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by

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Non-negative integers will be denoted by $i, j, \ldots, n$; real numbers by $t, u, v, x, y, \sigma, \tau, a_n, c_n$; positive numbers that are arbitrarily small by $\varepsilon, \varepsilon_1$; complex values by $z = x + iy$, $w = u + iv$, $s = \sigma + i\tau$. It will be understood that

$$(0.01) \qquad w^s = \exp\,(s \log |\, w \,| + is \arg w),$$

where the value of $\arg w$ has to be fixed. I shall write $x = \operatorname{Re} z$, $y = \operatorname{Im} z$, etc. The letter χ will stand for a bounded non-decreasing function of a non-negative argument; χ will be normalized by the conditions

$$(0.02) \qquad \chi(0) = 0, \quad \chi(t) = \tfrac{1}{2}\{\chi(t+0) + \chi(t-0)\},$$

and the same will apply to $\chi_n, \overline{\chi}, \overline{\chi}_n$. If $\chi(t+\varepsilon) > \chi(t-\varepsilon)$ for a fixed value of t and for every $\varepsilon > 0$, t will be called a point of increment of χ. An open interval $a < x < b$ will be denoted by (a, b), a closed interval $a \leqq x \leqq b$ by $\langle a, b\rangle$. An empty sum will be put equal to zero, an empty product equal to unity. If different integrals of the same integrand occur in the same formula, the integrand may be written only once.

A function $f(x)$ is said to be completely monotonic in (a, b) if it has derivatives of all orders there, and if

$$(0.03) \qquad (-)^k f^{(k)}(x) \geqq 0 \qquad (a < x < b,\ k = 0, 1, 2, \ldots);$$

f is said to be completely monotonic in $\langle a, b\rangle$ if it is continuous in $\langle a, b\rangle$ and completely monotonic in (a, b).

For the sake of concision no attempt has been made to make this paper correct in the sense of intuistionistic mathematics. I shall speak e.g. of the class $\{F\}$ of all functions F, while one might doubt whether this is quite correct. Yet I have tried to make proofs correct in this respect wherever I could. Thus, the use of a well-known theorem of Helly [1] [1]) has been avoided; but, to achieve this, a much longer proof of Theorem 33 had to

[1]) Numbers in brackets refer to the bibliography.

be given. On the contrary, the theorem of Porter-Vitali has been used throughout. In the proofs where it has been applied (of Theorems 1, 31, 33), it would have been easy to deduce the uniform convergence of a certain sequence $\{f_n(z)\}_1^\infty$ within a fixed domain of the z-plane by giving explicit upper bounds of $|f_n(z) - f_{n+\nu}(z)|$, but I left this out, as it seemed to be of little interest. Properly speaking, we could do without this theorem.

I am indebted to Prof. van der Corput for Lemma 2, which greatly simplified my own proof of Lemma 3. My thanks are also due to Prof. van der Waerden for his critical remarks. With the exception of Theorems 8—16 and 42—47, this paper was finished in 1943, when it has been discussed with Prof. van der Corput; by various circumstances publication has been delayed till now.

§ 1. Introduction.

The main problem of this paper belongs to the field of interpolation theory or rather to that of integral equations of the first kind. This problem is a special case of the next one:

Problem (*a*). Let $\{x_n\}_1^\infty$ and $\{a_n\}_1^\infty$ be two given sequences. Let

$$(1.01) \quad x_1 > 0; \quad x_{n+1} > x_n \ (n = 1, 2, \ldots); \quad x_n \to \infty \quad \text{as} \quad n \to \infty; \quad a_n > 0 \ (n = 1, 2, \ldots).$$

Let $K(x, t)$ be a given kernel, and let $K \geqq 0$ for $x \geqq 0$, $0 \leqq t \leqq 1$. Put

$$(1.02) \qquad f(x) = \int_0^1 K(x, t)d\chi(t).$$

To determine the functions χ that satisfy the set of equations

$$(1.03) \qquad f(x_n) = a_n \qquad (n = 1, 2, \ldots).$$

Several cases of problem (a) have been treated in literature. I mention the following, including the one that is dealt with here, but I am not sure the list is complete.

Problem (*b*). If $x_n = n$, $K(x, t) = (t^{-1} - 1)^x$, we have a problem that is equivalent to the moment problem of Stieltjes [1].

Problem (*c*). If $x_n = n$, $K(x, t) = t^x$, the problem is equivalent to the moment problem of Hausdorff [1], [2].

Problem (*d*). If we only add to (1.01) the condition

$$(1.04) \qquad \sum_1^\infty \frac{1}{x_n} = \infty,$$

and if $K(x, t) = t^x$, we get a generalization of (c) that has been treated by Hausdorff [1] and Feller [1].

Problem (*e*). If the sequence $\{x_n\}_1^\infty$ is subjected to no other conditions than (1.01), and if $K(x, t) = (1 - t + tx)^{-1}$, we get the problem that will occupy us here. In this particular case

$$F(z) = \int_0^1 \frac{d\chi(t)}{1 - t + tz} \tag{1.05}$$

will be written instead of (1.02). The integral (1.05) is convergent for all values of z, with the possible exception of the values $z \leqq 0$. For the present, the *function* F will be made one-valued by excluding the values $z \leqq 0$, so that $F(z)$ can always be represented by (1.05). The class of all functions F will be denoted by $\{F\}$.

The next problem, which has been solved by R. Nevanlinna [1], is closely related to the type (a), though somewhat different from it.

Problem (*f*). Let $\{z_n\}_1^\infty$ and $\{w_n\}_1^\infty$ be two given sequences of complex numbers. Let $|z_n| < 1$, $|w_n| < 1$ $(n = 1, 2, \ldots)$. To determine the functions $w(z)$ holomorphic in the interior of the unit circle, which satisfy the conditions

$$|w(z)| \leqq 1 \ (|z| < 1; \ w(z_n) = w_n \qquad (n = 1, 2, \ldots).$$

Obviously the theory of the cases (b) . . . (e) will have many traits in common. A necessary and sufficient condition for the existence of at least one solution consists, in each of these cases, of a set of inequalities

$$A(x_1, \ldots x_n, a_1, \ldots a_n) \geqq 0 \qquad (n = 1, 2, \ldots). \tag{1.06}$$

In the cases (c) and (d) there are, in addition to (1.01), $n - 1$ inequalities (1.06) that correspond to a single value n. In the cases (b) and (e) there is just one such inequality required for every value of n. The explicit conditions (1.06) that correspond to problem (e) will be given later; these will be shown to be necessary (§ 3) as well as sufficient (§ 5).

Stieltjes distinguished a *determined* moment problem, which has a unique solution, from an *indeterminate* one with an infinity of solutions. The terms have also been applied by R. Nevanlinna in the case (f). The same distinction will be made here. If solvable, the problems (c) and (d) are always determined. The solvable cases (b) and (e) may be either determined or indeterminate. Perhaps this second resemblance between (b) and (e) points to a deeper analogy; at any rate, the discussions of §§ 4—6 are much like corresponding ones of Stieltjes. A necessary and sufficient condition for the uniqueness of a solution of (e) will be given in § 6, where a further classification of the determined cases of problem (e) will be made too.

Different connections between the problems (b), . . ., (f) can be stated:

(α) If all numbers x_n tend to a given value $x \geqq 0$, (d) tends to the moment problem of Stieltjes as a limit case.

(β) If all numbers z_n tend to a given value $z_0 = \exp(\varphi i)$, (f) tends to a problem equivalent to Hamburger's generalization of the moment problem of Stieltjes. As Nevanlinna [1] has shown, the solutions of the moment problem of Hamburger [1], [2] can be obtained from the theory of (f).

(γ) If all numbers x_n tend to a given value $x > 0$, (e) tends to the moment problem of Hausdorff as a limit case.

Since various problems are contained in Nevanlinna's problem (f), the question must be raised whether (e) is also in some way contained in it. The question is too vague to be denied with certainty, but as yet I see no way to solve (e) by means of Nevanlinna's formulae. On the other hand, if we add to problem (e) the condition $|F(z)| \leqq 1$ for $|z - 1| < 1$, we get a problem that is certainly contained in (f). For, let $f(w)$ be holomorphic and $|f(w)| \leqq 1$ within the circle $|w| < 1$; let f also be real when w is real. Then, by the transformation

$$z = \frac{4w}{(1-w)^2}, \quad F(z+1) = \tfrac{1}{2}(1-w)\frac{1-f(w)}{1+wf(w)},$$

there is a one-to-one correspondence between the functions f and F (Wall [1]).

Now, the condition $|w(z)| \leqq 1$ in problem (f) has been replaced by Lokki [1] by the less restrictive one

$$\frac{1}{\pi}\int_0^1 \int_0^{2\pi} |w'(z)|^2 \, r \, dr \, d\varphi \leqq I_0 < \infty,$$

and it may well be that problem (e) can be subsumed under Lokki's, or even that the two problems are equivalent. This is a question that still has to be decided. However, it is most probable that generalizations of problem (e) can be treated by the method of Stieltjes used here. If we replace the particular functions F

in the expressions (3.27) ... (3.30) by more general functions f as defined by (1.02), results might be obtained that are analogous to the basic Theorem 17, but I am sorry I had as yet no opportunity of investigating this question. Of course all these problems are very closely related, but it still remains doubtful whether the methods of Nevanlinna and Stieltjes are equally powerful, or which of these is the most powerful.

