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On exceptional values of entire functions

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On Exceptional Values of Entire Functions

by

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1. Let $f(z)$ be an entire function of finite order ϱ. A value α is said to be an exceptional value (e.v.) B if [1])

$$\limsup_{r\to\infty} \frac{\overset{+}{\log}\, n(r,\alpha)}{\log r} = \varrho_1(\alpha) < \varrho$$

e.v. N if [1,78—107; 2, 254—269]

$$\delta(\alpha) = 1 - \limsup_{r\to\infty} \frac{N(r,\alpha)}{T(r)} > 0,$$

and e.v. V (in the sense of Valiron [2]) if

$$\Delta(\alpha) = 1 - \liminf_{r\to\infty} \frac{N(r,\alpha)}{T(r)} > 0.$$

2. Let E denote the set of positive non-decreasing functions $\varphi(x)$ such that [3])

$$\int_A^\infty \frac{dx}{x\varphi(x)}$$

is convergent. It is known that for functions of non-integral and zero order and for a class of functions of integral order, including all functions of maximum or minimum type, we have [4 (i), (ii)]

$$\liminf_{r\to\infty} \frac{\log M(r)}{n(r,\alpha)\varphi(r)} = 0$$

where $\varphi(x)$ is any function of E, for every α. Hence it is natural to define a value α $(0 \leqq |\alpha| < \infty)$ *e. E* for $f(z)$ if

$$(1) \qquad \liminf_{r\to\infty} \frac{\log M(r)}{n(r,\alpha)\varphi(r)} > 0$$

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1) For notations see [1] chapter 1.

2) See [9] where further references will be found. It is known that $\delta(\alpha)$ is not invariant with respect to a change of the origin [12]. To overcome this difficulty Valiron has suggested another definition for $\delta(\alpha)$ [13].

3) In what follows, A denotes a positive constant not necessarily the same at each occurrence.

for some $\varphi \subset E$. Let $n_1(r, \alpha)$ denote the number of simple zeros of $f - \alpha$ in $|z| \leqq r$. We define α to be an e.v. E for simple zeros if

$$R_1(\alpha) = \liminf_{r \to \infty} \frac{\log M(r)}{n_1(r, \alpha)\varphi(r)} > 0$$

for some $\varphi \subset E$, and normal E for simple zeros if $R_1(\alpha) = 0$ for every $\varphi \subset E$.

3. We prove the following results. Theorem 1 generalises a well known result of Borel [5,279]. Theorems 2,3 and 4 give results analogus to those [3,75—78] for a v.e. B for simple zeros. We note however that a v.e. B for simple zeros may not be a v.e. E for simple zeros [1]).

THEOREM 1. (i) *If α is a v.e. B then it is also a v.e. E but the converse is not true.*

(ii) *If α is a v.e. E, then it is also a v.e. N but the converse is not true.*

(iii) *If $f(z)$ has a v.e. E, then ϱ is necessarily an integer and $f(z)$ is of perfectly regular growth order ϱ; also $f(z)$ can have no other v.e. E or N.*

THEOREM 2. *If for a function, the deficiency sum (excluding $\alpha = \infty$) $\Sigma\, \delta(\alpha) = 1$, then there cannot be two values e. E for simple zeros.*

COROLLARY. *If a function has a v.e. E for the whole aggregate of zeros, then there can be no other v.e. E for simple zeros.*

THEOREM 3. *Let $f(z)$ be of order ϱ and suppose that either ϱ is non-integer, or when ϱ is integer or zero then $f(z)$ satisfies the condition*

$$\limsup_{r \to \infty} \frac{\log M(r)}{r^{\varrho} L(r)} = 1, \tag{2}$$

where $L(r)$ is any positive continuous and monotone function for all large r and satisfies the condition $L(kr) \sim L(r)$, as $r \to \infty$, for any fixed positive k. If $\varrho = 0$, suppose further that $\log r = o(L(r))$. Then there cannot be more than two values e. E for simple zeros.

THEOREM 4. *Let $f(z)$ satisfy the conditions of Theorem 3. Then there cannot be more than one v.e. E for the joint sequence of simple and double zeros and if such a value exists, the sequence of simple zeros is normal E for every other value.*

It is known that a v.e. N for a proper meromorphic function $f(z)$ (that is, $n(r, \infty) > 0$) may not be an asymptotic value of

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[1]) See § 7 below.

