak{H}^2(e^{i\lambda})$ the function $F(z_1, z_2)$, so that

[6]) A transformation of a four-dimensional domain in the (z_1, z_2)-space by a pair of analytic functions of two complex variables $w_k = w_k(z_1, z_2)$, $k = 1, 2$, $\partial(w_1, w_2)/\partial(z_1, z_2) \neq 0$, will be called a pseudo-conformal transformation.

(2.4) $$F^{\dagger}(e,^{i\lambda}\, z_2)$$

is a harmonic function of x_2, y_2 in $x_2{}^2 + y_2{}^2 < 1$, i.e., we write

$$(2.5) \qquad F^{\dagger}(e^{i\lambda}, z_2) = \frac{1}{2\pi}\int_{\varphi_2=0}^{2\pi} f(e^{i\lambda}, e^{i\varphi_2})\frac{1-|z_2|^2}{|1-z_2e^{-i\varphi_2}|^2}d\varphi_2.$$

By (2.5), $F(z_1, z_2)$ is defined in $\mathfrak{h}^3$.

We now proceed to define the function $F(z_1, z_2)$ inside $\mathfrak{D}$. The intersection of $\mathfrak{D}$ with $z_2 = z_2{}^0$ is the domain $\mathfrak{D}^2(z_2{}^0)$ bounded by the curve $\mathfrak{d}^1(z_2{}^0)$. According to (1.2b), (1.1b), (1.1), every point $(z_1{}^0, z_2{}^0) \in \mathfrak{d}^1(z_2{}^0)$, lies in $\bar{\mathfrak{h}}^3$, and therefore, according to our previous considerations, the function $F(z_1, z_2)$ is determined at every point of $\mathfrak{d}^1(z_2{}^0)$. We now determine that function of x_1, y_1 which in $\mathfrak{D}^2(z_2{}^0)$ is harmonic in x_1, y_1 and which assumes on the boundary curve $d^1(z_2{}^0)$ the prescribed values. By (1.3), this procedure defines $F(z_1, z_2)$ in $\mathfrak{D}$.

We obtain for $t_1 \in \mathfrak{D}^2(t_2)$

$$(2.6) \quad F(t_1, t_2) = \frac{1}{4\pi^2}\int_{\mathfrak{d}^1(t_2)}\int_{\varphi_2=0}^{2\pi} f(e^{i\lambda}, e^{i\varphi_2})P(e^{i\varphi_2}, t_2)\frac{\partial\gamma(t_1, h(e^{i\lambda}, t_2); t_2)}{\partial n_1}ds_1 d\varphi_2,$$

where $P(e^{i\varphi_2}, t_2)$ is the Poisson kernel, and n_1 and ds_1 the interior normal and line-element of $\mathfrak{d}^1(t_2)$, respectively.

Using the theorems of Carathéodory, Courant, and Radó, it can easily be shown that $F(z_1, z_2)$ assumes on the distinguished boundary surface $\mathfrak{S}^2$, the boundary values $f(e^{i\lambda}, e^{i\varphi_2})$.

We now proceed to the description of functions of the class $\breve{E}(\mathfrak{D})$. In order to do this, it is useful to introduce the notion of a function $F^*(z_1, z_2)$, *pseudo-conjugate* to $\breve{F}(z_1, z_2)$. Let $f(e^{i\lambda}, e^{i\varphi_2})$ have continuous partial derivatives with respect to each argument. Then there exists a function $f^*(e^{i\lambda}, e^{i\varphi_2}) + C_1(e^{i\varphi_2})$, such that $f(e^{i\lambda}, e^{i\varphi_2})$ and $f^*(e^{i\lambda}, e^{i\varphi_2}) + C_1(e^{i\varphi_2})$ are the boundary values of functions harmonic in $\mathfrak{D}^2(e^{i\varphi_2})$ and (considered as functions of x_1 and y_1) conjugate to each other there. $C_1(e^{i\varphi_2})$ is an arbitrary function of φ_2, which we shall assume continuous.

REMARK. Let $Z = s(z_1, e^{i\varphi_2})$ be that analytic function which maps the unit circle in the Z-plane onto $\mathfrak{D}^2(e^{i\varphi_2})$. Then

$$f^*[s(e^{i\lambda}, e^{i\varphi_2}), e^{i\varphi_2}] = \frac{1}{2\pi}\int_{t=0}^{\pi}\frac{f[s(e^{i(\lambda+t)}, e^{i\varphi_2}), e^{i\varphi_2}] - f[s(e^{i(\lambda-t)}, e^{i\varphi_2}), e^{i\varphi_2}]}{2\tan\frac{1}{2}t}dt.$$

The function f^* will be continuous in φ_2 for every fixed λ. We may then determine the function $F^*(z_1, z_2)$ of $E(\mathfrak{D})$ which assumes the values $f^*(e^{i\lambda}, e^{i\varphi_2})$ on $\mathfrak{S}^2$. For each z_2^0, let $\overset{\frown}{F}(z_1, z_2^0) + C_2(z_2^0)$ be conjugate to $F^*(z_1, z_2^0)$, considered as a function of x_1, y_1.

It is clear that $\overset{\frown}{F}(z_1, e^{i\varphi_2})$ is, to within a function of φ_2, that harmonic function of x_1, y_1, in $\mathfrak{D}^2(e^{i\varphi_2})$ which assumes the boundary values $f(e^{i\lambda}, e^{i\varphi_2})$. Thus, by proper choice of $C_2(z_2^0)$, we may insure that $F(z_1, z_2^0)$ has the boundary values $f(e^{i\lambda}, e^{i\varphi_2})$ on $\mathfrak{S}^2 \cap [z_2^0 = e^{i\varphi_2}]$ and that $C_2(z_2)$ is a harmonic function in $|z_2| < 1$ which assumes the boundary values $C_2(e^{i\varphi_2})$. Applying, then the theorems of Carathéodory, Radó, and Courant, it is possible to show that $\overset{\frown}{F}(z_1, z_2)$ assumes on $\mathfrak{S}^2$ the boundary values $f(e^{i\lambda}, e^{i\varphi_2})$.

Definition 2.3. $\overset{\frown}{F}(z_1, z_2)$, which has been obtained in the above manner, will be denoted as "*the function of the class* $\overset{\frown}{E}(\mathfrak{D})$ *corresponding to* $f(e^{i\lambda}, e^{i\varphi_2})$." $F^*(z_1, z_2) + C_1(z_2)$ is a function of the class $E(\mathfrak{D})$. It will be called *pseudo-conjugate* to $\overset{\frown}{F}(z_1, z_2)$. For every fixed z_2, they are conjugate, considered as functions of x_1, y_1.

We wish to prove a property of $\overset{\frown}{F}(z_1, z_2)$ which leads to a new possibility of a determination of $\overset{\frown}{F}(z_1, z_2) \in \overset{\frown}{E}(\mathfrak{D})$, important in some applications. Since, in a sufficiently small neighborhood of $\zeta = e^{i\lambda}$, $h(\zeta, z_2)$ is an analytic function of ζ, $\overset{\frown}{F}{}^\dagger(\varrho e^{i\lambda}, z_2)$ and $F^{*\dagger}(\varrho e^{i\lambda}, z_2) + C_1(z_2)$ are (considered as functions of $\log \varrho$ and λ) conjugate to each other. Therefore

$$(2.8)\qquad \boldsymbol{D}_1[\overset{\frown}{F}{}^\dagger(\varrho e^{i\lambda}, z_2)] = \frac{\partial F^{*\dagger}(e^{i\lambda}, z_2)}{\partial \lambda}$$

holds, the right-hand side existing for $\varrho = 1$ by definition of the class $\overset{\frown}{E}(\mathfrak{D})$. By construction $F^{*\dagger}(e^{i\lambda}, z_2)$ is a harmonic function in x_2, y_2. Therefore, there follows

Lemma 2.1. *The function* $\overset{\frown}{F}(z_1, z_2) \in \overset{\frown}{E}(\mathfrak{D})$ *is harmonic in* x_1, y_1 *in every* $\mathfrak{D}^2(z_2)$, $|z_2| \leqq 1$, *and possesses the property that in* $|z_2| < 1$, *the derivative* $\boldsymbol{D}_1[\overset{\frown}{F}{}^\dagger(\varrho e^{i\lambda}, z_2)]$ *is a harmonic function of* x_2, y_2.

Remark. If on the other hand $\boldsymbol{D}_1[\overset{\frown}{F}(\varrho e^{i\lambda}, z_2)]$ is prescribed, then obviously $\overset{\frown}{F}$ is determined only within an arbitrary function $C(z_2)$.