In connection with his moment problem, Stieltjes examined integrals of the type

$$f(z) = \int_0^\infty \frac{d\bar{\chi}(u)}{z+u}, \tag{1.07}$$

where $\bar{\chi}(\infty) < \infty$. By the transformation

$$u = t^{-1} - 1,\ \bar{\chi}(u) = \int_t^1 \frac{d\chi(t)}{t}, \tag{1.08}$$

we get an integral of the type (1.05). Hence the functions (1.07) belong to the class $\{F\}$, and they are characterized by

$$\int_0^1 \frac{d\chi(t)}{t} < \infty. \tag{1.09}$$

As it has been shown by Feller [1], the Newton series represents the solution of problem (d). The same holds in certain cases of (e), and Theorem 48 states a result that is much like Feller's. There is also a remarkable similarity between the determinants, defined in (3.44) ... (3.51), and those studied by Barkley Rosser [1], and one might be inclined to look for more general connections here.

A solution of problem (a) will be called *degenerate*, if χ only increases for a finite number of values t, and the problem (a) itself will be called so if it has a degenerate solution. Perhaps the study of degenerate problems is not *quite* uninteresting. In the case of a degenerate moment problem of Stieltjes, only a finite number of the usual expressions $A(x_1, \dots a_n)$ in (1.06) is positive. On the contrary, the solution of a degenerate moment problem of Hausdorff satisfies a set of inequalities (1.06) with all left hand members positive, except for the very special case when χ only increases for the value $t = 1$. A degenerate solution of problem (e) is rational. Conversely, it will be shown that any rational solution of (e) is degenerate (Theorem 16). The degenerate cases of (b) and (e) are always determined.

Without loss of generality we can add to (e) the conditions

$$(1.10) \qquad x_1 = a_1 = 1.$$

For, if we put

$$(1.11) \qquad \bar{x}_n = \frac{x_n}{x_1},\ \bar{a}_n = \frac{a_n}{a_1} \quad (n = 1, 2, \ldots),$$

the sequences $\{\bar{x}_n\}_1^\infty$, $\{\bar{a}_n\}_1^\infty$ will satisfy (1.01) and (1.10). Now if $F(x)$ is a solution of (e), and if we put

$$(1.12) \qquad u = \frac{tx_1}{1 - t + tx_1},\ \bar{\chi}(u) = a_1^{-1}\int_0^t \frac{d\chi(t)}{1 - t + tx_1},$$

the function

$$\bar{F}(x) = a_1^{-1}F(xx_1) = \int_0^1 \frac{d\bar{\chi}(u)}{1 - u + ux}$$

will satisfy the conditions $\bar{F}(\bar{x}_n) = \bar{a}_n$ $(n = 1, 2, \ldots)$. For this reason the restrictions (1.10) will always be made, unless the contrary is expressed. By (0.01) (1.05) and (1.10) we then have

$$(1.13) \qquad \chi(1) = 1.$$

Before solving problem (e), some generalities concerning the functions F will be discussed in the next section.

§ 2. Elementary Properties of the Functions F.

Any function $F(z)$ is bounded in a half plane $x \geqq \xi > 0$. It is easy to prove that F is bounded in a much bigger part of the z-plane.

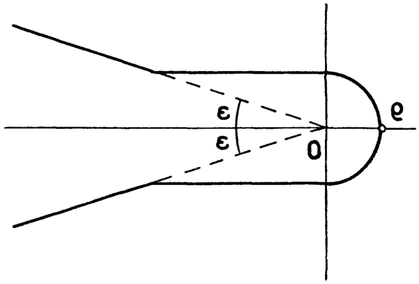

Fig. 1.

Lemma 1. Let the closed region $G(\varepsilon, \varrho)$ in the z-plane be defined by the inequalities.

$$(2.01) \qquad |\varphi| \leqq \pi - \varepsilon,$$
$$(2.02) \qquad |y| \geqq \varrho \text{ if } x \leqq 0,$$
$$(2.03) \qquad |z| \geqq \varrho \text{ if } x > 0,$$

where $z = x + iy = r \exp(\varphi i)$, $0 < \varepsilon < \frac{\pi}{2}$, $\varrho > 0$. Let $0 \leqq t \leqq 1$. Then

$$| 1 - t + tz | \geqq \frac{\varrho \sin \varepsilon}{2(\varrho + \sin \varepsilon)}$$

for any z in G.

PROOF. The lemma is true when $t = 0$; let us first take $0 < t \leqq \frac{1}{2}$. By (2.01) we have $\cos \varphi \geqq - \cos \varepsilon$; hence, if $\lambda > 0$, we have

$$| \lambda + re^{\varphi i} |^2 = \lambda^2 + 2\lambda r \cos \varphi + r^2 \geqq \lambda^2 - 2\lambda r \cos \varepsilon + r^2 = | \lambda - re^{-\varepsilon i} |^2,$$

and

$$| \lambda + z | \geqq | \lambda - re^{-\varepsilon i} | = | \lambda e^{\varepsilon i} - r | = | \lambda (\cos \varepsilon + i \sin \varepsilon) - r | \geqq \lambda \sin \varepsilon,$$

or, putting $\lambda = t^{-1} - 1$,

$$| 1 - t + tz | \geqq t\lambda \sin \varepsilon \geqq \tfrac{1}{2} \sin \varepsilon,$$

hence the lemma is true. Next, take $\frac{1}{2} < t \leqq 1$. If $x \leqq 0$ we have

$$| 1 - t + tz | \geqq t | y | > \frac{\varrho}{2}$$

by (2.02); if $x > 0$ we have

$$| t - t + tz | = \{(1 - t + tx)^2 + t^2 y^2\}^{1/2} > t \sqrt{x^2 + y^2} = t | z | > \frac{\varrho}{2}$$

by (2.03); hence the lemma is true again.

THEOREM 1. The functions $F(z)$ are uniformly bounded in a given domain $G(\varepsilon, \varrho)$. We have

$$| F(z) | \leqq \frac{2(\varrho + \sin \varepsilon)}{\varrho \sin \varepsilon}.$$

PROOF. By Lemma 1 and by (1.13) we have, for any z in $G(\varepsilon, \varrho)$,

$$| F(z) | \leqq \int_0^1 \frac{d\chi(t)}{| 1 - t + tz |} \leqq \frac{2(\varrho + \sin \varepsilon)}{\varrho \sin \varepsilon} \int_0^1 d\chi(t) = \frac{2(\varrho + \sin \varepsilon)}{\varrho \sin \varepsilon}.$$

THEOREM 2. Any function F is holomorphic for all values of z, with the possible exception of the values $z \leqq 0$.

PROOF. If we put

$$F_n(z) = \sum_{\nu=0}^{n-1} \frac{\chi\left(\frac{\nu+1}{n}\right) - \chi\left(\frac{\nu}{n}\right)}{1 + \frac{\nu}{n}(z-1)} \qquad (n = 1, 2, \ldots),$$

the sequence $\{F_n(z)\}_1^\infty$ converges uniformly, by the Porter-Vitali theorem, in any domain $G(\varepsilon, \varrho)$. For, this sequence converges to $F(z)$ for any z different from the values $z \leqq 0$, and the expressions $F_n(z)$ belong to the class $\{F\}$, hence they are uniformly bounded in $G(\varepsilon, \varrho)$.

Since $F_n^{(k)}(z) \to F^{(k)}(z)$ as $n \to \infty$ we also have

$$\text{(2.04)} \quad F^{(k)}(z) = (-)^k k! \int_0^1 \frac{t^k d\chi(t)}{(1-t+tz)^{k+1}} \qquad (k = 0, 1, 2, \ldots)$$

for every z different from the values $z \leqq 0$.

THEOREM 3. Any function $F(x)$ is completely monotonic for $x > 0$.

PROOF. By (2.04) we have $(-)^k\, F^{(k)}(x) > 0$ for any k and $x > 0$.

The converse of this theorem does not hold. The inequality

$$\frac{1}{1-t+2tx} \geqq \frac{1}{2(1-t+tx)} \qquad (x > 0,\ 0 \leqq t \leqq 1)$$

yields $F(2x) \geqq \frac{1}{2}F(x)$ for any function F. Now, when $f(x) = x^{-2}$, we have $f(2x) = \frac{1}{4}f(x)$. Hence f does not belong to $\{F\}$, though it is completely monotonic for $x > 0$.

THEOREM 4. In order that a function $f(z)$ be contained in the class $\{F\}$, it is necessary and sufficient that an expansion

$$\text{(2.05)} \qquad f(z) = \sum_{k=0}^{\infty} (-)^k c_k (z-1)^k,$$

where

$$\text{(2.06)} \qquad c_k = \int_0^1 t^k d\chi(t) \qquad (k = 0, 1, 2, \ldots),$$

be valid within the circle $|z-1| < 1$.

PROOF. First let f belong to $\{F\}$. Since $f(z)$ will be holomorphic within the circle $|z-1| < 1$, it can be expanded in a Taylor series (2.05), where

$$c_k = (-)^k \frac{f^{(k)}(1)}{k!}.$$

Now (2.06) will hold, by (2.04); hence the conditions are necessary. Next, let (2.05) and (2.06) hold. Substituting we have

$$f(z) = \Sigma\ (-)^k(z-1)^k \int_0^1 t^k d\chi(t) = \int_0^1 \frac{d\chi(t)}{1-t+tz},$$

hence the conditions are sufficient.

Theorem 5. The function $\chi(t)$ in (1.05) is uniquely determined by $F(z)$.

Proof. According to (2.05) the sequence $\{c_k\}_0^\infty$ is uniquely determined by $F(z)$. Now, by (2.06), $\{c_k\}_0^\infty$ is a sequence of moments of Hausdorff, and the corresponding moment problem is determined.

Hence there is a one-to-one correspondence between the functions F and χ. Two functions F and χ, connected by (1.05), will henceforth be called *corresponding*.

Theorem 6. Im $F(z) < 0$ for $y > 0$, unless $F(z) \equiv 1$.