$f(z)$ [14]. If $f(z)$ be an entire function and $\delta(\alpha) > 0$ for $f(z)$, then it is not known whether α is necessarily an asymptotic value of $f(z)$. For a v.e. E we have

Theorem 5. (i) *If α is a v.e. E for an entire function $f(z)$, then it is also an asymptotic value but the converse is not true.*

(ii) *A v.e. E is 'invariant' with respect to the displacement of the origin; that is, if*

$$\liminf_{r\to\infty} \frac{\log M(r)}{n(r, \alpha)\varphi(r)} > 0$$

for some $\varphi \subset E$, and if $M_A(r)$, $n_A(r, \alpha)$ refer to another 'origin' A then

$$\liminf_{r\to\infty} \frac{\log M_A(r)}{n_A(r, \alpha)\varphi(r)} > 0.$$

We now give two theorems of a different type. Theorem 6 extends a result of Polya and Pfluger [7].

Theorem 6. *If a function of finite order ϱ has a v.e. E, its power series has a density equal to one of the fractions*

$$\frac{1}{\varrho}, \frac{2}{\varrho}, \ldots, \frac{\varrho}{\varrho}.$$

Theorem 7. *Suppose $f(z)$ is of order 1 and has a v.e. E. If* [1] $\lim_{r\to\infty} \log M(r)/r = T$ *and if $f(z)$ has an asymptotic period β then* $|\beta| \geqq 2\pi/T$.

We suppose here β to be Whittaker [6,84] period. If we follow the definition of an asymptotic period as given by S. S. Macintyre [15] then $|\beta| \geqq \pi/T$.

4. **Proof of Theorem 1.** (i) If α is a v.e. B then $\log M(r) \sim Tr^{\varrho}$ $(0 < T < \infty)$ and $\varrho_1(\alpha) < \varrho$. Hence

$$\lim_{r\to\infty} \frac{\log M(r)}{n(r, \alpha) r^{\beta}} = \infty, \; 0 < \beta < \varrho - \varrho_1(\alpha).$$

To show that the converse is not true, we consider

$$f(z) = e^z P(z) = e^z \prod_{2}^{\infty} \left\{1 + \frac{z}{n\,(\log n)^2}\right\}; \; \alpha = 0. \tag{3}$$

(ii) If $\varrho > 0$ is non-integer then [3,69]

$$\liminf_{r\to\infty} \frac{\log M(r)}{n(r, \alpha)} < A$$

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[1]) This limit exists. See Theorem 1 (iii) above.

for every α. If $\varrho = 0$ then [4(i), 29—30]

$$\liminf_{r\to\infty} \frac{\log M(r, f)}{n(r, 0)\varphi(r)} = 0$$

and hence

$$\liminf_{r\to\infty} \frac{\log M(r, f)}{n(r, \alpha)\varphi(r)} = 0.$$

Hence if α is a v.e. E, ϱ must be integer and we will have

$$f(z) - \alpha = z^n \exp\{Q(z, \alpha)\}P(z, \alpha) \tag{4}$$

where $Q(z, \alpha)$ is a polynomial of degree $q(\alpha)$ (say) and $P(z, \alpha)$ is the canonical product (c.p.) of genus $p(\alpha)$ (say). We have either [4 (ii) 186—187] $\varrho_1(\alpha) < \varrho$ or $\varrho_1(\alpha) = q(\alpha) = \varrho$; $p(\alpha) = \varrho - 1$. In either case we have $\log M(r) \sim Tr^\varrho$ $(0 < T < \infty)$ for we have

LEMMA. *If* $f(z) = z^N e^{Q(z)} P(z)$ *is of order* ϱ, ϱ *integer and* $q = \varrho$, $p \leqq \varrho - 1$, *then*

$$\log M(r, f) \sim Tr^\varrho \quad (0 < T < \infty) \tag{5}$$

PROOF. Let $Q(z) = az^\varrho + \ldots$, $|a| = T$. Then

$$\log M(r, f) < O(\log r) + (T + 0(1))r^\varrho + 0(r^\varrho) \sim Tr^\varrho.$$

Suppose if possible

$$\liminf_{r\to\infty} \frac{\log M(r, f)}{r^\varrho} = l < T$$

and let $l < L < T$, $1 < k < (T/L)^{1/\varrho}$. Then for a sequence of values of $r = r_n$ $(n = 1, 2, \ldots)$, $\log M(r, f) < Lr^\varrho$.