The above property leads to another construction of $\overset{\frown}{F}(z_1, z_2)$. Let $\overset{\frown}{f}(e^{i\lambda}, e^{i\varphi_2})$ be given as before, i.e., have continuous partial derivatives with respect to both variables. We determine that harmonic function $\breve{F}(z_1, e^{i\varphi_2})$ which on $\mathfrak{b}^1(e^{i\varphi_2})$ assumes the values $\breve{F}^\dagger(e^{i\lambda}, e^{i\varphi_2}) = \breve{f}(e^{i\lambda}, e^{i\varphi_2})$. Let, further $\dfrac{\partial \breve{F}(z_1, e^{i\varphi_2})}{\partial n_{z_1}}$ be the values of the normal derivative, which, in view of our assumptions, exist for every φ_2 and λ. For $1-\varepsilon \leqq |\zeta| \leqq 1$, $h(\zeta, z_2)$ is analytic in ζ, and therefore

$$(2.9) \quad -\boldsymbol{D}_1[\breve{F}^\dagger(\varrho e^{i\lambda}, e^{i\varphi_2})] = \frac{\partial \breve{F}(z_1, e^{i\varphi_2})}{\partial n_{z_1}} \left| \frac{\partial h(\zeta, e^{i\varphi_2})}{\partial \zeta} \right|_{\zeta = e^{i\lambda}} = \chi(\lambda, e^{i\varphi_2}).$$

According to Lemma 2.1, $\boldsymbol{D}_1[F^\dagger(\varrho e^{i\lambda}, z_2)]$ is a harmonic function of x_2, y_2. Let $\chi(\lambda, z_2)$, λ fixed, be that harmonic function in x_2, y_2 which assumes the boundary value $\chi(\lambda, e^{i\varphi_2})$ for $|z_2| = 1$. We determine now for every z_2^0, $|z_2^0| < 1$, that harmonic function $\breve{F}(z_1, z_2^0)$, $z_1 \in \mathfrak{D}^2(z_2^0)$, which, on the boundary $\mathfrak{b}^1(z_2^0)$, satisfies the condition

$$(2.10) \quad \frac{\partial \breve{F}(z_1, z_2^0)}{\partial n_{z_1}} = \chi(\lambda, z_2^0) \bigg/ \left| \frac{\partial h(\zeta, z_2^0)}{\partial \zeta} \right|_{\zeta = e^{i\lambda}}, \quad z_1 = h(e^{i\lambda}, z_2^0).$$

In order that this is possible, we have to show that

$$(2.11) \qquad \int_{\mathfrak{b}^1(z_2)} \frac{\chi(\lambda, z_2)}{|\partial h(\zeta, z_2)/\partial \zeta|_{\zeta = e^{i\lambda}}} ds_1 = 0.$$

The mapping $z_1 = h(\zeta, z_2)$, $z_2 = \text{const.}$ is conformal on the boundary curve $\mathfrak{b}^1(z_2)$, and we have, in particular, the relation

$$\frac{ds_1}{|\partial h(\zeta, z_2)/\partial \zeta|_{\zeta = e^{i\lambda}}} = d\lambda,$$

where $d\lambda$ denotes the line element of the unit circle $|\zeta| = 1$. Therefore, it suffices to show that

$$(2.12) \qquad \int_{\lambda=0}^{2\pi} \boldsymbol{D}_1[\breve{F}^\dagger(\varrho e^{i\lambda}, z_2)] d\lambda = 0,$$

for $|z_2| \leqq 1$.

We remark at first that for every $z_2 = e^{i\varphi_2}$, $0 \leqq \varphi_2 \leqq 2\pi$, $\breve{F}$ is a harmonic function of x_1, y_1, and therefore we have the identity

$$(2.13) \qquad \int_{\mathfrak{b}^1(e^{i\varphi_2})} \frac{\partial \breve{F}(z_1, e^{i\varphi_2})}{\partial n_{z_1}} ds_{z_1} = 0.$$

We may write, in view of (2.9), this identity also in the form

$$(2.14)\quad -\int_{\lambda=0}^{2\pi} \boldsymbol{D}_1[\breve{F}^{\dagger}(\varrho e^{i\lambda}, e^{i\varphi_2})]d\lambda = \int_{\lambda=0}^{2\pi} \chi(\lambda, e^{i\varphi_2})d\lambda = 0.$$

or

$$(2.14a)\quad -\int_{\lambda=0}^{2\pi} \boldsymbol{D}_1[\breve{F}^{\dagger}(\varrho e^{i\lambda}, z_2)]d\lambda = \int_{\lambda=0}^{2\pi} \chi(\lambda, z_2)d\lambda = 0, \ | z_2 | = 1.$$

Thus, the required identity has been proved for $| z_2 | = 1$. But the left-hand integral in (2.14a) is a bounded harmonic function of z_2 for $| z_2 | < 1$, and since it vanishes on the boundary of the unit circle, it must vanish identically. This proves our assertion. Since (2.13) is fulfilled, we are able to solve the boundary value problem required, and to determine the harmonic function $\breve{F}(z_1, z_2)$ with a prescribed normal derivative on $\mathfrak{d}^1(z_2)$. In this procedure, $\breve{F}(z_1, z_2)$ is determined within a function $C_2(z_2)$. If we require in addition that $C_2(z_2)$ is a harmonic function of x_2, y_2, then both determinations lead to the same function $\breve{F}(z_1, z_2)$ of the class $\breve{E}(\mathfrak{D})$ corresponding to a given $f(e^{i\lambda}, e^{i\varphi_2})$.

In the special case where $\mathfrak{D}$ is a bicylinder $\mathfrak{B}$, the classes $E(\mathfrak{B})$ and $\breve{E}(\mathfrak{B})$ become doubly-harmonic functions [7]).

Indeed, if the values of F on $z_1 = e^{i\varphi_1}$, $z_2 = e^{i\varphi_2}$ are $f(e^{i\varphi_1}, e^{i\varphi_2})$, we obtain for the functions $E(\mathfrak{B})$ the representation

$$(2.15)\quad F(z_1, z_2) = \frac{1}{4\pi^2}\int_{\varphi_1=0}^{2\pi}\int_{\varphi_2=0}^{2\pi} f(e^{i\varphi_1}, e^{i\varphi_2})\left.\frac{\partial\gamma(\tau e^{i\varphi_2}, z_2)}{\partial\tau}\right|_{\tau=1} \times \left.\frac{\partial\gamma(\varrho e^{i\varphi_1}, z_1)}{\partial\varrho}\right|_{\varrho=1} d\varphi_1 d\varphi_2$$

where γ is Green's function for the unit circle. If, further, for $\breve{F} \in \breve{E}(\mathfrak{B})$

$$-\boldsymbol{D}_1[\breve{F}(\varrho e^{i\varphi_1}, e^{i\varphi_2}] = \chi(e^{i\varphi_1}, e^{i\varphi_2})$$

then we obtain

$$(2.16)\quad \breve{F}(z_1, z_2) = \frac{1}{4\pi^2}\int_{\varphi_1=0}^{2\pi}\int_{\varphi_2=0}^{2\pi} \chi(e^{i\varphi_1}, e^{i\varphi_2})\left.\frac{\partial\gamma(\tau e^{i\varphi_2}, z_2)}{\partial\tau}\right|_{\tau=1} \mu(e^{i\varphi_1}, z_1)d\varphi_2 d\varphi_1 + C(z_2).$$

Since in (2.15) and (2.16), in the integrals on the right-hand sides, the f and χ are multiplied with two factors, the first of which is harmonic in x_2, y_2, and the second, in x_1, y_1, the functions $F(z_1, z_2)$ are in both cases doubly harmonic.

If the domain $\mathfrak{D}$ is a pseudo-conformal image of a bicylinder

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[7]) It should be stressed however that the function F and $\breve{F}$ corresponding to the same $f(e^{i\lambda}, e^{i\varphi_2})$ need not be identical.

(in particular if $\mathfrak{D}$ is a product domain), the classes E and $\overset{\frown}{E}$ coincide: they are pseudo-conformal transforms of doubly harmonic functions.

If the general case (i.e., if $\mathfrak{D}$ is not a pseudo-conformal image of a bicylinder $E(\mathfrak{D})$ differs from $\overset{\frown}{E}(\mathfrak{D})$. The corresponding functions $F \in E(\mathfrak{D})$ and $\overset{\frown}{F} \in \overset{\frown}{E}(\mathfrak{D})$, with the same values on $\mathfrak{S}^2$, coincide if $F(z_1, z_2)(= \overset{\frown}{F}(z_1, z_2))$ is a $\boldsymbol{B}$-harmonic function.

§ 3. Representation of functions of the class $E(\mathfrak{D})$ and $\overset{\frown}{E}(\mathfrak{D})$ by means of generalized Green's functions in terms of boundary data on $\mathfrak{S}^2$.