Proof. Since

$$\text{Im } F(z) = -y\int_0^1 \frac{td\chi(t)}{|\,1-t+tz\,|^2},$$

the theorem holds whenever the integral in the right hand member differs from zero. Now this integral can only be equal to zero if $\chi(t)$ is a constant for $t > 0$, or, by (1.13), if $\chi(t) = 1$ for $t > 0$, i.e. if $F(z) \equiv 1$.

On the other hand, a function may be contained in the class I of Nevanlinna [1], i.e. be holomorphic and satisfy Im $f(z) \leqq 0$ in the upper half plane, without belonging to $\{F\}$. An example is furnished by $f(z) = z^{-1} - z$. Hence $\{F\}$ is a subclass in the strict sense of I, and this also points to a difference between the problems (e) and (f).

By Theorem 6, a function F that is not identically unity can take no real values in both half planes $y > 0$ and $y < 0$. Since $F(z)$ is positive if $z > 0$, we have as a special case:

Theorem 7. Any function $F(z)$ is different from zero outside the half line $z \leqq 0$.

Another proof of this theorem is as follows. It will be shown in § 7 that to any function F there is a function F^* of $\{F\}$ with the property $F(z)F^*(z^{-1}) = 1$. Now $F^*(z^{-1})$ is holomorphic for all values of z outside the half line $z^{-1} \leqq 0$, i.e. outside the half line $z \leqq 0$. Hence F can have no zeros for these values of z.

I now proceed to the inversion of (1.05). Any algorithm that solves the moment problem of Hausdorff will yield an inversion formula, which is clear by the proof of Theorem 5. However, $\chi(t)$ can only be expressed in this way by means of the values $F^{(k)}_{(1)}$ $(k = 0, 1, \ldots)$. Of course the formulae may be transformed afterwards into results of a more general type. The formulae given here are of a different kind. Theorems 8 and 9 are results of Stieltjes [1] and Hilbert [2]), extended to the class $\{F\}$. Though Theorem 9 may be considered as a limit case of Theorem 8, an independent proof of Theorem 9 will be given.

THEOREM 8. If $0 < r < \infty$, $\vartheta = (1 + r)^{-1}$, then

$$\int_{\vartheta}^{1-0} \frac{d\chi(t)}{t} + \tfrac{1}{2}\int_{1-0}^{1} \frac{d\chi(t)}{t} = -\frac{1}{\pi} \lim_{\varepsilon \to +0} \operatorname{Im} \int_{-r+i\varepsilon}^{i\varepsilon} F(z)dz$$

for any function F; the limit in the right hand member exists for any r.

PROOF. Using the proof of Theorem 2, it can immediately be shown that the inversion of the order of integrations:

$$\int_{-r+i\varepsilon}^{i\varepsilon} F(z)dz = \int_0^1 d\chi(t) \int_{-r+i\varepsilon}^{i\varepsilon} \frac{dz}{1 - t + tz}$$

is legitimate. Hence we may write

$$(2.07) \quad J(\varepsilon) = -\operatorname{Im} \int_{-r+i\varepsilon}^{i\varepsilon} F(z)dz = \varepsilon \int_0^1 t d\chi(t) \int_{-r}^{0} \frac{dx}{|1 - t + tx + it\varepsilon|^2}.$$

Putting

$$(2.08) \qquad u = \frac{1 - t + tx}{t\varepsilon}, \; \varphi(t, \varepsilon) = \int_{\frac{1-t\vartheta^{-1}}{t\varepsilon}}^{\frac{1-t}{t\varepsilon}} \frac{du}{1 + u^2},$$

we have, by (2.07),

$$(2.09) \quad J(\varepsilon) = \int_0^1 \frac{\varphi(t, \varepsilon)}{t} d\chi(t)$$

$$= \int_0^{(1-\sqrt{\varepsilon})\vartheta} + \int_{(1-\sqrt{\varepsilon})\vartheta}^{\vartheta-0} + \int_{\vartheta-0}^{\vartheta+0} + \int_{\vartheta+0}^{(1+\sqrt{\varepsilon})\vartheta} + \int_{(1+\sqrt{\varepsilon})\vartheta}^{1-\sqrt{\varepsilon}} + \int_{1-\sqrt{\varepsilon}}^{1-0} + \int_{1-0}^{1}.$$

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[1]) A. WINTNER [1], while speaking of the „Hilbertsche Residuenformel", probably refers to Hilbert [1].

Let the integrals in the right hand member of (2.09) be denoted by $J_1, \ldots, J$. If $t < \vartheta$, we have, by (2.08),

$$\varphi(t, \varepsilon) < \int\limits_{\frac{1-t\vartheta^{-1}}{t\varepsilon}}^{\infty} \frac{du}{1+u^2} < \int\limits_{\frac{1-t\vartheta^{-1}}{t\varepsilon}}^{\infty} \frac{du}{u^2} = \frac{t\varepsilon}{1-t\vartheta^{-1}},$$

and hence

$$(2.10)\quad J_1 < \varepsilon \int\limits_0^{(1-\sqrt{\varepsilon})\vartheta} \frac{d\chi(t)}{1-t\vartheta^{-1}} \leqq \sqrt{\varepsilon} \int\limits_0^{(1-\sqrt{\varepsilon})\vartheta} d\chi(t) = O(\sqrt{\varepsilon}).$$

We also have $\varphi(t, \varepsilon) < \frac{\pi}{2}$, hence

$$(2.11)\qquad J_2 < \frac{\pi}{2} \int\limits_{(1-\sqrt{\varepsilon})\vartheta}^{\vartheta-0} \frac{d\chi(t)}{t} = 0(1).$$

By (2.08) we have

$$J_3 = \frac{\chi(\vartheta+0)-\chi(\vartheta-0)}{\vartheta} \int\limits_0^{\frac{1-\vartheta}{\vartheta\varepsilon}} \frac{du}{1+u^2}$$

$$= \frac{\chi(\vartheta+0)-\chi(\vartheta-0)}{\vartheta} \left\{\frac{\pi}{2} - \int\limits_{\frac{1-\vartheta}{\vartheta\varepsilon}}^{\infty} \frac{du}{1+u^2}\right\} = \frac{\pi}{2} \frac{\chi(\vartheta+0)-\chi(\vartheta-0)}{\vartheta} + O(\varepsilon),$$

and this may be written, by (0.01),

$$(2.12)\quad J_3 = \pi \frac{\chi(\vartheta+0)-\chi(\vartheta)}{\vartheta} + O(\varepsilon) = \pi \int\limits_{\vartheta}^{\vartheta+0} \frac{d\chi(t)}{t} + O(\varepsilon).$$

If $t > \vartheta$, we have $\varphi(t, \varepsilon) < \pi$, hence

$$(2.13)\qquad J_4 < \pi \int\limits_{\vartheta+0}^{(1+\sqrt{\varepsilon})\vartheta} \frac{d\chi(t)}{t} = o(1),$$

$$(2.14)\qquad J_6 < \pi \int\limits_{1-\sqrt{\varepsilon}}^{1-0} \frac{d\chi(t)}{t} = o(1).$$

If in addition we have $(1+\sqrt{\varepsilon})\vartheta \leqq t \leqq 1-\sqrt{\varepsilon}$, then

$$\varphi(t, \varepsilon) = \pi - \int\limits_{\frac{1-t}{t\varepsilon}}^{\infty} \frac{du}{1+u^2} - \int\limits_{-\infty}^{\frac{1-t\vartheta^{-1}}{t\varepsilon}} \frac{du}{1+u^2}.$$

Now

$$\int\limits_{\frac{1-t}{t\varepsilon}}^{\infty} \frac{du}{1+u^2} < \int\limits_{\frac{1-t}{t\varepsilon}}^{\infty} \frac{du}{u^2} = \frac{t\varepsilon}{1-t} < \sqrt{\varepsilon},$$

$$\int_{-\infty}^{\frac{1-t\vartheta^{-1}}{t\varepsilon}} \frac{du}{1+u^2} = \int_{\frac{t\vartheta^{-1}-1}{t\varepsilon}}^{\infty} \frac{du}{1+u^2} < \int_{\frac{t\vartheta^{-1}-1}{t\varepsilon}}^{\infty} \frac{du}{u^2} = \frac{t\varepsilon}{t\vartheta^{-1}-1} \leqq \frac{(1+\sqrt{\varepsilon})\vartheta\varepsilon}{\sqrt{\varepsilon}} = O(\sqrt{\varepsilon}),$$

hence

$$\varphi(t, \varepsilon) = \pi + O(\sqrt{\varepsilon})$$

and

$$J_5 = \pi \int_{(1+\sqrt{\varepsilon})\vartheta}^{1-\sqrt{\varepsilon}} \frac{d\chi(t)}{t} + O(\sqrt{\varepsilon}). \tag{2.15}$$

Finally we have

$$J_7 = \{\chi(1) - \chi(1-0)\} \int_{\frac{1-\vartheta^{-1}}{\varepsilon}}^{0} \frac{du}{1+u^2} \tag{2.16}$$

$$= \frac{\pi}{2}\{\chi(1) - \chi(1-0)\} + O(\varepsilon) = \frac{\pi}{2}\int_{1-0}^{1} \frac{d\chi(t)}{t} + O(\varepsilon).$$

Now by (2.09) . . . (2.16) we obtain

$$J(\varepsilon) = \pi\left\{\int_{\vartheta}^{\vartheta+0} \frac{d\chi(t)}{t} + \int_{(1+\sqrt{\varepsilon})\vartheta}^{1-\sqrt{\varepsilon}} \frac{d\chi(t)}{t} + \frac{1}{2}\int_{1-0}^{1} \frac{d\chi(t)}{t}\right\} + o(1),$$

which proves the theorem.