For any R_n $(n = 1, 2, \ldots)$ such that $r_n/k \leqq R_n \leqq r_n$ we have

$$\log M(R_n, f) \leqq \log M(r_n, f) < Lr_n^\varrho \leqq Lk^\varrho R_n^\varrho.$$

Further $\log M(r, P) = 0(r^\varrho)$ and there is always a circle $|z| = R_n$ in the annulus $r_n/k \leqq |z| \leqq r_n$ on which [3,89]

$$|P(z)| > \{M(kr, P)\}^{-H}.$$

Hence for $r = R_n$ $(n > N_0)$

$$\log\left|\frac{1}{P(z)}\right| < H \log M(kr, P) < H\epsilon k^\varrho r^\varrho$$

$$e^{\{RQ(z)\}} = \left|\frac{f(z)}{z^N P(z)}\right| < \exp\{Lk^\varrho R_n^\varrho + 2H\epsilon k^\varrho R_n^\varrho\}.$$

Hence

$$\liminf_{r\to\infty} \frac{\max\limits_{|z|=R_n} R\{Q(z)\}}{R_n^{\varrho}} \leqq Lk^{\varrho}.$$

But the left hand expression has the limit $T > k^{\varrho}L$.

Hence we have a contradiction and so $l = T$ which proves the lemma.

If α is a v.e. E, then $\log M(r) \backsim Tr^{\varrho}$ and so

$$T(r) > A \log M(r) > An(r, \alpha)\varphi(r)$$

and since $\log r = o(\varphi(r))$ we have

$$\lim_{r\to\infty} T(r)/N(r, \alpha) = \infty;\ \delta(\alpha) = \Delta(\alpha) = 1. \tag{6}$$

To show that $\delta(\alpha)$ may be equal to unity but α may not be a v.e. E we need consider the c.p. $P(z)$ defined in (3). For this c.p. $\delta(0) = 1$ and

$$\lim_{r\to\infty} \frac{\log M(r)}{n(r, 0)\varphi(r)} = 0.$$

(iii) To complete the proof of (iii) we note that $\sum \delta(\alpha) \leqq 1$ the summation being over all finite values of a. Since $\delta(\alpha) = 1$ there can be no other v.e. N and a fortiori e. E.

5. Proof of Theorem 2. Since $\sum \delta(\alpha) = 1$, ϱ is integer [8,92—94] Let $\varrho(r)$ be a proximate order. Then

$$\lim_{r\leftarrow\infty} \varrho(r) = \varrho,\ \lim_{r\to\infty} r\varrho'(r) \log r = 0$$

$$\log M(r) \leqq r^{\varrho(r)} \text{ for all } r > r_0$$

$$= r^{\varrho(r)} \text{ for an infinity of } r.$$

Further [8,94] $N(r, f') = o(r^{\varrho(r)})$.

If α and β ($\alpha \neq \beta$) be v.e. E for simple zeros then

$$N(r, \alpha) + N(r, \beta) > A(k) \log M(r/k) > Ar^{\varrho(r)}$$

for an infinity of r, say $r = r_n$. Also if N_1 refers to simple zeros then

$$N_1(r, \alpha) + N_1(r, \beta) + 4N(r, f') + O(\log r) > N(r, \alpha) + N(r, \beta)$$

and so for $r = r_n$ ($n > n_0$)

$$N_1(r, \alpha) + N_1(r, \beta) > Ar^{\varrho(r)}.$$

Now

$$\log M(r) > An_1(r, \alpha)\varphi_1(r),\ \varphi_1(r) \subset E,\ r > R_0$$

$$\log M(r) > An_1(r, \beta)\varphi_2(r),\ \varphi_2(r) \subset E,\ r > R_0.$$

Let $\varphi(x) = \min\{\varphi_1(x), \varphi_2(x)\}$. Then it is easily seen that $\varphi(x) \subset E$ and we have for $r > R_0$