In analogy to the procedures of [5] in the present paper, we shall consider two different boundary value problems for functions of the class $E(\mathfrak{D})$, namely by requiring

1.1) that the value of $F = f_{11}(\lambda, \varphi_2)$;

1.2) that the values of $\left.\dfrac{\partial F[h(e^{i\lambda}, z_2), z_2]}{\partial n_2}\right|_{z_2=e^{i\varphi_2}} = f_{12}(\lambda, \varphi_2)$; $n_2 \equiv n_{z_2}$;

are prescribed on $\mathfrak{S}^2$. In an analogues way, we shall consider the following two boundary value problems for functions belonging to $\overset{\frown}{E}(\mathfrak{D})$ by requiring

2.1) that the values of $\left.\dfrac{\partial \overset{\frown}{F}(z_1, e^{i\varphi_2})}{\partial n_1}\right|_{z_1=h(e^{i\lambda}, e^{i\varphi_2})} = f_{21}(\lambda, \varphi_2)$; $n_1 \equiv n_{z_1}$;

2.2) that the values of $\left.-\dfrac{\partial^2 \overset{\frown}{F}^\dagger}{\partial \varrho \partial n_2}\right|_{\varrho=1,\ z_2=e^{i\varphi_2}} = f_{22}(\lambda, \varphi_2)$,

are prescribed on $\mathfrak{S}^2$. In the present section we shall assume that $f_{nk}(\lambda, \varphi_2)$ are continuous functions of λ and φ_2. Naturally, the prescribed boundary data $f_{nk}(\lambda, \varphi_2)$ must satisfy the following conditions:

In the case 1.2): Since $F^\dagger(e^{i\lambda}, z_2)$, is supposed to be a harmonic function of x_2, y_2,

$$\int\limits_{\mathfrak{h}^1(e^{i\lambda})} \frac{\partial F^\dagger}{\partial n_2} ds_2 = \int_{\varphi_2=0}^{2\pi} f_{12}(\lambda, \varphi_2) d\varphi_2 = 0 \text{ for every } \lambda. \tag{3.1}$$

In the case 2.1): Since $\overset{\frown}{F}(z_1, z_2)$ is a harmonic function of x_1, y_1 and $ds_1 \equiv ds_{z_1} = \left| h_\zeta(\zeta, e^{i\varphi_2}) \right|_{\zeta=e^{i\lambda}} d\lambda$,

$$(3.2) \qquad \int\limits_{\mathfrak{d}^1(e^{i\varphi_2})} \frac{\partial \overset{\frown}{F}}{\partial n_1} ds_1 = \int\limits_{\lambda=0}^{2\pi} f_{21}(\lambda, \varphi_2) \Big| h_\zeta(\zeta, e^{i\varphi_2}) \Big|_{\zeta=e^{i\lambda}} d\lambda = 0,$$

for every φ_2.

In the case 2.2): $\overset{\frown}{F}(z_1, z_2)$ is a harmonic function of x_1, y_1 for every $z_2 = \tau e^{i\varphi_2}$, and therefore

$$\int\limits_{\mathfrak{d}^1(z_2)} \frac{\partial \overset{\frown}{F}(z_1, z_2)}{\partial n_1} ds_1 = -\int\limits_{\lambda=0}^{2\pi} \boldsymbol{D}_1[\overset{\frown}{F}^\dagger(\varrho e^{i\lambda}, \tau e^{i\varphi_2})] d\lambda = 0,$$

for every τ and φ_2. Differentiating the last expression with respect to τ, we obtain

$$(3.3) \qquad \int\limits_{\lambda=0}^{2\pi} \boldsymbol{T}_1 \boldsymbol{D}_1[\overset{\frown}{F}^\dagger(\varrho e^{i\lambda}, \tau e^{i\varphi_2})] d\lambda = \int\limits_{\lambda=0}^{2\pi} f_{22}(\lambda, \varphi_2) d\lambda = 0,$$

for every φ_2.

Since

$$\boldsymbol{D}_1[\overset{\frown}{F}^\dagger(\varrho e^{i\lambda}, \tau e^{i\varphi_2})]$$

is a harmonic function of x_2, y_2, for every λ, we have

$$\int\limits_{\varphi_2=0}^{2\pi} \boldsymbol{T}_1 \boldsymbol{D}_1[\overset{\frown}{F}^\dagger(\varrho e^{i\lambda}, \tau e^{i\varphi_2})] d\varphi_2 = \int\limits_{\varphi_2=0}^{2\pi} f_{22}(\lambda, \varphi_2) d\varphi_2 = 0,$$

for every λ.

On the other hand, if the above conditions are satisfied in the case of the problems 1.2), 2.1), and 2.2), the functions F and $\overset{\frown}{F}$ are only determined up to certain functions. We exclude this ambiguity by introducing a normalization in the classes E and $\overset{\frown}{E}$.

$$(3.5) \qquad F^\dagger(e^{i\lambda}, e^{i\varphi_2}) = f(\lambda, \varphi_2)$$

is a function which is defined almost everywhere [8]) on the distinguished boundary surface $\mathfrak{S}^2$. The functions F_1 of the classes E and $\overset{\frown}{E}$, respectively, will be normalized by the following requirements:

1. To a given $F_1 \in E(\mathfrak{D})$, we form first the harmonic function

$$(3.6) \qquad S_1(z_2) = \frac{1}{2\pi} \int\limits_{\lambda=0}^{2\pi} F_1^\dagger(e^{i\lambda}, z_2) d\lambda$$

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[8]) We remind the reader that according to our definition, functions of the class E and $\overset{\frown}{E}$ exist almost everywhere on $\mathfrak{S}^2$.

and define

$$(3.7) \qquad F_2(z_1, z_2) = F_1(z_1, z_2) - S_1(z_2).$$

Obviously, $S_1 \epsilon E$ and consequently $F_2 \epsilon E$.

We now determine a second function $S_2(z_1, z_2)$ which has the property that it is harmonic in x_1, y_1 in every intersection $\mathfrak{D}^2(z_2)$, $z_2 = \text{const.}$, and for $z_1 = h(e^{i\lambda}, z_2)$, $z_2 = z_2$, $\lambda = \text{const.}$, assumes the values

$$\frac{1}{2\pi}\int_{\varphi_2=0}^{2\pi} F_2^{\dagger}(e^{i\lambda}, e^{i\varphi_2})d\varphi_2,$$

i.e., we determine

$$(3.8) \quad S_2(z_1, z_2) = \frac{1}{4\pi^2}\int_{\mathfrak{b}^1(z_2)}\int_{\varphi_2=0}^{2\pi}\left(F_2^{\dagger}(e^{i\lambda}, e^{i\varphi_2})d\varphi_2\right)\frac{\partial\gamma[h(e^{i\lambda}, z_2), z_1; z_2]}{\partial n_1}ds_1$$

where γ $(Z, z_1; z_2)$ is the Green's function of $\mathfrak{D}^2(z_2)$. Since S_2 is constant in every lamina $\bar{\mathfrak{H}}^2(e^{i\lambda})$, $S_2 \epsilon E(\mathfrak{D})$. Thus

$$(3.9) \qquad F = F_2 - S_2 = F_1 - S_1 - S_2 \epsilon E \equiv E(\mathfrak{D}).$$

The function F obtained in this manner is said to be normalized. The totality of functions of E which are normalized form the class E_n. [9])

REMARK 3.1. $S_1(z_1, z_2) \equiv S_1(z_2)$ is constant in every intersection z_2 const. and the same harmonic function of x_2, y_2 in every lamina $\bar{\mathfrak{H}}^2(e^{i\lambda}) = [z_1 = h(e^{i\lambda}, z_2), |z_2| \leqq 1]$. $S_2(z_1, z_2)$ is constant in every lamina $\bar{\mathfrak{H}}^2(e^{i\lambda})$ and harmonic in every intersection $\mathfrak{D}^2(z_2)$, $z_2 = \text{const.}$ We note further that

$$(3.10) \qquad \int_{\varphi_2=0}^{2\pi} F^{\dagger}(e^{i\lambda}, e^{i\varphi_2})d\varphi_2 = 0,$$

for every λ, since, according to (3.8), the corresponding expressions for F_2 and S_2 coincide.

2. Let $\breve{F}_1 \epsilon \breve{E}(\mathfrak{D})$. We note that

$$(3.11) \quad \int_{\lambda=0}^{2\pi} \boldsymbol{D}_1[\breve{F}_1^{\dagger}(\varrho e^{i\lambda}, z_2)]d\lambda = -\int_{\mathfrak{b}^1(z_2)} \frac{\partial \breve{F}_1(z_1, z_2)}{\partial n_1}\partial s_1 = 0.$$

Hence

$$\boldsymbol{D}_1\left[\int_{\lambda=0}^{2\pi} \breve{F}_1^{\dagger}(\varrho e^{i\lambda}, e^{i\varphi_2})\right]d\lambda = 0$$

[9]) The reader should note that this is a different normalization from that which is described in § 2.

for every φ_2. Therefore, for every z_2 we can determine a function $T(z_1, z_2)$ which is harmonic in x_1, y_1, such that

$$(3.12)\quad \frac{\partial T(z_1, z_2)}{\partial n_1} = -D_1\left[\int_{\varphi_2=0}^{2\pi} \overset{\frown}{F}_1{}^{\dagger}(\varrho e^{i\lambda}, e^{i\varphi_2})d\varphi_2\right][|D_1[h(\varrho e^{i\lambda}, z_2)]|]^{-1}.$$

Indeed,

$$\int_{\mathfrak{b}^1(z_2)} \frac{\partial T(z_1, z_2)}{\partial n_1} ds_1 = -\int_{\lambda=0}^{2\pi} D_1[T^{\dagger}(\varrho e^{i\lambda}, z_2)]d\lambda = 0,$$

and therefore such a function exists. Obviously $T \in \overset{\frown}{E}(\mathfrak{D})$.