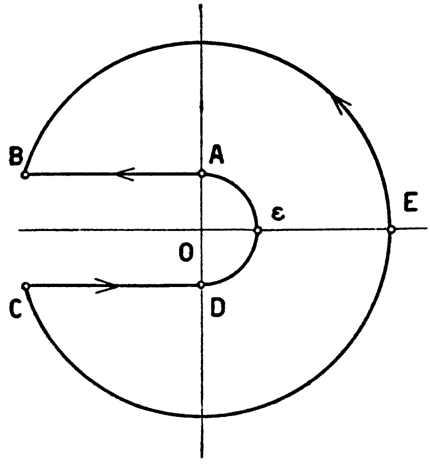

Fig. 2.

The proof of Theorem 8 was based on an estimation of the integral

$$\int F(z)dz$$

extended along the path AB + CD (fig. 2). If this path is replaced by CEB, a slightly simpler result can be obtained, viz.

$$(2.17)\qquad \int_{\vartheta}^{1} \frac{d\chi(t)}{t} = \frac{r}{2\pi} \lim_{\varepsilon \to +0} \int_{-\pi+\varepsilon}^{\pi-\varepsilon} F(re^{\varphi i}) e^{\varphi i} d\varphi.$$

The proof can best be given in a direct way.

THEOREM 9. Suppose (0.02) to hold for $0 < t < 1$ only, and define $\chi(1+0) = \chi(1)$. Now if $0 \leqq r < \infty$, and if we put $\vartheta = (1+r)^{-1}$, then

$$\frac{\chi(\vartheta+0) - \chi(\vartheta-0)}{\vartheta} = \lim_{\varepsilon \to +0} i\varepsilon F(-r+i\varepsilon),$$

The limit in the right hand member exists for any r.

PROOF. First take $r > 0$. Putting

$$(2.18)\qquad \psi(t, \varepsilon) = \frac{i\varepsilon}{1 - t(1+r) + it\varepsilon},$$

we have, by (1.05),

$$(2.19)\quad i\varepsilon F(-r+i\varepsilon) = \int_0^1 \psi(t,\varepsilon) d\chi(t)$$

$$= \int_0^{(1-\sqrt{\varepsilon})\vartheta} + \int_{(1-\sqrt{\varepsilon})\vartheta}^{(1-\varepsilon\sqrt{\varepsilon})\vartheta} + \int_{(1-\varepsilon\sqrt{\varepsilon})\vartheta}^{(1+\varepsilon\sqrt{\varepsilon})\vartheta} + \int_{(1+\varepsilon\sqrt{\varepsilon})\vartheta}^{(1+\sqrt{\varepsilon})\vartheta} + \int_{(1+\sqrt{\varepsilon})\vartheta}^{1}$$

Let the integrals in the right hand member of (2.19) be denoted by $J_1 \cdots J_5$. If

$$|1 - t(1+r)| \geqq \sqrt{\varepsilon},$$

we have, by (2.18),

$$|\psi(t,\varepsilon)|^{-1} = \left| \frac{1-t(1+r)}{i\varepsilon} + t \right| \geqq \frac{|1-t(1+r)|}{\varepsilon} \geqq \frac{1}{\sqrt{\varepsilon}},$$

hence $|\psi(t,\varepsilon)| \leqq \sqrt{\varepsilon}$ and

$$(2.20)\qquad \begin{cases} |J_1| \leqq \sqrt{\varepsilon} \displaystyle\int_0^{(1-\sqrt{\varepsilon})\vartheta} d\chi(t) = O(\sqrt{\varepsilon}), \\ |J_5| \leqq \sqrt{\varepsilon} \displaystyle\int_{(1+\sqrt{\varepsilon})\vartheta}^{1} d\chi(t) = O(\sqrt{\varepsilon}). \end{cases}$$

Next, if ε is constant, $|\psi(t,\varepsilon)|$ is a maximum in $0 \leqq t \leqq 1$ when

$$t = \frac{1+r}{(1+r)^2 + \varepsilon^2},$$

hence

$$|\psi(t,\varepsilon)|^2 \leqq (1+r)^2+\varepsilon^2,$$

$$(2.21)\quad \begin{cases} |J_2| \leqq \sqrt{(1+r)^2+\varepsilon^2}\displaystyle\int_{(1-\sqrt{\varepsilon})\vartheta}^{(1-\varepsilon\sqrt{\varepsilon})\vartheta} d\chi(t) = o(1),\\ |J_4| \leqq \sqrt{(1+r)^2+\varepsilon^2}\displaystyle\int_{(1+\varepsilon\sqrt{\varepsilon})\vartheta}^{(1+\sqrt{\varepsilon})\vartheta} d\chi(t) = o(1). \end{cases}$$

Finally, if

$$|1-t(1+r)| \leqq \varepsilon\sqrt{\varepsilon},$$

and we have

$$\left|\frac{1}{\psi(t,\varepsilon)}-\frac{1}{1+r}\right| = \left|\frac{1-t(1+r)}{i\varepsilon}+t-\frac{1}{1+r}\right| \leqq \frac{|1-t(1+r)|}{\varepsilon}+\frac{|t(1+r)-1|}{1+r} = O(\sqrt{\varepsilon}$$

and since $\psi(t,\varepsilon)=O(1)$ as $\varepsilon\to 0$ we now obtain

$$\psi(t,\varepsilon) = 1+r+O(\sqrt{\varepsilon}),$$

hence

$$J_3 = \int_{(1-\varepsilon\sqrt{\varepsilon})\vartheta}^{(1+\varepsilon\sqrt{\varepsilon})\vartheta} \{1+r+O(\sqrt{\varepsilon})\}\,d\chi(t) = (1+r)\int_{(1-\varepsilon\sqrt{\varepsilon})\vartheta}^{(1+\varepsilon\sqrt{\varepsilon})\vartheta} d\chi(t)+O(\sqrt{\varepsilon}),$$

or

$$(2.22)\qquad J_3 = \frac{\chi(\vartheta+0)-\chi(\vartheta-0)}{\vartheta}+o(1).$$

Now, by (2.19) . . . (2.22) we have

$$i\varepsilon F(-r+i\varepsilon) = \frac{\chi(\vartheta+0)-\chi(\vartheta-0)}{\vartheta}+o(1),$$

which proves the theorem if $r>0$. If $r=0$ the proof is similar.

Using (2.17), we can now discuss some elementary properties of the functions F on the half line $z \leqq 0$. If, in the folowing theorems, $F(z)$ is investigated within a domain D of the z-plane that also contains a set of values $z \leqq 0$, it will be understood that appropriate intervals of the half line $z \leqq 0$ have been excluded from D in order to make F one-valued.

Theorem 10. Let $0 \leqq r_1 < r_2 \leqq \infty$, $\vartheta_1 = (1+r_1)^{-1}$, $\vartheta_2 = (1+r_2)^{-1}$, and hence $0 \leqq \vartheta_2 < \vartheta_1 \leqq 1$. In order that $F(z)$ be holomorphic in the interval $(-r_2, -r_1)$ it is necessary and sufficient that $\chi(t)$ be constant in $(\vartheta_2, \vartheta_1)$.

Proof. First let F be holomorphic in $(-r_2, -r_1)$. Take $r_1 < r < r_2$ and put $\vartheta = (1+r)^{-1}$. By (2.17) we have

$$\int_{\vartheta}^{1} \frac{d\chi(t)}{t} = \frac{1}{2\pi i} \int_{|z|=r} F(z) dz, \tag{2.23}$$

where the integral in the right hand member is taken in the positive sense. This integral must be independent of r, hence $\chi(t)$ is constant if $\vartheta_2 < t < \vartheta_1$. The converse is an immediate consequence of the definition of the Stieltjes integral.

Theorem 11. Let $r_1 > 0$, $\vartheta_1 = (1+r_1)^{-1}$. In order that $F(z)$ be holomorphic within the circle $|z| < r_1$ it is necessary and sufficient that $\chi(t)$ be constant in $(\vartheta_1, 1\rangle$.

Proof. First let F be holomorphic when $|z| < r_1$. Take $0 < r < r_1$ and put $\vartheta = (1+r)^{-1}$. According to the former proof, (2.23) holds; hence

$$\int_{\vartheta}^{1} \frac{d\chi(t)}{t} = 0,$$

which proves the condition to be necessary. The converse is trivial.

Theorem 12. Let $r > 0$, $\vartheta = (1+r)^{-1}$. In order that $F(z)$ be holomorphic when $|z| > r$, it is necessary and sufficient that $\chi(t)$ be constant in $(0, \vartheta)$.

Proof. According to Theorem 10 the condition is necessary. Next, if $\chi(t)$ is constant in $(0, \vartheta)$, we can write, by (1.05),

$$F(z) = \chi(+0) + \int_{\vartheta-0}^{1} \frac{d\chi(t)}{1-t+tz}.$$

Since the integral in the right hand member is holomorphic for $|z| > r$, the condition is sufficient.

Theorem 13. Let $0 \leqq r < \infty$. Put $\vartheta = (1+r)^{-1}$ and $\chi(1+0) = \chi(1)$. As in Theorem 9, (0.02) is supposed to be valid only for $0 < t < 1$. In order that the value $z = -r$ be a pole of F, it is necessary and sufficient that $t = \vartheta$ be an isolated point of increment of χ.

Proof. First suppose $r > 0$. When $z = -r$ is a pole, the function F is holomorphic in the intervals $(-r-\varepsilon, -r)$ and $(-r, -r+\varepsilon)$, if ε is sufficiently small. According to Theorem 10, $\chi(t)$ can have no points of increment, say in an interval $(\vartheta - \varepsilon_1, \vartheta + \varepsilon_1)$, and different from $t = \vartheta$. We then have, by (1.05),

$$(2.24)\quad F(z) = \frac{\chi(\vartheta+0)-\chi(\vartheta-0)}{1+(z-1)\vartheta} + \int_0^{\vartheta-\varepsilon_1} \frac{d\chi(t)}{1-t+tz} + \int_{\vartheta+\varepsilon_1}^{1}$$

Since both integrals in the right hand member are holomorphic in the point $z = -r$, the remaining term must have a pole there. Hence χ increases when $t = \vartheta$, so the condition is necessary. The converse is trivial.