$$\log M(r) > A\{n_1(r, \alpha) + n_1(r, \beta)\}\varphi(r).$$

Hence for $r = r_n$ $(n > N_2 > n_0)$

$$r^{\varrho(r)} \geqq \log M(r) > A\{n_1(r, \alpha) + n_1(r, \beta)\}\varphi(r)$$
$$> \frac{A\varphi(r)}{\log r}\{N_1(r, \alpha) + N_1(r, \beta)\} > \frac{A\varphi(r)}{\log r} r^{\varrho(r)}.$$

Hence we have a contradiction and so the theorem is proved.

PROOF OF COROLLARY. Let α be a value exceptional E for the whole aggregate of zeros. Then $\delta(\alpha) = 1$ and so by the theorem there cannot be two values e. E for simple zeros. Since α is a fortiori a v.e. E for simple zeros, there can be no other v.e. E for simple zeros.

6. PROOF OF THEOREM 3. Suppose if possible there are three such values a, b, c $(a \neq b \neq c)$. Let $P(z, a) = P_a$ denote the c.p. formed with the simple zeros of $f(z) - a$ and denote by $p_1(a)$ its genus and by $\varrho_{11}(a)$ its order. Similarly for $P(z, b)$ and $P(z, c)$. Then

$$\theta(z) = \frac{P(z, a)P(z, b)P(z, c)\{f'(z)\}^2}{\{f(z) - a\}\{f(z) - b\}\{f(z) - c\}}$$

is an entire function. [3,76].

(i) Consider first when $\varrho > 0$ is non-integer. We have

$$(7) \qquad n_1(r, a) < \frac{A \log M(r, f)}{\varphi(r)} \leqq \frac{A r^{\varrho(r)}}{\varphi(r)}, \; r > r_0.$$

We prove that

$$(8) \qquad \log M(r, P_a) = o(r^{\varrho(r)}).$$

If $\varrho_{11}(a) < \varrho$ then (8) follows. Suppose therefore $\varrho_{11}(a) = \varrho$, $p_1(a) < \varrho < 1 + p_1(a)$. Writing $p_1(a) = p$ and $n_1(x, a) = n(x)$ we have

$$(9) \qquad \log M(r, P_a) < A\left\{r^p \int_0^r \frac{n(t)\,dt}{t^{p+1}} + r^{p+1}\int_r^\infty \frac{n(t)\,dt}{t^{p+2}}\right\}.$$

Now for all $x > x_0$, $p < \varrho(x) < 1 + p$ and so $x^{\varrho(x)-p}$ is increasing and $x^{\varrho(x)-p-1}$ is decreasing for $x > x_1$. Hence from (7) and (9) we obtain (8). Similarly for P_b and P_c. Let the zeros of $f - a$, $f - b$, $f - c$ be respectively

$$(a_n)_1^\infty, \; (b_n)_1^\infty, \; (c_n)_1^\infty;$$

and denote by S the set of circles

$$|z-a_n| = |a_n|^{-h}, \quad |z-b_n| = |b_n|^{-h}, \quad |z-c_n| = |c_n|^{-h};$$
$$(|a_n| \geqq 1, \quad |b_n| \geqq 1, \quad |c_n| \geqq 1, \quad h > \varrho)$$

Then in the domain D exterior to the circles S we have [3,74] for $r > r_0$

$$\left|\frac{f'(z)}{f(z)-a}\right| \left|\frac{f'(z)}{f(z)-b}\right| < r^{2K}$$

and hence in D

$$\log M\{r, (f-c)\theta\} = o(r^{\varrho(r)}) + O(\log r) = o(r^{\varrho(r)}).$$

Similarly for $(f-b)\theta$ and hence in D

$$\log M(r, \theta) = o(r^{\varrho(r)}).$$

Now

$$\log M(r, f-c) > Ar^{\varrho(r)}$$

for a sequence of values of $r = r_n \to \infty$. Let $k > 1$ be a fixed positive constant and let $r_n \leqq r \leqq kr_n$. Then for $n > n_0$

$$\log M(r, f-c) \geqq \log M(r_n, f-c) > A_1 r_n^{\varrho(r_n)} > Ar^{\varrho(r)}.$$