Let

$$(3.13)\qquad \overset{\frown}{F}_2(z_1, z_2) = \overset{\frown}{F}_1(z_1, z_2) - T(z_1, z_2).$$

We form now the function

$$(3.14)\qquad \overset{\frown}{S}_1(z_2) = \frac{1}{l(z_2)} \int_{\mathfrak{b}^1(z_2)} \overset{\frown}{F}_2(z_1, z_2)ds_1,$$

where $l(z_2)$ denotes the length of $\mathfrak{b}^1(z_2)$. This function belongs to $\overset{\frown}{E}$, for it is constant in every $\mathfrak{D}^2(z_2)$, and therefore $\dfrac{\partial \overset{\frown}{S}_1}{\partial n_1}$ and *a fortiori* $\dfrac{\partial}{\partial \varrho}\overset{\frown}{S}_1(z_2) = 0$. The function

$$(3.15)\qquad \overset{\frown}{F}(z_1, z_2) = \overset{\frown}{F}_2(z_1, z_2) - \overset{\frown}{S}_1(z_2)$$

is said to be a "normalized function of the class $\overset{\frown}{E}(\mathfrak{D})$", and the subclass of such functions will be denoted by $\overset{\frown}{E}_n(\mathfrak{D})$.

Remark 3.2. From (3.14) and (3.15), it follows that for $\overset{\frown}{F} \in \overset{\frown}{E}_n(\mathfrak{D})$, we have

$$(3.16)\qquad \int_{\mathfrak{b}^1(z_2)} \overset{\frown}{F}(z_1, z_2)ds_1 = 0, \text{ for every } z_2,$$

and from (3.12), (3.13), and the fact that $\overset{\frown}{S}_1(z_2)$ is independent of x_1, y_1, (and therefore $(\partial\overset{\frown}{S}_1/\partial\varrho) = 0$) it follows that

$$(3.17)\qquad \int_{\varphi_2=0}^{2\pi} D_1[\overset{\frown}{F}^{\dagger}(\varrho e^{i\lambda}, e^{i\varphi_2})]d\varphi_2 = 0,$$

for every λ.

Notation. $\Omega(z_1, z_2)$ will denote the analytic function of the complex variable z_1 which transforms the domain $\mathfrak{D}^2(z_2)$ into the unit circle, with $\Omega(0, z_2) = 0$, $\Omega_{z_1}(0, z_2) > 0$.

Let further $\gamma(Z, Y) = \log\left|\dfrac{1-\overline{Y}Z}{Y-Z}\right|$ denote Green's function of the unit circle, and let $\mu(z_1, Y; z_2)$, $\int\limits_{\mathfrak{b}^1(z_2)} \mu(z_1, Y; z_2)ds_{z_1} = 0$, denote the (normalized) Neumann's function [10]) of $\mathfrak{D}^2(z_2)$.

In the following, we shall use the abbreviations

$$(3.18)\quad \begin{aligned} dA(t_1, t_2, \lambda) &= \boldsymbol{D}_1[\gamma(\Omega^{\dagger}(\varrho e^{i\lambda}, t_2);\ \Omega(t_1, t_2))]d\lambda = \\ &= P[\Omega^{\dagger}(e^{i\lambda}, t_2),\ \Omega(t_1, t_2)]\ |\ \boldsymbol{D}_1[\Omega^{\dagger}(\varrho e^{i\lambda}, t_2)]\ |\ d\lambda, \end{aligned}$$

$$(3.19)\quad dT(t_1, t_2, \lambda) = \mu(h(e^{i\lambda}, t_2), t_1; t_2)d\lambda.$$

With this notation, (2.6) gives us the following:

THEOREM 3.1. *Let $F(z_1, z_2) \in E(\mathfrak{D})$ and possess piecewise continuous boundary values. Then*

$$(3.20)\quad F(t_1, t_2) = \frac{1}{4\pi^2}\int\limits_{\varphi_2=0}^{2\pi}\int\limits_{\lambda=0}^{2\pi} F^{\dagger}(e^{i\lambda}, e^{i\varphi_2})dA(t_1, t_2, \lambda)P(e^{i\varphi_2}, t_2)d\varphi_2.$$

THEOREM 3.2. *Let $F(z_1, z_2) \in E(\mathfrak{D})$, and possess piecewise continuous derivatives $\dfrac{\partial F^{\dagger}}{\partial \tau}$ on $\mathfrak{S}^2$, $z_2 = \tau e^{i\varphi_2}$. Then*

$$(3.21)\quad \begin{aligned} F(t_1, t_2) = &-\frac{1}{4\pi^2}\int\limits_{\varphi_2=0}^{2\pi}\int\limits_{\lambda=0}^{2\pi} \boldsymbol{T}_1[F^{\dagger}(e^{i\lambda}, \tau e^{i\varphi_2})]\mu(e^{i\varphi_2}, t_2)dA(t_1, t_2, \lambda)d\varphi_2 + \\ &+\frac{1}{4\pi^2}\int\limits_{\varphi_2=0}^{2\pi}\int\limits_{\lambda=0}^{2\pi} F^{\dagger}(e^{i\lambda}, e^{i\varphi_2})dA(t_1, t_2, \lambda)d\varphi_2. \end{aligned}$$

REMARK 3.3. If $F \in E_n(\mathfrak{D})$, then, according to (3.10), the last integral on the right-hand side of (3.2) vanishes.

PROOF. The formula (3.21) is derived in exactly the same manner as (2.6); we merely have to replace (2.5) by

$$(3.22)\quad F^{\dagger}(e^{i\lambda}, t_2), = -\frac{1}{2\pi}\int\limits_{\varphi_2=0}^{2\pi} \frac{\partial F^{\dagger}}{\partial \tau}\mu(e^{i\varphi_2}, t_2)d\varphi_2 + \frac{1}{2\pi}\int\limits_{\varphi_2=0}^{2\pi} F^{\dagger}(e^{i\lambda}, e^{i\varphi_2})d\varphi_2.$$

Applying Green's formula to (3.22) and proceeding in the same way as in deriving (2.6), we obtain (3.21) which proves Theorem 3.2.

THEOREM 3.3. *Let $\overleftrightarrow{F}(z_1, z_2) \in \overleftrightarrow{E}(\mathfrak{D})$ and let $\dfrac{\partial \overleftrightarrow{F}}{\partial n_1}$ be continuous*

---

10) In the case of the unit circle $\mu(z_1, Y) = -\log|(1-z_1\overline{Y})(z_1-Y)|$.

on $\mathfrak{S}^2$. *Then*

$$\tilde{F}(t_1, t_2) = -\frac{1}{4\pi^2}\int_{\lambda=0}^{2\pi}\int_{\varphi_2=0}^{2\pi} \boldsymbol{D}_1[\tilde{F}^\dagger(\varrho e^{i\lambda}, e^{i\varphi_2})]P(e^{i\varphi_2}, t_2)dT(t_1, t_2, \lambda)d\varphi_2 +$$

$$(3.23) \qquad + \frac{1}{l(t_2)}\int_{\mathfrak{b}^1(t_2)} \tilde{F}(\eta, t_2)ds_\eta,$$

where $l(t_2)$ *is the length of* $\mathfrak{b}^1(t_2)$.

REMARK 3.4. If $\tilde{F} \in \tilde{E}_n(\mathfrak{D})$, then according to (3.16), the last integral on the right-hand side of (3.23) vanishes.

PROOF. (3.23) is derived exactly in the same manner as (3.22).