For $r = 0$ the proof is similar.

THEOREM 14. Any pole of $F(z)$ is of the first order with a positive residue.

PROOF. Let $z = -r$ be a pole of F; by Theorem 2 we have $r \geqq 0$. If we put $\vartheta = (1+r)^{-1}$, the value $t = \vartheta$ will be an isolated point of increment of $\chi(t)$, by Theorem 13. Hence (2.24) holds, which proves that $z = -r$ is a pole of the first order with a residue $\{\chi(\vartheta+0)-\chi(\vartheta-0)\}\vartheta^{-1}$, which is positive.

THEOREM 15. In order that $F(z)$ be meromorphic it is necessary and sufficient that the set of points of increment of χ be denumbrable and have a single cluster point $t = 0$. The formula $\vartheta_n = (1+r_n)^{-1}$, $r_n = -z_n$ determines a one-to-one correspondence between the poles z_n of F and the points of increment ϑ_n of χ.

PROOF. If F is meromorphic, F has an infinity of poles z_n $(n = 1, 2, \ldots)$ on the half line $z \leqq 0$, and the sequence $\{z_n\}_1^\infty$ has the value $z = \infty$ as a single cluster point. According to Theorem 13, a jump of $\chi(t)$ for the value $t = \vartheta_n = (1+r_n)^{-1}$ corresponds to the pole $z_n = -r_n$, and the value $t = 0$ is a single cluster point of the sequence $\{\vartheta_n\}_1^\infty$. According to Theorem 10, χ increases for no other values of t. Hence the condition is necessary. The converse is trivial.

THEOREM 16. Any function F that is meromorphic can be represented by the series

$$(2.25)\qquad F(z) = \chi(+0) + \sum_{(n)} \frac{\chi(\vartheta_n+0)-\chi(\vartheta_n-0)}{1+(z-1)\vartheta_n},$$

where the summation has to be extended over all poles $z_n = -r_n$ of F, and where $\vartheta_n = (1+r_n)^{-1}$, $\chi(1+0) = \chi(1)$.

PROOF. The theorem is an immediate consequence of Theorem 15, and of the notion of Stieltjes integral.

The series (2.25) converges absolutely and uniformly in any domain D of the z-plane, if we exclude terms that have a pole in D. Evidently the representation (2.25) also holds when F is rational. Hence any rational solution of problem (e) is degenerate.

§ 3. Existence of a Solution: necessary Conditions.

Let us first discuss the determinants

(3.01) $D\begin{pmatrix}\xi_1 \dots \xi_{2m}\\ t_1 \dots t_m\end{pmatrix}$

$$= \left| 1, \xi_i, \dots, \xi_i^{m-1}, \frac{1}{1-t_1+t_1\xi_i}, \frac{\xi_i}{1-t_2+t_2\xi_i}, \dots, \frac{\xi_i^{m-1}}{1-t_m+t_m\xi_i} \right|_{(i=1\dots 2m)},$$

(3.02) $D^*\begin{pmatrix}\xi_1 \dots \xi_{2m}\\ t_1 \dots t_m\end{pmatrix}$

$$= \left| 1, \xi_i, \dots, \xi_i^{m-1}, \frac{\xi_i}{1-t_1+t_1\xi_i}, \frac{\xi_i^2}{1-t_2+t_2\xi_i}, \dots, \frac{\xi_i^{m}}{1-t_m+t_m\xi_i} \right|_{(i=1\dots 2m)},$$

(3.03) $D\begin{pmatrix}\xi_1 \dots \xi_{2m+1}\\ t_1 \dots t_{m+1}\end{pmatrix}$

$$= \left| 1, \xi_i, \dots, \xi_i^{m-1}, \frac{1}{1-t_1+t_1\xi_i}, \frac{\xi_i}{1-t_2+t_2\xi_i}, \dots, \frac{\xi_i^{m}}{1-t_{m+1}+t_{m+1}\xi_i} \right|_{(i=1\dots 2m+1)}$$

(3.04) $D^*\begin{pmatrix}\xi_1 \dots \xi_{2m+1}\\ t_1 \dots t_m\end{pmatrix}$

$$= \left| 1, \xi_i, \dots, \xi_i^{m}, \frac{\xi_i}{1-t_1+t_1\xi_i}, \frac{\xi_i^2}{1-t_2+t_2\xi_i}, \dots, \frac{\xi_i^{m}}{1-t_m+t_m\xi_i} \right|_{(i=1\dots 2m+1)}.$$

The following formulae hold:

(3.05) $D\begin{pmatrix}\xi_1 \dots \xi_{2m}\\ t_1 \dots t_m\end{pmatrix}$

$$= d_{2m}(k) \left| 1, \xi_i, \dots, \xi_i^{m+k-1}, \frac{\xi_i^k}{1-t_{k+1}+t_{k+1}\xi_i}, \dots, \frac{\xi_i^{m-1}}{1-t_m+t_m\xi_i} \right|_{(i=1\dots 2m)},$$

(3.06) $D^*\begin{pmatrix}\xi_1 \dots \xi_{2m}\\ t_1 \dots t_m\end{pmatrix}$

$$= d_{2m}^*(k) \left| 1, \xi_i, \dots, \xi_i^{m+k-1}, \frac{\xi_i^{k+1}}{1-t_{k+1}+t_{k+1}\xi_i}, \dots, \frac{\xi_i^{m}}{1-t_m+t_m\xi_i} \right|_{(i=1\dots 2m)},$$

(3.07) $D\begin{pmatrix}\xi_1 \dots \xi_{2m+1}\\ t_1 \dots t_{m+1}\end{pmatrix}$

$$= d_{2m+1}(k) \left| 1, \xi_i, \dots, \xi_i^{m+k-1}, \frac{\xi_i^k}{1-t_{k+1}+t_{k+1}\xi_i}, \dots, \frac{\xi_i^{m}}{1-t_{m+1}+t_{m+1}\xi_i} \right|_{(i=1\dots 2m+1)},$$

(3.08) $D^*\begin{pmatrix}\xi_1 \dots \xi_{2m+1}\\ t_1 \dots t_m\end{pmatrix}$

$$= d_{2m+1}^*(k) \left| 1, \xi_i, \dots, \xi_i^{m+k}, \frac{\xi_i^{k+1}}{1-t_{k+1}+t_{k+1}\xi_i}, \dots, \frac{\xi_i^{m}}{1-t_m+t_m\xi_i} \right|_{(i=1\dots 2m+1)},$$

where $0 \leqq k \leqq m$, except in (3.07), where $0 \leqq k \leqq m+1$, and where the expressions $d_n(k)$, $d_n^*(k)$ are defined by

$$(3.09)\quad d_{2m}(k) = (-)^{km} \frac{\prod_{\lambda=1}^{k} t_\lambda^m (1-t_\lambda)^{\lambda-1}}{\prod_{\lambda=1}^{k} \prod_{i=1}^{2m} (1-t_\lambda+t_\lambda \xi_i)} \prod_{l=1}^{k} \prod_{\lambda=l+1}^{m} \frac{t_\lambda - t_l}{t_\lambda},$$

$$(3.10)\quad d_{2m}^*(k) = (-)^{k(m-1)} \frac{\prod_{\lambda=1}^{k} t_\lambda^{m-1} (1-t_\lambda)^{\lambda}}{\prod_{\lambda=1}^{k} \prod_{i=1}^{2m} (1-t_\lambda+t_\lambda \xi_i)} \prod_{l=1}^{k} \prod_{\lambda=l+1}^{m} \frac{t_\lambda - t_l}{t_\lambda},$$

$$(3.11)\quad d_{2m+1}(k) = (-)^{km} \frac{\prod_{\lambda=1}^{k} t_\lambda^m (1-t_\lambda)^{\lambda-1}}{\prod_{\lambda=1}^{k} \prod_{i=1}^{2m+1} (1-t_\lambda+t_\lambda \xi_i)} \prod_{l=1}^{k} \prod_{\lambda=l+1}^{m+1} \frac{t_\lambda - t_l}{t_\lambda},$$

$$(3.12)\quad d_{2m+1}^*(k) = (-)^{km} \frac{\prod_{\lambda=1}^{k} t_\lambda^m (1-t_\lambda)^{\lambda}}{\prod_{\lambda=1}^{k} \prod_{i=1}^{2m+1} (1-t_\lambda+t_\lambda \xi_i)} \prod_{l=1}^{k} \prod_{\lambda=l+1}^{m} \frac{t_\lambda - t_l}{t_\lambda}.$$