Further

$$\begin{aligned}\log M(r, f-c) &\leqq AT(2r, f-c)\\ &\leqq A\left[T\{2r, (f-c)\theta\} + T\left\{2r, \frac{1}{\theta}\right\}\right]\\ &\leqq A\ [\log M\{2r, (f-c)\theta\} + \log M(2r, \theta) + O(1)]\\ &< \epsilon\, r^{\varrho(r)}\end{aligned}$$

for all $r > R_1$, such that $2r \subset D$. Let $r_n > R_1$, $n > n_0$. Since we can always draw a circle $|z| = r$ in the annulus $r_n \leqq |z| \leqq kr_n$ such that $2r \subset D$, we have for a sequence of values of $r \to \infty$,

$$Ar^{\varrho(r)} < \log M(r, f-c) < \epsilon\, r^{\varrho(r)}$$

which leads to a contradiction and so the theorem is proved.

(ii) ϱ integer. We prove first that

$$\text{(10)} \qquad \log M(r, P_a) = o(r^{\varrho} L(r)).$$

We have

$$n_1(r, a) = n(r) \text{ (say) } < \frac{Ar^{\varrho}L(r)}{\varphi(r)}.$$

It is known that [10] $r^c L(r) \to \infty$, $r^{-c}L(r) \to 0$, for every constant $c > 0$, as $r \to \infty$. Further

$$\int_1^r L(t)dt \sim rL(r), \quad \int_r^\infty \gamma^{-2}L(t)dt \sim r^{-1}L(r).$$

If $\varrho_{11}(a) < \varrho$ then (10) is obvious. Suppose therefore

$$\varrho_{11}(a) = \varrho,\ p_1(a) = p \text{ (say)} = \varrho - 1 \text{ or } \varrho.$$

(a) Consider first when $p = \varrho - 1$ and $L(r) \downarrow$. We divide the interval of integration (0, r) of the first integral on the right hand side of (9) in the intervals (0, $\sqrt{r}$), ($\sqrt{r}$, r). Then each of these three integrals is $o(r^\varrho L(r))$.

(b) $p = \varrho - 1$, $L(r) \uparrow$.

Here $\log M(r, P_a) = o(r^\varrho) = o(r^\varrho L(r))$

(c) $p = \varrho$, $L(r) \overset{\infty}{\uparrow}$. We choose $\lambda = \lambda(r)$, $(0 < \lambda < r)$ tending to infinity with r so slowly that $L(\lambda(r)) = o(L(r))$ and divide the interval of integration (0, r) in the intervals (0, λ) (λ, r). Then each of these three integrals is $o(r^\varrho L(r))$.

(d) $p = \varrho$, $L(r) \overset{A}{\uparrow}$ or $\downarrow$. This alternative is not possible since it would make the integral $\int_1^\infty \{n(x/x^{p+1})\}dx$ convergent.

Hence in all cases (10) holds and the rest of the argument is similar to that given in (i).

(iii) $\varrho = 0$. The proof is similar to that given in (i). The proof of Theorem 4 is similar to that of Theorem 3.

7. Example. Let $G(z)$ be any entire function of order $\varrho > 1$ and lower order $\lambda < 1$ and let

$$f(z) = \{G(z)\}^2 P(z)$$

where $P(z)$ is c.p. defined in (3). Then it is easily seen that 0 is a v.e. B for the simple zeros of $f(z)$. But

$$n_1(r, 0) \sim r/\log^2 r$$

$$\log M(r, f) \leqq 2 \log M(r, G) + \log M(r, P).$$

Hence for a sequence of values of r tending to infinity we have

$$\log M(r, f) < Ar/\log r,$$

$$\liminf_{r\to\infty} \frac{\log M(r, f)}{n_1(r, 0)\varphi(r)} = 0.$$

Hence 0 is not a v.e. E for simple zeros of $f(z)$.