Since $\tilde{F}(z_1, z_2) \in \tilde{E}(\mathfrak{D})$, $\boldsymbol{D}_1[\tilde{F}^\dagger(\varrho e^{i\lambda}, z_2)]$ is a harmonic function of x_2, y_2 for $|z_2| < 1$, and therefore it can be represented for $|t_2| < 1$ and fixed λ in the form

$$(3.24) \quad \boldsymbol{D}_1[\tilde{F}^\dagger(\varrho e^{i\lambda}, t_2)] = \frac{1}{2\pi}\int_{\varphi_2=0}^{2\pi} \boldsymbol{D}_1[\tilde{F}^\dagger(\varrho e^{i\lambda}, e^{i\varphi_2})]P(e^{i\varphi_2}, t_2)d\varphi_2.$$

According to (2.9), we have

$$\int_{\mathfrak{b}^1(t_2)} \frac{\partial \tilde{F}(z_1, t_2)}{\partial n_1}ds_1 = -\int_{\lambda=0}^{2\pi} \boldsymbol{D}_1[\tilde{F}^\dagger(\varrho e^{i\lambda}, t_2)]d\lambda.$$

(3.23) follows from (2.9) and

$$\tilde{F}(t_1, t_2) = \frac{1}{2\pi}\int_{\mathfrak{b}^1(t_2)} \frac{\partial \tilde{F}(z_1, t_2)}{\partial n_1}\mu[h(e^{i\lambda}, t_2), t_1; t_2]ds_{z_1} + \frac{1}{l(t_2)}\int_{\mathfrak{b}^1(t_2)} \tilde{F}(z_1, t_2)ds_{z_1} =$$

$$= -\frac{1}{4\pi^2}\int_{\lambda=0}^{2\pi}\int_{\varphi_2=0}^{2\pi} \boldsymbol{D}_1[\tilde{F}^\dagger(\varrho e^{i\lambda}, e^{i\varphi_2})]P(e^{i\varphi_2}, t_2)dT(t_1, t_2, \lambda)d\varphi_2 +$$

$$+ \frac{1}{l(t_2)}\int_{\mathfrak{b}^1(t_2)} \tilde{F}(z_1, t_2)ds_{z_1}.$$

THEOREM 3.4. *Let* $\tilde{F}(t_1, t_2) \in \tilde{E}(\mathfrak{D})$ *and possess continuous derivatives* $\dfrac{\partial^2 \tilde{F}^\dagger}{\partial\varrho\partial\tau}$. *Then*

$$\tilde{F}(t_1 t_2) = \frac{1}{4\pi^2}\int_{\lambda=0}^{2\pi}\int_{\varphi_2=0}^{2\pi} \boldsymbol{T}_1\boldsymbol{D}_1[\tilde{F}^\dagger(\varrho e^{i\lambda}, \tau e^{i\varphi_2})]\mu(e^{i\varphi_2}, t_2)dT(t_1, t_2, \lambda)d\varphi_2.$$

$$-\frac{1}{4\pi^2}\int_{\lambda=0}^{2\pi}\int_{\varphi_2=0}^{2\pi} \boldsymbol{D}_1(\tilde{F}^\dagger(\varrho e^{i\lambda}, e^{i\varphi_2})]dT(t_1, t_2, \lambda)d\varphi_2 +$$

$$(3.25) \qquad + \frac{1}{l(t_2)}\int_{\mathfrak{b}^1(t_2)} \tilde{F}(z_1, t_2)ds_{z_1},$$

where $l(t_2)$ *is the length of* $\mathfrak{b}^1(t_2)$, *and* $dT(t_1, t_2, \lambda)$ *is defined in* (3.19).

REMARK 3.5. If $\overset{\frown}{F} \epsilon \overset{\frown}{E}_n(\mathfrak{D})$, then according to (3.17) and (3.16), the second and third integrals on the right-hand side of (3.25) vanish.

PROOF. Since $\overset{\frown}{F} \epsilon \overset{\frown}{E}(\mathfrak{D})$,

$$\boldsymbol{D}_1[\overset{\frown}{F}^\dagger(\varrho e^{i\lambda}, z_2)]$$

is a harmonic function of x_2, y_2, and therefore

$$\begin{aligned} \boldsymbol{D}_1[\overset{\frown}{F}^\dagger(\varrho e^{i\lambda}, t_2)] = &-\frac{1}{2\pi}\int_{\varphi_2=0}^{2\pi} \boldsymbol{T}_1\boldsymbol{D}_1[\overset{\frown}{F}^\dagger(\varrho e^{i\lambda}, \tau e^{i\varphi_2})]\mu(e^{i\varphi_2}, t_2)d\varphi_2 + \\ &+\frac{1}{2\pi}\int_{\varphi_2=0}^{2\pi} \boldsymbol{D}_1[\overset{\frown}{F}^\dagger(\varrho e^{i\lambda}, e^{i\varphi_2})]d\varphi_2. \end{aligned} \tag{3.26}$$

Since, further, $\overset{\frown}{F}(z_1, t_2)$ is a harmonic function of x_1, y_1, we have

$$\begin{aligned} \overset{\frown}{F}(t_1, t_2) &= \frac{1}{2\pi}\int_{\mathfrak{b}^1(t_2)} \frac{\partial \overset{\frown}{F}(z_1, t_2)}{\partial n_1}\mu(z_1, t_1; t_2)ds_1 + \frac{1}{l(t_2)}\int_{\mathfrak{b}^1(t_2)} \overset{\frown}{F}(z_1, t_2)ds_1 = \\ &= -\frac{1}{2\pi}\int_{\lambda=0}^{2\pi} \boldsymbol{D}_1[\overset{\frown}{F}^\dagger(\varrho e^{i\lambda}, t_2)]dT(t_1, t_2, \lambda) + \\ &+\frac{1}{l(t_2)}\int_{\lambda=0}^{2\pi} \overset{\frown}{F}^\dagger(e^{i\lambda}, t_2) \mid h_\zeta(e^{i\lambda}, t_2) \mid d\lambda. \end{aligned} \tag{3.27}$$

Substituting (3.27) into (3.26) we obtain (3.25).

In analogy to the case of one variable, we introduce the function (see (27) of [5])

$$\begin{aligned} \chi^*(Z, z_2; T, t_2) = g_1^* - g_2^* - g_3^* + g_4^* = \\ k(z_2, t_2)k[h(Z, t_2), h(T, t_2); t_2], \end{aligned} \tag{3.28}$$

defined for $|\, Z\,| \leqq 1$, $|\, z_2\,| \leqq 1$, $|\, T\,| \leqq 1$, $|\, t_2\,| \leqq 1$. Here T is defined by $t_1 = h(T, t_2)$. $k(Z_1, t; t_2)$ is the harmonic kernel function of $\mathfrak{D}^2(t_2)$. $g_\varkappa^*(Z, z_2; T, t_2) = g_\varkappa[h(Z, t_2), z_2; h(T, t_2), t_2]$, $\varkappa = 1, 2, 3, 4$, where $g_1 = \gamma^{(1)}\gamma^{(2)}$, $g_2 = \gamma^{(1)}\mu^{(2)}$, $g_3 = \mu^{(1)}\gamma^{(2)}$, $g_4 = \mu^{(1)}\mu^{(2)}$.

In analogy to the case of one variable in formulas (3.20), (3.21), (3.23), (3.25) the functions $g_\varkappa$, $\varkappa = 1, 2, 3, 4$, can be replaced by χ^*.

COROLLARY 3.1. *Let* $F(z_1, z_2) \in E_n(\mathfrak{D})$, *and* $\breve{F}(z_1, z_2) \in \breve{E}_n(\mathfrak{D})$. *Then*

$$
\begin{aligned}
&F(t_1, t_2) = F^\dagger(T, t_2) = \\
&= \frac{1}{4\pi^2}\int_{\lambda=0}^{2\pi}\int_{\varphi_2=0}^{2\pi} F^\dagger(e^{i\lambda}, e^{i\varphi_2})\boldsymbol{T}_1\boldsymbol{D}_1[\chi^*(\varrho e^{i\lambda}, \tau e^{i\varphi_2}; T, t_2)]d\lambda d\varphi_2 = \\
&= \frac{1}{4\pi^2}\int_{\lambda=0}^{2\pi}\int_{\varphi_2=0}^{2\pi} \boldsymbol{T}_1[F^\dagger(e^{i\lambda}, \tau e^{i\varphi_2})]\boldsymbol{D}_1[\chi^*(\varrho e^{i\lambda}, e^{i\varphi_2}; T, t_2)]d\lambda d\varphi_2,
\end{aligned}
\tag{3.29}
$$

and

$$
\begin{aligned}
&\breve{F}(t_1, t_2) = \breve{F}^\dagger(T, t_2) = \\
&= \frac{1}{4\pi^2}\int_{\lambda=0}^{2\pi}\int_{\varphi_2=0}^{2\pi} \boldsymbol{D}_1[\breve{F}^\dagger(\varrho e^{i\lambda}, e^{i\varphi_2})]\boldsymbol{T}_1[\chi^*(e^{i\lambda}, \tau e^{i\varphi_2}; T, t_2)]d\lambda d\varphi_2 = \\
&= \frac{1}{4\pi^2}\int_{\lambda=0}^{2\pi}\int_{\varphi_2=0}^{2\pi} \boldsymbol{T}_1\boldsymbol{D}_1[\breve{F}^\dagger(\varrho e^{i\lambda}, \tau e^{i\varphi_2})]\chi^*(e^{i\lambda}, e^{i\varphi_2}; T, t_2)d\lambda d\varphi_2,
\end{aligned}
\tag{3.30}
$$

It is of interest to represent the function χ^* in terms of kernel functions of one complex variable. See [5].