The proof will be given by induction; (3.05) is true when $k=0$; let (3.05) hold for an arbitrary value of k. By putting factors outside the determinants and by repeated substraction of columns we get

$$D\begin{pmatrix}\xi_1 \ldots \xi_{2m} \\ t_1 \ldots t_m\end{pmatrix} = \frac{d_{2m}(k)}{\prod_{i=1}^{2m} (1-t_{k+1}+t_{k+1}\xi_i)}$$

$$\Big| 1-t_{k+1}+t_{k+1}\xi_i,\ \xi_i(1-t_{k+1}+t_{k+1}\xi_i),\ \ldots,\ \xi_i^{k-1}(1-t_{k+1}+t_{k+1}\xi_i),$$

$$\xi_i^k(1-t_{k+1}+t_{k+1}\xi_i),\ \ldots,\ \xi_i^{m+k-1}(1-t_{k+1}+t_{k+1}\xi_i),\ \xi_i^k,$$

$$\xi_i^{k+1}\frac{1-t_{k+1}+t_{k+1}\xi_i}{1-t_{k+2}+t_{k+2}\xi_i},\ \ldots,\ \xi_i^{m-1}\frac{1-t_{k+1}+t_{k+1}\xi_i}{1-t_m+t_m\xi_i}\Big|_{(i=1\ldots 2m)}$$

$$= \frac{d_{2m}(k)t_{k+1}^m(1-t_{k+1})^k}{\prod_{i=1}^{2m} 1-t_{k+1}+t_{k+1}\xi_i)} \Big| 1,\ \xi_i,\ \ldots,\ \xi_i^{k-1},\ \xi_i^{k+1},\ \ldots,\ \xi_i^{m+k},\ \xi_i^k,$$

$$\xi_i^{k+1}\left(\frac{t_{k+1}}{t_{k+2}} + \frac{t_{k+2}-t_{k+1}}{t_{k+2}} \cdot \frac{1}{1-t_{k+2}+t_{k+2}\xi_i}\right),\ \ldots,$$

$$\xi_i^{m-1}\left(\frac{t_{k+1}}{t_m} + \frac{t_m-t_{k+1}}{t_m} \cdot \frac{1}{1-t_m+t_m\xi_i}\right)\Big|_{(i=1\ldots 2m)},$$

$$= \frac{(-)^m d_{2m}(k) t_{k+1}^m (1-t_{k+1})^k}{\prod_{i=1}^{2m}(1-t_{k+1}+t_{k+1}\xi_i)} \left| 1, \xi_i, \ldots, \xi_i^{m+k}, \frac{t_{k+2}-t_{k+1}}{t_{k+2}} \cdot \frac{\xi_i^{k+1}}{1-t_{k+2}+t_{k+2}\xi_i}, \right.$$

$$\left. \ldots, \frac{t_m - t_{k+1}}{t_m} \cdot \frac{\xi_i^{m-1}}{1-t_m+t_m\xi_i} \right|_{(i=1\ldots 2m)}$$

$$= \frac{(-)^m d_{2m}(k) t_{k+1}^m (1-t_{k+1})^k}{\prod_{i=1}^{2m}(1-t_{k+1}+t_{k+1}\xi_i)} \prod_{\lambda=k+2}^{m} \frac{t_\lambda - t_{k+1}}{t_\lambda} \cdot$$

$$\left| 1, \xi_i, \ldots, \xi_i^{m+k}, \frac{\xi_i^{k+1}}{1-t_{k+2}+t_{k+2}\xi_i}, \ldots, \frac{\xi_i^{m-1}}{1-t_m+t_m\xi_i} \right|_{(i=1\ldots 2m)}$$

$$= d_{2m}(k+1) \left| 1, \xi_i, \ldots, \xi_i^{m+k}, \frac{\xi_i^{k+1}}{1-t_{k+2}+t_{k+2}\xi_i}, \ldots, \frac{\xi_i^{m-1}}{1-t_m+t_m\xi_i} \right|_{(i=1\ldots 2m)},$$

hence (3.05) is true for any k. The formulae (3.06) ... (3.08) can be proved in a similar way. Giving k its maximum value m or $m+1$ we get

(3.13) $$D\begin{pmatrix}\xi_1 \ldots \xi_{2m} \\ t_1 \ldots t_m\end{pmatrix}$$

$$= (-)^m \frac{\left| 1, \xi_i, \ldots, \xi_i^{2m-1} \right|_{(i=1\ldots 2m)}}{\prod_{\lambda=1}^{m} \prod_{i=1}^{2m} (1-t_\lambda+t_\lambda\xi_i)} \prod_{\lambda=1}^{m} t_\lambda^{m-\lambda+1}(1-t_\lambda)^{\lambda-1} \prod_{1\leq\mu<\lambda\leq m} (t_\lambda - t_\mu),$$

(3.14) $$D^*\begin{pmatrix}\xi_i \ldots \xi_{2m} \\ t_1 \ldots t_m\end{pmatrix}$$

$$= \frac{\left| 1, \xi_i, \ldots \xi_i^{2m-1} \right|_{(i=1\ldots 2m)}}{\prod_{\lambda=1}^{m} \prod_{i=1}^{2m} (1-t_\lambda+t_\lambda\xi_i)} \prod_{\lambda=1}^{m} t_\lambda^{m-\lambda}(1-t_\lambda)^{\lambda} \prod_{1\leq\mu<\lambda\leq m} (t_\lambda - t_\mu),$$

(3.15) $$D\begin{pmatrix}\xi_1 \ldots \xi_{2m+1} \\ t_1 \ldots t_{m+1}\end{pmatrix}$$

$$= \frac{\left| 1, \xi_i, \ldots \xi_i^{2m} \right|_{(i=1\ldots 2m+1)}}{\prod_{\lambda=1}^{m+1} \prod_{i=1}^{2m+1} (1-t_\lambda+t_\lambda\xi_i)} \prod_{\lambda=1}^{m+1} t_\lambda^{m-\lambda+1}(1-t_\lambda)^{\lambda-1} \prod_{1\leq\mu<\lambda\leq m+1} (t_\lambda - t_\mu),$$

(3.16) $$D^*\begin{pmatrix}\xi_1 \ldots \xi_{2m+1} \\ t_1 \ldots t_m\end{pmatrix}$$

$$= (-)^m \frac{\left| 1, \xi_i, \ldots \xi_i^{2m} \right|_{(i=1\ldots 2m+1)}}{\prod_{\lambda=1}^{m} \prod_{i=1}^{2m+1} (1-t_\lambda+t_\lambda\xi_i)} \prod_{\lambda=1}^{m} t_\lambda^{m-\lambda+1}(1-t_\lambda)^{\lambda} \prod_{1\leq\mu<\lambda\leq m} (t_\lambda - t_\mu).$$

Let $(k_1, \dots k_\varrho)$ be a permutation of the set $(1, 2, \dots \varrho)$, and let sgn $(k_1 \dots k_\varrho)$ be equal to 1 or -1, whenever the permutation $(k_1, \dots k_\varrho)$ is even or odd. The product

$$\prod_{1 \leq \mu < \lambda \leq \varrho} (t_\lambda - t_\mu)$$

is transformed by the permutation $(t_1, \dots t_\varrho) \to (t_{k_1}, \dots t_{k_\varrho})$ into

$$\prod_{1 \leq \mu < \lambda \leq \varrho} (t_{k_\lambda} - t_{k_\mu}) = \operatorname{sgn}(k_1 \dots k_\varrho) \prod_{1 \leq \mu < \lambda \leq \varrho} (t_\lambda - t_\mu).$$

Hence we have, by (3.13) ... (3.16),

$$(3.17) \quad D\begin{pmatrix} \xi_1 \dots \xi_{2m} \\ t_{k_1} \dots t_{k_m} \end{pmatrix}$$

$$= (-)^m \operatorname{sgn}(k_1 \dots k_m) \frac{\prod\limits_{1 \leq i < j \leq 2m} (\xi_j - \xi_i)}{\prod\limits_{\lambda=1}^{m} \prod\limits_{i=1}^{2m} (1 - t_\lambda + t_\lambda \xi_i)} \prod_{\lambda=1}^{m} t_{k_\lambda}^{m-\lambda+1} (1 - t_{k_\lambda})^{\lambda-1} \prod_{1 \leq \mu < \lambda \leq m} (t_\lambda - t_\mu).$$

$$(3.18) \quad D^*\begin{pmatrix} \xi_1 \dots \xi_{2m} \\ t_{k_1} \dots t_{k_m} \end{pmatrix}$$

$$= \operatorname{sgn}(k_1 \dots k_m) \frac{\prod\limits_{1 \leq i < j \leq 2m} (\xi_j - \xi_i)}{\prod\limits_{\lambda=1}^{m} \prod\limits_{i=1}^{2m} (1 - t_\lambda + t_\lambda \xi_i)} \prod_{\lambda=1}^{m} t_{k_\lambda}^{m-\lambda} (1 - t_{k_\lambda})^{\lambda} \prod_{1 \leq \mu < \lambda \leq m} (t_\lambda - t_\mu).$$

$$(3.19) \quad D\begin{pmatrix} \xi_1 \dots \xi_{2m+1} \\ t_{k_1} \dots t_{k_{m+1}} \end{pmatrix}$$

$$= \operatorname{sgn}(k_1 \dots k_{m+1}) \frac{\prod\limits_{1 \leq i < j \leq 2m+1} (\xi_j - \xi_i)}{\prod\limits_{\lambda=1}^{m+1} \prod\limits_{i=1}^{2m+1} (1 - t_\lambda + t_\lambda \xi_i)} \prod_{\lambda=1}^{m+1} t_{k_\lambda}^{m-\lambda+1} (1 - t_{k_\lambda})^{\lambda-1} \prod_{1 \leq \mu < \lambda \leq m+1} (t_\lambda - t_\mu),$$

$$(3.20) \quad D^*\begin{pmatrix} \xi_1 \dots \xi_{2m+1} \\ t_{k_1} \dots t_{k_m} \end{pmatrix}$$

$$= (-)^m \operatorname{sgn}(k_1 \dots k_m) \frac{\prod\limits_{1 \leq i < j \leq 2m+1} (\xi_j - \xi_i)}{\prod\limits_{\lambda=1}^{m} \prod\limits_{i=1}^{2m+1} (1 - t_\lambda + t_\lambda \xi_i)} \prod_{\lambda=1}^{m} t_{k_\lambda}^{m-\lambda+1} (1 - t_{k_\lambda})^{\lambda} \prod_{1 \leq \mu < \lambda \leq m} (t_\lambda - t_\mu).$$