8. Proof of Theorem 5. (i) From (4) we have

$$|f(z)-\alpha| = r^n e^{RQ(z,\alpha)}|P(z,\alpha)|$$

Let $Q(z,\alpha) = az^\varrho + Q_1(z)$; $a = Te^{i\beta}$, $Q_1(z)$ a polynomial of degree $\leqq \varrho - 1$. Then

$$\log|f(z)-\alpha| = Tr^\varrho \cos(\varrho\theta+\beta) + RQ_1(z) + \log|P(z,\alpha)|$$

Let $0 < \delta < \pi/10$ and θ_0 be such that

$$\frac{\pi}{2} + \delta \leqq \varrho\theta_0 + \beta + 2k\pi \leqq \pi + \frac{\pi}{2} - \delta$$

(k integer or zero); and let $0 < \epsilon < -(T/4)\cos(\varrho\theta_0+\beta)$, $z = re^{i\theta_0}$. Choose r_0 so large that for all $r > r_0$ and all θ

$$RQ_1(z) < \epsilon r^\varrho,\ n\log r < \epsilon r^\varrho,\ \log|P(z,\alpha)| < \epsilon r^\varrho.$$

Then for $z = re^{i\theta_0}$, $r > r_0$.

$$\log|f(z)-\alpha| < r^\varrho\{T\cos(\varrho\theta_0+\beta)+3\epsilon\} \to -\infty \text{ as } r\to\infty.$$

Hence $f(z)\to\alpha$ as $z = re^{i\theta_0}\to\infty$; that is α is an asymptotic value.

To show that the converse is not true, we consider [2,160—161]

$$f(z) = \int_0^z e^{-t^\varrho}dt \quad \varrho \text{ integer, } 2 \leqq \varrho < \infty.$$

Let

$$a_\mu = \exp\left(\frac{2\mu\pi i}{\varrho}\right)\int_0^\infty e^{-r^\varrho}dr,\ \mu = 0, 1, 2, \ldots, \varrho-1.$$

Then for $a = a_0, a_1, \ldots, a_{\varrho-1}$.

$$T(r) \backsim \frac{r^\varrho}{\pi};\ n(r,a) > \frac{A_1 r^\varrho}{\log r};\ \lim_{r\to\infty}\frac{\log M(r)}{n(r,a)\varphi(r)} = 0.$$

Hence each of these numbers $a_0, a_1, \ldots, a_{\varrho-1}$ is an asymptotic value but not a value exceptional E.

(ii) We may suppose that the new 'origin' A is on the real positive axis at a distance h from 0. Then since

$$\log M(r) \backsim Tr^\varrho \qquad (0 < T < \infty)$$

$$M(r-h) \leqq M_A(r) \leqq M(r+h)$$

t follows that $\log M_A(r)$ lies between $A_1 r^\varrho$ and $A_2 r^\varrho$ for all

$r > r_0(h)$. Further

$$n(r - h, \alpha) \leqq n_A(r, \alpha) \leqq n(r + h, \alpha)$$
$$n(r + h, \alpha)\varphi(r + h) < A_3 r^{\varrho}.$$

Hence

$$n_A(r, \alpha)\varphi(r) \leqq n(r + h, \alpha)\varphi(r + h) < A_3 r^{\varrho} < A_4 \log M_A(r).$$

$$\liminf_{r\to\infty} \frac{\log M_A(r)}{n_A(r, \alpha)\varphi(r)} > 0.$$

We omit the proofs of Theorems 6 an 7 which can be proved by following the argument given by Whittaker [6,61—62; 84—87].

9. Meromorphic Functions. Let $F(z)$ be a meromorphic function of finite order ϱ. We define a number α $(0 \leqq |\alpha| \leqq \infty)$ to be an e.v. E for $F(z)$ if

$$\liminf_{r\to\infty} \frac{T(r)}{\{n(r, \alpha)\varphi(r)\}} > 0 \tag{11}$$

for some $\varphi \subset E$. It is easily seen that the two definitions of values e. E for entire functions are equivalent. Obviously ∞ is a v.e. E for entire functions according to (1) or (11). We can also prove that if α is a v.e. E for a meromorphic function $F(z)$ then it is a v.e. N, with deficiency $\delta(\alpha) = 1$ and $\Delta(\alpha) = 1$. To see that the converse is not true we consider the meromorphic function [1,91—93]