COROLLARY 3.2. Let $\Omega(z_1, t_2)$, $\Omega(0, t_2) = 0$, $\Omega'(0, t_2) > 0$, be the function which maps $\mathfrak{D}^2(t_2)$ into the unit circle. Then

$$
\begin{aligned}
&\chi^*(Z, z_2; T, t_2) = \\
&= \frac{1}{4\pi^2}[\log(1-\bar{t}_2 z_2) + \log(1-t_2\bar{z}_2)] \cdot [\log(1-\Omega^\dagger(Z, t_2)\overline{\Omega^\dagger(T, t_2)}) + \\
&\quad + \log(1-\Omega^\dagger(T, t_2)\overline{\Omega^\dagger(Z, t_2)}) + S^\dagger(Z, t_2) + S^\dagger(T, t_2)]
\end{aligned}
\tag{3.31}
$$

where $S(z_1, t_2)$ is a harmonic function of x_1, y_1 such that $-\log|\Omega(z_1, t_2)| + S(z_1, t_2)$ is Neumann's function of $\mathfrak{D}^2(t_2)$.

PROOF. The function

$$
-\frac{1}{2\pi}\log\left|\frac{(\Omega(z_1, t_2) - \Omega(t_1, t_2))(1-\overline{\Omega(t_1, t_2)}\Omega(z_1, t_2))}{\Omega(z_1, t_2)}\right|
$$

has a vanishing normal derivative $\partial/\partial n_{z_1}$ at the boundary. It becomes logarithmically infinite at the points $z_1 = t_1$ and $z_1 = 0$. Let $-\frac{1}{2\pi}[\log|\Omega(z_1, t_2)| + S(z_1, t_2) + S(t_1, t_2)]$ be Neumann's function of $\mathfrak{D}^2(t_2)$ with the logarithmic singularity at $z_1 = 0$. Here $S(z_1, t_2)$ is a conveniently chosen harmonic function of x_1, y_1. The kernel function of $\mathfrak{D}^2(t_2)$ is

$$
\begin{aligned}
&-\frac{1}{2\pi}[\log(1-\Omega(z_1, t_2)\overline{\Omega(t_1, t_2)}) + \log(1-\Omega(t_1, t_2)\overline{\Omega(z_1, t_2)}) \\
&\quad + S(z_1, t_2) + S(t_1, t_2)]
\end{aligned}
\tag{3.32}
$$

Substituting $z_1 = h(z, t_2)$, $t_1 = h(T, t_2)$ and multiplying the resulting expression by $-(2\pi)^{-1}[\log(1-\bar{t}_2 z_2) + \log(1-t_2\bar{z}_2)]$ we obtain (3.31).

REMARK 3.5. The classes $E_n(\mathfrak{D})$, $\overset{\frown}{E}_n(\mathfrak{D})$ of functions considered in the present paper have been normalised by the requirements (3.10), (3.16). If instead of these normalization conditions, we consider functions which are normalized by the requirements that they vanish on surfaces $z_2 = 0$ and $z_1 = 0$, then the corresponding "kernel function" $\chi_0^*(Z, z_2; T, t_2)$ is invariant with respect to pseudo-conformal transformation, and we obtain for $\chi_0^*(Z, z_2; T, t_2)$ the expression on the right hand side of (3.31) with $S=0$.

4. Inequalities for measures of geometrical objects introduced in § 1.

The analogy to the one-dimensional case suggests that we introduce a certain generalization of the classes $E(\mathfrak{D})$ and $\overset{\frown}{E}(\mathfrak{D})$.

DEFINITION 4.1: Suppose that

(1) $F^{\dagger}(\varrho e^{i\lambda}, z_2)$ for every fixed ϱ, $1-\varepsilon \leqq \varrho \leqq 1$, and for almost all λ is a harmonic function of z_2 in $|z_2| < 1$, except at finitely many points, say

(4.1) $[z_2 = a_{\nu\varkappa}(\varrho e^{i\lambda}),\ \nu = 1, 2, \ldots, M_\varkappa(\varrho e^{i\lambda}) \leqq M < \infty],\ \varkappa = 1, 2$

where it possesses logarithmic singularities,

$$-(-1)^\varkappa \log |z_2 - a_{\nu\varkappa}(\varrho e^{i\lambda})|, \text{ and}$$

(2) $F(z_1, z_2)$ is a harmonic function of x_1, y_1 in every $\mathfrak{D}^2(z_2)$ except at finitely many points

(4.2) $[z_1 = \alpha_{\nu\varkappa}(z_2),\ \nu = 1, 2, \ldots, m_\varkappa(z_2) \leqq M],\ \varkappa = 1, 2,$

where it becomes infinite as $-(-1)^\varkappa \log |z_1 - \alpha_{\nu\varkappa}(z_2)|$.

Here $a_{\nu\varkappa}(\varrho e^{i\lambda})$ and $\alpha_{\nu\varkappa}(z_2)$ are continuously differentiable functions of their respective arguments.

Further, on the distinguished boundary surface $F(z_1, z_2)$ may become logarithmically infinite at finitely many points.

Then $F(z_1, z_2)$ will be said to be of the class $M(\mathfrak{D})$.

The condition which we obtain by replacing $F^{\dagger}(\varrho e^{i\lambda}, z_2)$ in (1) by $\boldsymbol{D}_1[F^{\dagger}(\varrho e^{i\lambda}, z_2)]$ and $\log |z_2 - a_{\nu\varkappa}(\varrho e^{i\lambda})|$ by $\dfrac{\partial}{\partial \varrho}(\log |z_2 - a_{\nu\varkappa}(\varrho e^{i\lambda})|)$, will be denoted as condition $(\overset{\frown}{1})$.

If $\overset{\frown}{F}$ satisfies $(\overset{\frown}{1})$ and (2), we say it belongs to the class $\overset{\frown}{M}(\mathfrak{D})$.
If $f(z_1, z_2)$ is a quotient of two functions P_1, P_2, both of which (considered as functions of two complex variables) are regular

in $\overline{\mathfrak{D}}$, then $\log|f(z_1, z_2)|$ belongs simultaneously to the classes $M(\mathfrak{D})$ and $\widetilde{M}(\mathfrak{D})$.

THEOREM 4.1. *Let $f(z_1, z_2)$ be a function of two complex variables which is meromorphic in $\overline{\mathfrak{D}}$. Let $a_{\nu\varkappa}(\varrho e^{i\lambda})$, $\nu = 1, 2, \ldots, M_\varkappa(\varrho e^{i\lambda})$ and $\alpha_{\nu\varkappa}(t_2)$, $\nu = 1, 2, \ldots, m_\varkappa(t_2)$ be, for $\varkappa = 1$, the zeros, and for $\varkappa = 2$, the poles of $f(h(\varrho e^{i\lambda}, z_2), z_2)$ and of $f(z_1, t_2)$, respectively. We assume that on the distinguished boundary surface $\mathfrak{S}^2$ there are only finitely many points, say $(\lambda^{(k)}, \varphi_2^{(k)})$, $k = 1, \ldots, n$, where $f(z_1, z_2)$ vanishes or becomes infinite. Let, further, the point (t_1, t_2), $t_1 \in \mathfrak{D}^2(t_2)$, be chosen so that:*

1) at (t_1, t_2) and on the curves $[z_1 = h(e^{i\lambda}, t_2),\ 0 \leqq \lambda \leqq 2\pi,\ z_2 = t_2]$ and $[z_1 = h(e^{i\lambda}, 0),\ 0 \leqq \lambda \leqq 2\pi,\ z_2 = t_2]$ the function $f(z_1, z_2)$ is regular and does not vanish;

2) the points t_2 and 0 do not lie on the curves $[z_2 = a_{\nu\varkappa}(e^{i\lambda}),\ \nu = 1, 2, \ldots, M_\varkappa(e^{i\lambda}),\ 0 \leqq \lambda \leqq 2\pi,\ \varkappa = 1, 2]$.

Then the following relations hold:

$$
\begin{aligned}
\log|f(t_1, t_2)| &= \\
&= \sum_{\varkappa=1}^{2} (-1)^\varkappa \sum_{\nu=1}^{m_\varkappa(t_2)} \gamma(t_1, \alpha_{\nu\varkappa}(t_2); t_2) + \\
&+ \frac{1}{4\pi^2} \int_{\lambda=0}^{2\pi} \int_{\varphi_2=0}^{2\pi} \log|f^\dagger(e^{i\lambda}, e^{i\varphi_2})|\, P(e^{i\varphi_2}, t_2)\, dA(t_1, t_2, \lambda)\, d\varphi_2 + \\
&+ \frac{1}{2\pi} \sum_{\varkappa=1}^{2} (-1)^\varkappa \int_{\lambda=0}^{2\pi} \sum_{\nu=1}^{M_\varkappa(\exp(i\lambda))} \log\left|\frac{1 - t_2 \overline{a_{\nu\varkappa}(e^{i\lambda})}}{(t_2 - a_{\nu\varkappa}(e^{i\lambda}))}\right| dA(t_1, t_2; \lambda) =
\end{aligned} \tag{4.3}
$$