Let $\sum\limits_{(\varrho)}$ be a summation extended over all permutations $(k_1, \dots k_\varrho)$ of the numbers $(1, 2, \dots \varrho)$. According to the definition of a determinant we have

$$\sum_{(\varrho)} \operatorname{sgn}(k_1 \ldots k_\varrho) \prod_{\lambda=1}^{\varrho} t_{k_\lambda}^{\varrho-\lambda+\zeta} (1-t_{k_\lambda})^{\lambda-1+\eta}$$
$$= \left| t_\lambda^{\varrho-1+\zeta} (1-t_\lambda)^\eta, t_\lambda^{\varrho-2+\eta} (1-t_\lambda)^{1+\eta}, \ldots, t_\lambda^\zeta (1-t_\lambda)^{\varrho-1+\eta} \right|_{(\lambda=1\ldots\varrho)}$$
$$= \left| t_\lambda^{\varrho-1}, t_\lambda^{\varrho-2}, \ldots 1 \right|_{(\lambda=1\ldots\varrho)} \prod_{\lambda=1}^{\varrho} t_\lambda^\zeta (1-t_\lambda)^\eta$$
$$= (-)^{\frac{1}{2}\varrho(\varrho-1)} \left| 1, t_\lambda, \ldots t_\lambda^{\varrho-1} \right|_{(\lambda=1\ldots\varrho)} \prod_{\lambda=1}^{\varrho} t_\lambda^\zeta (1-t_\lambda)^\eta$$
$$= (-)^{\frac{1}{2}(\varrho\varrho-1)} \prod_{\lambda=1}^{\varrho} t_\lambda^\zeta (1-t_\lambda)^\eta \prod_{1\leqq\mu<\lambda\leqq\varrho} (t_\lambda - t_\mu).$$

Hence, putting (ϱ, ζ, η) equal to $(m, 1, 0)$, $(m, 0, 1)$, $(m+1, 0, 0)$, $(m, 1, 1)$ successively, we obtain

$$\sum_{(m)} \operatorname{sgn}(k_1 \ldots k_m) \prod_{\lambda=1}^{m} t_{k_\lambda}^{m-\lambda+1} (1-t_{k_\lambda})^{\lambda-1} = (-)^{\frac{1}{2}m(m-1)} \prod_{\lambda=1}^{m} t_\lambda \prod_{1\leqq\mu<\lambda\leqq m} (t_\lambda - t_\mu),$$
$$\sum_{(m)} \operatorname{sgn}(k_1 \ldots k_m) \prod_{\lambda=1}^{m} t_{k_\lambda}^{m-\lambda} (1-t_{k_\lambda})^{\lambda} = (-)^{\frac{1}{2}m(m-1)} \prod_{\lambda=1}^{m} (1-t_\lambda) \prod_{1\leqq\mu<\lambda\leqq m} (t_\lambda - t_\mu),$$
$$\sum_{(m+1)} \operatorname{sgn}(k_1 \ldots k_{m+1}) \prod_{\lambda=1}^{m+1} t_{k_\lambda}^{m-\lambda+1} (1-t_{k_\lambda})^{\lambda-1} = (-)^{\frac{1}{2}m(m+1)} \prod_{1\leqq\mu<\lambda\leqq m+1} (t_\lambda - t_\mu),$$
$$\sum_{(m)} \operatorname{sgn}(k_1 \ldots k_m) \prod_{\lambda=1}^{m} t_{k_\lambda}^{m-\lambda+1} (1-t_{k_\lambda})^{\lambda} = (-)^{\frac{1}{2}m(m-1)} \prod_{\lambda=1}^{m} t_\lambda(1-t_\lambda) \prod_{1\leqq\mu<\lambda\leqq m} (t_\lambda - t_\lambda),$$

and, by (3.17) ... (3.20), we finally have the identities

$$(3.21) \quad \sum_{(m)} D\begin{pmatrix} \xi_1 \ldots \xi_{2m} \\ t_{k_1} \ldots t_{k_m} \end{pmatrix}$$
$$= (-)^{\frac{1}{2}m(m+1)} \frac{\prod\limits_{1\leqq i<j\leqq 2m} (\xi_j - \xi_i)}{\prod\limits_{\lambda=1}^{m} \prod\limits_{i=1}^{2m} (1 - t_\lambda + t_\lambda \xi_i)} \prod_{\lambda=1}^{m} t_\lambda \prod_{1\leqq\mu<\lambda\leqq m} (t_\lambda - t_\mu)^2,$$

$$(3.22) \quad \sum D^*\begin{pmatrix} \xi_1 \ldots \xi_{2m} \\ t_{k_1} \ldots t_{k_m} \end{pmatrix}$$
$$= (-)^{\frac{1}{2}m(m-1)} \frac{\prod\limits_{1\leqq i<j\leqq 2m} (\xi_j - \xi_i)}{\prod\limits_{\lambda=1}^{m} \prod\limits_{i=1}^{2m} (1 - t_\lambda + t_\lambda \xi_i)} \prod_{\lambda=1}^{m} (1-t_\lambda) \prod_{1\leqq\mu<\lambda\leqq m} (t_\lambda - t_\mu)^2,$$

$$(3.23) \quad \sum_{(m+1)} D\begin{pmatrix} \xi_1 \ldots \xi_{2m+1} \\ t_{k_1} \ldots t_{k_{m+1}} \end{pmatrix}$$
$$= (-)^{\frac{1}{2}m(m+1)} \frac{\prod\limits_{1\leqq i<j\leqq 2m+1} (\xi_j - \xi_i)}{\prod\limits_{\lambda=1}^{m+1} \prod\limits_{i=1}^{2m+1} (1 - t_\lambda + t_\lambda \xi_i)} \prod_{1\leqq\mu<\lambda\leqq m+1} (t_\lambda - t_\mu)^2,$$

$$(3.24)\quad \sum_{(m)} D^*\begin{pmatrix}\xi_1 \dots \xi_{2m+1}\\ t_{k_1} \dots t_{k_m}\end{pmatrix} = (-)^{\frac{1}{2}m(m+1)} \frac{\prod\limits_{1\leq i<j\leq 2m+1}(\xi_j-\xi_i)}{\prod\limits_{\lambda=1}^{m}\prod\limits_{i=1}^{2m+1}(1-t_\lambda+t_\lambda\xi_i)} \prod_{\lambda=1}^{m} t_\lambda(1-t_\lambda) \prod_{1\leq\mu<\lambda\leq m}(t_\lambda-t_\mu)^2.$$

From these identities it is easy to obtain a set of necessary conditions for the existence of a solution of problem (e).

THEOREM 17. Let $N = N(\chi)$ denote the number of values, for which the corresponding function χ of F increases; hence $N < \infty$ if and only if $F(z)$ is rational. If $N < \infty$, the set $\{\tau_i\}_{i=1}^{N}$ of values $t = \tau_i$, where $\chi(t)$ increases, is supposed to be decreasing:

$$(3.25)\qquad 1 \geqq \tau_1 > \tau_2 > \dots > \tau_N \geqq 0.$$

Let $\xi_i > 0$ $(i = 1, 2, \dots)$, $\xi_i \neq \xi_j$ if $i \neq j$; put

$$(3.26)\qquad F(\xi_i) = \alpha_i,$$

$$(3.27)\quad \Delta(\xi_1 \dots \xi_{2m}) = (-)^{\frac{1}{2}m(m+1)} \left| 1, \xi_i, \dots, \xi_i^{m-1}, \alpha_i, \alpha_i\xi_i, \dots, \alpha_i\xi_i^{m-1} \right|_{(i=1\dots 2m)},$$

$$(3.28)\quad \Delta^*(\xi_1 \dots \xi_{2m}) = (-)^{\frac{1}{2}m(m-1)} \left| 1, \xi_i, \dots, \xi_i^{m-1}, \alpha_i\xi_i, \alpha_i\xi_i^2, \dots, \alpha_i\xi_i^{m} \right|_{(i=1\dots 2m)},$$

$$(3.29)\quad \Delta(\xi_1 \dots \xi_{2m+1}) = (-)^{\frac{1}{2}m(m+1)} \left| 1, \xi_i, \dots, \xi_i^{m-1}, \alpha_i, \alpha_i\xi_i, \dots, \alpha_i\xi_i^{m} \right|_{(i=1\dots 2m+1)};$$

$$(3.30)\quad \Delta^*(\xi_1 \dots \xi_{2m+1}) = (-)^{\frac{1}{2}m(m+1)} \left| 1, \xi_i, \dots, \xi_i^{m}, \alpha_i\xi_i, \alpha_i\xi_i^2, \dots, \alpha_i\xi_i^{m} \right|_{(i=1\dots 2m+1)}.$$

If no ambiguity is to be feared, we shall write

$$(3.31)\quad \Delta_n = \Delta(\xi_1 \dots \xi_n),\ \Delta_n^* = \Delta^*(\xi_1 \dots \xi_n).$$