$$f_\lambda(z) = \sum_{\nu=0}^{2\lambda-1} \eta^\nu f(\eta^\nu z)$$

where $\lambda > 1$ is an odd integer, $\eta = \exp.\ (\pi i/\lambda)$ and $f(z) = e^z/(e^z—)1$. This function $f_\lambda(z)$ is a meromorphic function of order 1 and has 2λ values e. N; $\eta^\nu\alpha$ $(\nu = 0, 1, 2, \ldots, 2\lambda - 1)$ each with deficiency $\frac{1}{\lambda}\left(1 - \cos\frac{\pi}{2\lambda}\right) < 1$.

Hence none of these 2λ values can be a v.e. E.

We note also that if α be a v.e. B for a meromorphic function $F(z)$ then it may not be a v.e. E. In fact Valiron has shown that [13] a value α e. B may have deficiency $\delta(\alpha) = 0$.

THEOREM 8. *If $F(z)$ is a meromorphic function of finite order ϱ, then there cannot be more than two values e. E for $F(z)$ and if $F(z)$ has two values e. E then ϱ is necessarily an integer and $T(r, F)/r^{\varrho}$ tends to a finite non-zero limit as r tends to infinity.*

PROOF. If α be a v.e. E then $\delta(\alpha) = 1$. Since $\Sigma\delta(\alpha) \leqq 2$ there cannot be more than two values e. E. Suppose then α, β $(\alpha \neq \beta$,

$0 \leqq |\alpha| \leqq \infty$, $0 \leqq |\beta| \leqq \infty$) be two values e. E. Then for all $r > r_0$

$$T(r) > \delta\{n(r, \alpha) + n(r, \beta)\}\varphi(r) > \delta_1\{N(r, \alpha) + N(r, \beta)\}\frac{\varphi(r)}{\log r},$$

$$\frac{N(r, \alpha) + N(r, \beta)}{T(r)} < \frac{\log r}{\delta_1 \varphi(r)}.$$

But if $\varrho > 0$ is non-integer then [1,51—54]

$$\limsup_{r \to \infty} \frac{N(r, \alpha) + N(r, \beta)}{T(r)} > 0$$

and if $\varrho = 0$ then [11,67—69]

$$\limsup_{r \to \infty} \frac{N(r, \alpha) + N(r, \beta)}{T(r)} \geqq 1.$$

Hence ϱ must be integer. Further since

$$T\left(r, \frac{AF + B}{CF + D}\right) = T(r) + O(1),$$

we may suppose that 0 and ∞ are values e.E. Write

$$F(z) = z^k e^{Q(z)} P_1(z)/P_2(z)$$

where P_1 is c.p. of genus p_1 (say) formed with zeros a_n($|a_n| > 0$) of $F(z)$ and P_2 is c.p. of genus p_2 (say) formed with poles b_n ($|b_n| > 0$) of $F(z)$. $Q(z)$ is a polynomial of degree q (say). We know that [4 (ii) 188]

$$\liminf_{r \to \infty} \frac{T(r, F)}{\{n(r, 0) + n(r, \infty)\}\varphi(r)} = 0$$

for every $\varphi \subset E$, except when $q > \max(p_1, p_2)$. Hence $q = \varrho$, $p_1 < \varrho$, $p_2 < \varrho$. So

$$\limsup_{r \to \infty} \frac{T(r, P_a)}{r^\varrho} \leqq \limsup_{r \to \infty} \frac{\log M(r, P_a)}{r^\varrho} = 0; \ a = 1,2.$$

$$\lim_{r \to \infty} \frac{T(r, F)}{r^\varrho} = \lim_{r \to \infty} \frac{T(r, e^Q)}{r^\varrho}.$$

Now $T(r) \backsim \operatorname{Max}_a N(r, a)$. Hence if $Q(z) = bz^\varrho + \ldots$ then

$$T(r, e^Q) \backsim \frac{r^q |b|}{\pi}; \quad \lim_{r \to \infty} \frac{T(r, F)}{r^\varrho} = \frac{|b|}{\pi}$$

and the theorem is proved.

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