$$
\begin{aligned}
&= \sum_{\varkappa=1}^{2} (-1)^\varkappa \sum_{\nu=1}^{m_\varkappa(t_2)} \gamma(t_1, \alpha_{\nu\varkappa}(t_2); t_2) - \\
&- \frac{1}{4\pi^2} \int_{\lambda=0}^{2\pi} \int_{\varphi_2=0}^{2\pi} T_1[\log|f^\dagger(e^{i\lambda}, \tau e^{i\varphi_2})|]\mu(e^{i\varphi_2}, t_2)\, dA(t_1, t_2, \lambda)\, d\varphi_2 \\
&- \frac{1}{2\pi} \sum_{\varkappa=1}^{2} (-1)^\varkappa \int_{\lambda=0}^{2\pi} \sum_{\nu=1}^{M_\varkappa(\exp(i\lambda))} \log\left|\frac{(1 - t_2\overline{a_{\nu\varkappa}(e^{i\lambda})})(t_2 - a_{\nu\varkappa}(e^{i\lambda}))}{a_{\nu\varkappa}(e^{i\lambda})}\right| dA(t_1, t_2, \lambda) +
\end{aligned} \tag{4.4}
$$

$$
\begin{aligned}
&+ \frac{1}{2\pi} \int_{\lambda=0}^{2\pi} \log|f^\dagger(e^{i\lambda}, 0)|\, dA(t_1, t_2, \lambda) = \sum_{\varkappa=1}^{2} (-1)^\varkappa \sum_{\nu=1}^{m_\varkappa(t_2)} N(\alpha_{\nu\varkappa}(t_2); t_1, 0; t_2) - \\
&- \frac{1}{4\pi^2} \int_{\lambda=0}^{2\pi} \int_{\varphi_2=0}^{2\pi} D_1[\log|f^\dagger(\varrho e^{i\lambda}, e^{i\varphi_2})|]\, P(e^{i\varphi_2}, t_2)\, dS(t_1, t_2, \lambda)\, d\varphi_2 +
\end{aligned} \tag{4.5}
$$

$$+\sum_{\varkappa=1}^{2}(-1)^{\varkappa}\int_{\lambda=0}^{2\pi}\sum_{\nu=1}^{M_{\varkappa}(\exp(i\lambda))} \boldsymbol{D}_1\left[\log\left|\frac{1-t_2\overline{a_{\nu\varkappa}(\varrho e^{i\lambda})}}{t_2-a_{\nu\varkappa}(\varrho e^{i\lambda})}\right|\right]dS(t_1,t_2,\lambda)+$$

$$+\log|f(0,t_2)|$$

where

$$dS(t_1,t_2,\lambda)=N(h(e^{i\lambda},t_2);t_1,0;t_2)d\lambda= \tag{4.6}$$

$$=[\mu(h(e^{i\lambda},t_2),t_1;t_2)-\mu(h(e^{i\lambda},t_2,0;t_2)]d\lambda,$$

$\gamma(z_1,t_1;t_2)$ *and* $\mu(z_1,t_1;t_2)$, $\int_{\mathfrak{b}^1(t_2)}\mu(z_1,t_1;t_2)ds_{z_1}=0$, *denote the Green and the Neumann functions of* $\mathfrak{D}^2(t_2)$, *respectively.*

Proof. We proceed to the proof of the relation (4.4). (The proof of (4.3) is somewhat simpler and proceeds exactly in the same manner). For every $\lambda\in T'=T-\sum_{k=1}^{n}\lambda^{(k)}$, $T=[0\leqq\lambda\leqq 2\pi]$, $\log|f^{\dagger}(e^{i\lambda},z_2)|$ is regular for $|z_2|=1$, and therefore it holds that

$$\log|f^{\dagger}(e^{i\lambda},t_2)|=-\frac{1}{2\pi}\int_{\varphi_2=0}^{2\pi}\boldsymbol{T}_1[\log|f^{\dagger}(e^{i\lambda},\tau e^{i\varphi_2})|]\mu(e^{i\varphi_2},t_2)d\varphi_2+$$

$$+\sum_{\varkappa=1}^{2}(-1)^{\varkappa}\sum_{\nu=1}^{M_{\varkappa}(\exp(i\lambda))}N(a_{\nu\varkappa}(e^{i\lambda});t_2,0)+\log|f^{\dagger}(e^{i\lambda},0)|. \tag{4.7}$$

Here we use the fact

$$\log|f^{\dagger}(e^{i\lambda},0)|-\sum_{\varkappa=1}^{2}(-1)^{\varkappa}\sum_{\nu=1}^{M_{\varkappa}(\exp(i\lambda))}\mu(a_{\nu\varkappa}(e^{i\lambda}),0)=$$

$$=\frac{1}{2\pi}\int_{\varphi_2=0}^{2\pi}\log|f^{\dagger}(e^{i\lambda},e^{i\varphi_2})|\,d\varphi_2,$$

where

$$N(a_{\nu\varkappa}(e^{i\lambda});t_2,0)=\mu(a_{\nu\varkappa}(e^{i\lambda}),t_2)-\mu(a_{\nu\varkappa}(e^{i\lambda}),0),$$

$$\mu(a,t)=-\log|(1-\bar{a}t)(t-a)|.$$

Since $\log|f(z_1,t_2)|$ is a harmonic function of x_1, y_1, it holds

$$\log|f(t_1,t_2)|=\sum_{\lambda=0}^{2}(-1)^{\varkappa}\gamma(t_1,\alpha_{\nu\varkappa}(t_2);t_2)+ \tag{4.8}$$

$$+\frac{1}{2\pi}\int_{\lambda=0}^{2\pi'}\log|f^{\dagger}(e^{i\lambda},t_2)|\,dA(t_1,t_2,\lambda).$$

(In carrying the integration we omit finitely many points $\lambda^{(k)}$, which is obviously admissible). Substituting (4.7) into (4.8), taking into account that

$$N(a_{\nu\varkappa}(e^{i\lambda}); t_2, 0) = -\log\left|\frac{(1-t_2\overline{a_{\nu\varkappa}(e^{i\lambda})})(t_2-a_{\nu\varkappa}(e^{i\lambda}))}{a_{\nu\varkappa}(e^{i\lambda})}\right|$$

and that in consequence of 2), $\int_{\lambda=0}^{2\pi'}$ can be replaced by $\int_{\lambda=0}^{2\pi}$ we obtain (4.4).

We proceed now to the derivation of (4.5): For every λ ($\neq \lambda^{(k)}$) since $\log|f(z_1, z_2)| \in \widetilde{M}(D)$ we have

$$(4.9)\quad \boldsymbol{D}_1[\log|f^{\dagger}(\varrho e^{i\lambda}, t_2)]| = \frac{1}{2\pi}\int_{\varphi_2=0}^{2\pi}\boldsymbol{D}_1[\log|f^{\dagger}(\varrho e^{i\lambda}, e^{i\varphi_2})|]P(e^{i\varphi_2}, t_2)d\varphi_2 +$$
$$+\sum_{\varkappa=1}^{2}(-1)^{\varkappa}\sum_{\nu=1}^{M_{\varkappa}(\exp(i\lambda))}\left(\frac{\partial}{\partial\varrho}\left[\log\left|\frac{1-t_2\overline{a_{\nu\varkappa}(\varrho e^{i\lambda})}}{t_2-a_{\nu\varkappa}(\varrho e^{i\lambda})}\right|\right]\right)_{\varrho=1}.$$

Further, for every function $g(z_1)$ which is harmonic in $\mathfrak{D}^2(t_2)$, we have

$$(4.10)\qquad g(t_1)-g(0) = \frac{1}{2\pi}\int_{\mathfrak{d}^1(t_2)}\frac{\partial g}{\partial n_{z_1}}N(z_1; t_1, 0; t_2)ds_{z_1},$$

where

$$N(z_1; t_1, 0; t_2) = \mu(z_1, t_1; t_2) - \mu(z_1, 0; t_2).$$

Substituting

$$g(z_1) = \log|f(z_1, t_2)| - \sum_{\varkappa=1}^{2}(-1)^{\varkappa}\sum_{\nu=1}^{m_{\varkappa}(t_2)}\mu(z_1, \alpha_{\nu\varkappa}(t_2); t_2)$$

into (4.10), we obtain

$$(4.11)\qquad \log|f(t_1, t_2)| - \log|f(0, t_2)| -$$
$$-\sum_{\varkappa=1}^{2}(-1)^{\varkappa}\sum_{\nu=1}^{m_{\varkappa}(t_2)}N(\alpha_{\nu\varkappa}(t_2); t_1, 0; t_2) =$$
$$= -\frac{1}{2\pi}\int_{\lambda=0}^{2\pi}\boldsymbol{D}_1[\log|f^{\dagger}(\varrho e^{i\lambda}, t_2)]\,N(h(e^{i\lambda}, t_2); t_1, 0; t_2)d\lambda.$$

Here we use the fact that

$$\frac{\partial\mu(z_1, \alpha_{\nu\varkappa}(t_2); t_2)}{\partial n_{z_1}} = \text{constant},\quad z_1 \in \mathfrak{d}^1(t_2),\ \alpha_{\nu\varkappa}(t_2)\in\mathfrak{D}^2(t_2)$$

and therefore (since $\int_{\mathfrak{d}^1(t_2)}\mu(z_1, Z; t_2)ds_{z_1} = 0$)

$$\frac{1}{2\pi} \int\limits_{\mathfrak{b}^1(t_2)} \frac{\partial \mu(z_1, \alpha_{\nu\varkappa}(t_2); t_2)}{\partial n_{z_1}} N(z_1; t_1, 0; t_2) ds_{z_1} = 0.$$

Since $| f(h(e^{i\lambda}, t_2), t_2) |$, $0 \leqq \lambda \leqq 2\pi$, is finite and does not vanish, we can replace the integrals $\int\limits_{\lambda=0}^{2\pi}$ in the last expression in (4.11) by $\int\limits_{\lambda=0}^{2\pi'}$. Substituting then (4.9) into (4.11), we obtain a formula which differs from (4.5) only by the fact that instead of integrals $\int\limits_{\lambda=0}^{2\pi} \int\limits_{\varphi_2=0}^{2\pi}$ and $\int\limits_{\lambda=0}^{2\pi}$ we have $\int\limits_{\lambda=0}^{2\pi'} \int\limits_{\varphi_2=0}^{2\pi}$ and $\int\limits_{\lambda=0}^{2\pi'}$, respectively. Since the integrand of the double integral becomes infinite of first order at finitely many points and since by hypothesis 2) the integrand of the line integral remains bounded, the improper integrals can be replaced by ordinary ones and (4.5) follows.

Corollary 4.1. *Let v be a (finite) constant. For every function $[f(z_1, z_2) - v]$ satisfying hypotheses of the Theorem* 4.1, *we have*

$$- \overset{+}{\log} | f(t_1, t_2) | + \boldsymbol{g}(v, t_1, t_2) + \boldsymbol{G}(v, t_2, 1, \Lambda) +$$

$$+ \frac{1}{4\pi^2} \int\limits_{\lambda=0}^{2\pi} \int\limits_{\varphi_2=0}^{2\pi} \left[\overset{+}{\log} \frac{1}{| f^\dagger(e^{i\lambda}, e^{i\varphi_2}) - v |} \right] P(e^{i\varphi_2}, t_2) d\Lambda(t_1, t_2, \lambda) d\varphi_2$$

$$\leqq \frac{1}{4\pi^2} \int\limits_{\lambda=0}^{2\pi} \int\limits_{\varphi_1=0}^{2\pi} \left[\overset{+}{\log} | f^\dagger(e^{i\lambda}, e^{i\varphi_2}) | \right] P(e^{i\varphi_2}, t_2) d\Lambda(t_1, t_2, \lambda) d\varphi_2$$

$$+ \boldsymbol{g}(\infty, t_1, t_2) + \boldsymbol{G}(\infty, t_2, \Lambda) + \overset{+}{\log} v + \log 2, \ \Lambda = \Lambda(t_1, t_2, \lambda).$$

See (1.8), (1.10) and (3.18).

Proof. Substituting $\log | f(z_1, z_2) - v |$, instead of $\log | f(z_1, z_2) |$, into (4.3), and using definitions (1.9), (1.12), we obtain (4.12) in the usual manner. See also [3].

Corollary 4.2. *Suppose that the function $[f(z_1, z_2) - v]$ where v is a constant, in addition to the hypothesis of Theorem* 4.1, *satisfies the inequalities*

$$(4.13) \quad \sum_{\varkappa=1}^{2} \frac{(-1)^\varkappa}{8\pi^2} \int\limits_{\lambda=0}^{2\pi} \int\limits_{\varphi_2=0}^{2\pi} [\mathrm{R}_\varkappa (f - v)^{-1}] \cdot [\mathrm{R}_\varkappa \boldsymbol{T}_1 [f^\dagger(e^{i\lambda}, \tau e^{i\varphi_2})] \times$$

$$\mu(e^{i\varphi_2}, t_2) d\Lambda(t_1, t_2, \lambda) d\varphi_2 \leqq Q(v), \quad \varkappa = 1, 2$$

where $\mathrm{R}_1 = \mathrm{Re}$, $\mathrm{R}_2 = \mathrm{Im}$. *Then*

$$
\begin{aligned}
&\boldsymbol{g}(\infty, t_1; t_2) + \boldsymbol{G}(\infty, t_2, 1, \Lambda) + \boldsymbol{G}(\infty, 0, 1, \Lambda) + \\
(4.14)\quad &+ 2\boldsymbol{L}(\infty, t_2, \Lambda) \leqq 2Q(v) + \boldsymbol{g}(v, t_1; t_2) + \\
&+ \boldsymbol{G}(v, t_2, 1, \Lambda) + \boldsymbol{G}(v, 0, 1, \Lambda) + 2\boldsymbol{L}(v, t_2, \Lambda) \\
&+ \overset{+}{\log} | f(t_1, t_2) | + \frac{1}{2\pi} \int_{\lambda=0}^{2\pi} \overset{+}{\log} | f(e^{i\lambda}, 0) | \, d\Lambda \\
&+ (\overset{+}{\log} v + \log 2)\Big(1 + \frac{1}{2\pi} \int_{\lambda=0}^{2\pi} | d\Lambda |\Big), \; \Lambda \equiv \Lambda(t_1, t_2, \lambda).
\end{aligned}
$$

See (1.8), (1.11), (1.10).

Proof. Using definitions (1.8), (1.12), (1.10) and the fact that

$$
\log \left| \frac{(1 - t_2 \overline{a_{\nu\varkappa}(e^{i\lambda})})(t_2 - a_{\nu\varkappa}(e^{i\lambda}))}{a_{\nu\varkappa}(e^{i\lambda})} \right| =
$$

$$
= \log \left| \frac{1 - t_2 \overline{a_{\nu\varkappa}(e^{i\lambda})}}{t_2 - a_{\nu\varkappa}(e^{i\lambda})} \right| + \log \left| \frac{1}{a_{\nu\varkappa}(e^{i\lambda})} \right| + 2 \log | t_2 - a_{\nu\varkappa}(e^{i\lambda}) |
$$

we have

$$
(4.15)\quad \frac{1}{2\pi} \int_{\lambda=0}^{2\pi} \left(\sum_{\nu=1}^{M_\varkappa \exp(i\lambda))} \log \left| \frac{(1 - t_2 \overline{a_{\nu\varkappa}(e^{i\lambda})})(t_2 - a_{\nu\varkappa}(e^{i\lambda}))}{a_{\nu\varkappa}(e^{i\lambda})} \right| \right) d\Lambda(t_1, t_2, \lambda) =
$$

$$
= \boldsymbol{G}(v, t_2, 1, \Lambda) + \boldsymbol{G}(v, 0, 1, \Lambda) + 2\boldsymbol{L}(v, t_2, \Lambda), \; \Lambda \equiv \Lambda(t_1, t_2, \lambda), \; \varkappa = 1.
$$

Replacing in (4.4) $\log | f(z_1, z_2) |$ by $\log | f(z_1, z_2) - v |$, and applying the usual considerations we obtain the inequality (4.14).

Corollary 4.3. *Suppose that the function* $[f(z_1, z_2) - v]$, $v =$ constant, *in addition to the hypothesis of Theorem* 4.1, *satisfies the inequalities*

$$
\sum_{\varkappa=1}^{2} \frac{(-1)^{\varkappa+1}}{4\pi^2} \int_{\lambda=0}^{2\pi} \int_{\varphi_2=0}^{2\pi} [\mathrm{R}_\varkappa (f - v)^{-1}] \cdot
$$

$$
\cdot [\mathrm{R}_\varkappa \boldsymbol{D}_1 [f^\dagger (\varrho e^{i\lambda}, e^{i\varphi_2})]] \, P(e^{i\varphi_2}, t_2) dS(t_1, t_2, \lambda) d\varphi_2 \leqq 2H(v), \; \varkappa = 1, 2
$$

and $dS(t_1, t_2, \lambda)$ *is defined in* (4.6). *Then*

$$
\begin{aligned}
(4.16)\quad &\boldsymbol{g}(\infty, t_1, t_2) + \boldsymbol{g}(\infty, 0; t_2) + 2\boldsymbol{l}(\infty, t_1, t_2) + \boldsymbol{D}_1[\boldsymbol{G}(\infty, t_2, \varrho, S)] \\
&\leqq 2H(v) + \boldsymbol{g}(v, t_1, t_2) + \boldsymbol{g}(v, 0, t_2) + 2\boldsymbol{l}(v, t_1, t_2) \\
&+ \boldsymbol{D}_1[\boldsymbol{G}(v, t_2, \varrho, S)] + \overset{+}{\log} | f(t_1, t_2) | + \overset{+}{\log} | f(t_1, 0) | + 2 \overset{+}{\log} v + \log 4.
\end{aligned}
$$

Proof. Using the notation (1.9), (1.10), and (1.12), we obtain from (4.5) the inequality (4.16).

REMARK 4.1. It should be noted that modifying slightly the considerations we obtain *lower* bounds for the expressions on the left hand side in (4.14) and (4.16).

(Oblatum 2-7-51).

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