We then have the equalities

$$(3.32)\quad \Delta_{2m} = \frac{\prod\limits_{1\leq i<j\leq 2m}(\xi_j-\xi_i)}{m!}\int_0^1 \dots \int_0^1 \frac{\prod\limits_{\lambda=1}^{m} t_\lambda \prod\limits_{1\leq\mu<\lambda\leq m}(t_\lambda-t_\mu)^2}{\prod\limits_{\lambda=1}^{m}\prod\limits_{i=1}^{2m}(1-t_\lambda+t_\lambda\xi_i)}\, d\chi(t_1)\dots d\chi(t_m),$$

$$(3.33)\quad \Delta_{2m}^* = \frac{\prod\limits_{1\leq i<j\leq 2m}(\xi_j-\xi_i)}{m!}\int_0^1 \dots \int_0^1 \frac{\prod\limits_{\lambda=1}^{m}(1-t_\lambda) \prod\limits_{1\leq\mu<\lambda\leq m}(t_\lambda-t_\mu)^2}{\prod\limits_{\lambda=1}^{m}\prod\limits_{i=1}^{2m}(1-t_\lambda+t_\lambda\xi_i)}\, d\chi(t_1)\dots d\chi(t_m),$$

$$(3.34)\quad \Delta_{2m-1} = \frac{\prod\limits_{1\leq i<j\leq 2m-1}(\xi_j-\xi_i)}{m!}\int_0^1 \dots \int_0^1 \frac{\prod\limits_{1\leq\mu<\lambda\leq m}(t_\lambda-t_\mu)^2}{\prod\limits_{\lambda=1}^{m}\prod\limits_{i=1}^{2m-1}(1-t_\lambda+t_\lambda\xi_i)}\, d\chi(t_1)\dots d\chi(t_m),$$

$$(3.35)\quad \Delta_{2m+1}^* = \frac{\prod\limits_{1\leq i<j\leq 2m+1}(\xi_j-\xi_i)}{m!}\int_0^1 \dots \int_0^1 \frac{\prod\limits_{=1}^{m} t_\lambda(1-t_\lambda) \prod\limits_{1\leq\mu<\lambda\leq m}(t_\lambda-t_\mu)^2}{\prod\limits_{\lambda=1}^{m}\prod\limits_{i=1}^{2m+1}(1-t_\lambda+t_\lambda\xi_i)}\, d\chi(t_1)\dots d\chi(t_m),$$

and in particular

(3.36) $\Delta_{2m} = \Delta^*_{2m} = \Delta_{2m-1} = \Delta^*_{2m+1} = 0$ if $N(\chi) < m$,

(3.37) $\Delta_{2m} = \Delta^*_{2m+1} = 0$ if $N(\chi) = m,\ \tau_m = 0$,

(3.38) $\Delta^*_{2m} = \Delta^*_{2m+1} = 0$ if $N(\chi) = m,\ \tau_1 = 1$,

(3.39) $\Delta^*_{2m+1} = 0$ if $N(\chi) = m+1,\ \tau_{m+1} = 0,\ \tau_1 = 1$

Moreover, if

(3.40) $$\xi_i < \xi_{i+1} \qquad (i = 1, 2, \ldots),$$

we have the inequalities

(3.41) $$\Delta_n > 0, \quad \Delta^*_n > 0 \qquad (n = 1, 2, \ldots),$$

in all cases different from (3.36) . . . (3.39).

PROOF. By (1.05), (3.26), (3.27) and (3.31) we have

$$\Delta_{2m} = (-)^{\frac{1}{2}m(m+1)} \left| 1, \xi_i, \ldots, \xi_i^{m-1}, \int_0^1 \frac{d\chi(t_{k_1})}{1-t_{k_1}+t_{k_1}\xi_i}, \right.$$
$$\left. \xi_i \int_0^1 \frac{d\chi(t_{k_2})}{1-t_{k_2}+t_{k_2}\xi_i}, \ldots, \xi_i^{m-1} \int_0^1 \frac{d\chi(t_{k_m})}{1-t_{k_m}+t_{k_m}\xi_i} \right|_{(i=1\ldots 2m)},$$

where $k_1, \ldots, k_m$ can be any numbers. Hence

$$\Delta_{2m} = \frac{(-)^{\frac{1}{2}m(m+1)}}{m!} \sum_{(m)} \left| 1, \xi_i, \ldots, \xi_i^{m-1}, \int_0^1 \frac{d\chi(t_{k_1})}{1-t_{k_1}+t_{k_1}\xi_i}, \right.$$
$$\left. \xi_i \int_0^1 \frac{d\chi(t_{k_2})}{1-t_{k_2}+t_{k_2}\xi_i}, \ldots, \xi_i^{m-1} \int_0^1 \frac{d\chi(t_{k_m})}{1-t_{k_m}+t_{k_m}\xi_i} \right|_{(i=1\ldots 2m)},$$

where the summation is extended over all permutations $(k_1 \ldots k_m)$ of the numbers $(1 \ldots m)$. The right hand member can be written as an m-fold Stieltjes integral:

$$\Delta_{2m} = \frac{(-)^{\frac{1}{2}m(m+1)}}{m!} \int_0^1 \cdots \int_0^1 d\chi(t_1) \ldots d\chi(t_m) \sum_{(m)} \left| 1, \xi_i, \ldots, \xi_i^{m-1}, \right.$$
$$\left. \frac{1}{1-t_{k_1}+t_{k_1}\xi_i}, \frac{\xi_i}{1-t_{k_2}+t_{k_2}\xi_i}, \ldots, \frac{\xi_i^{m-1}}{1-t_{k_m}+t_{k_m}\xi_i} \right|_{(i=1\ldots 2m)},$$

hence, by (3.01) and (3.21) we obtain (3.32). The proof of (3.33)... (3.35) is similar.

The integrals in the right hand members of (3.32) . . . (3.35) are non-negative. They can only be equal to zero (as they actually are), if the corresponding integrands vanish for all combinations $(t_1 \ldots t_m) = (\tau_{i_1} \ldots \tau_{i_m})$, where $\tau_{i_1} \ldots \tau_{i_m}$ are any values, different from one another or not, for which $\chi(t)$ increases. This is always true, whether F is rational or not. If $N(\chi) < m$, every combination of m values τ must contain at least two of them that are equal, hence, owing to the factor $\Pi(t_\lambda - t_\mu)^2$, these integrands vanish for all combinations $(\tau_{i_1} \ldots \tau_{i_m})$, which yields (3.36). If $N(\chi) \geqq m$, there are always possible combinations where the values τ are different from one another. In this case, it is only owing to a factor Πt_λ or $\Pi(1 - t_\lambda)$ that an integrand can vanish for such a combination $(\tau_{i_1} \ldots \tau_{i_m})$, and this requires that one value τ be equal to zero or unity. Evidently this leads to the cases (3.37) ... (3.39). In all other cases (3.40) implies (3.41).

I now return to the conditions (1.01) and $F(x_n) = a_n$ $(n = 1, 2, \ldots)$ of problem (e). In the rest of this section the notations of Theorem 17 will be used throughout. Let $F(x)$ be a solution of (e), and let us put

(3.42) $$D_n(x|F) = \Delta(x, x_1 \ldots x_n),\quad D_n^*(x|F) = \Delta^*(x, x_1 \ldots x_n)\ (n = 1, 2, \ldots).$$

By (3.27) ... (3.30) we can write

(3.43) $$D_n(x|F) = -P_n(x) + Q_n(x)F(x),\quad D_n^*(x|F) = P_n^*(x) - xQ_n^*(x)F(x),$$

where the expressions $P_n, \ldots, Q_n^*$ denote the following polynomials:

(3.44) $$P_{2m}(x) = (-)^{\frac{1}{2}m(m+1)+1} \left| \begin{matrix} 1, x, \ldots, x^{m-1}, 0, 0, \ldots, 0 \\ 1, x_i, \ldots, x_i^{m-1}, a_i, a_i x_i, \ldots, a_i x_i^m \end{matrix} \right|_{(i=1\ldots 2m)},$$

(3.45) $$Q_{2m}(x) = (-)^{\frac{1}{2}m(m+1)} \left| \begin{matrix} 0, 0, \ldots, 0, 1, x, \ldots, x^m \\ 1, x_i, \ldots, x_i^{m-1}, a_i, a_i x_i, \ldots, a_i x_i^m \end{matrix} \right|_{(i=1\ldots 2m)},$$

(3.46) $$P_{2m}^*(x) = (-)^{\frac{1}{2}m(m+1)} \left| \begin{matrix} 1, x, \ldots, x^m, 0, 0, \ldots, 0, \\ 1, x_i, \ldots, x_i^m, a_i x_i, a_i x_i^2, \ldots, a_i x_i^m \end{matrix} \right|_{(i=1\ldots 2m)},$$

(3.47) $$Q_{2m}^*(x) = (-)^{\frac{1}{2}m(m+1)+1} \left| \begin{matrix} 0, 0, \ldots, 0, 1, x, \ldots, x^{m-1} \\ 1, x_i, \ldots, x_i^m, a_i x_i, a_i x_i^2, \ldots, a_i x_i^m \end{matrix} \right|_{(i=1\ldots 2m)},$$

(3.48) $$P_{2m+1}(x) = (-)^{\frac{1}{2}m(m-1)} \left| \begin{matrix} 1, x, \ldots, x^m, 0, 0, \ldots, 0, \\ 1, x_i, \ldots, x_i^m, a_i, a_i x_i, \ldots, a_i x_i^m \end{matrix} \right|_{(i=1\ldots 2m+1)},$$

(3.49) $$Q_{2m+1}(x) = (-)^{\frac{1}{2}m(m-1)+1} \left| \begin{matrix} 0, 0, \ldots, 0, 1, x, \ldots, x^m \\ 1, x_i, \ldots, x_i^m, a_i, a_i x_i, \ldots, a_i x_i^m \end{matrix} \right|_{(i=1\ldots 2m+1)},$$

(3.50) $$P_{2m+1}^*(x) = (-)^{\frac{1}{2}m(m+1)} \left| \begin{matrix} 1, x, \ldots, x^m, 0, 0, \ldots, 0, \\ 1, x_i, \ldots, x_i^m, a_i x_i, a_i x_i^2, \ldots, a_i x_i^{m+1} \end{matrix} \right|_{(i=1\ldots 2m+1)},$$

$$(3.51)\quad Q^*_{2m+1}(x)=(-)^{\frac{1}{2}m(m+1)+1}\begin{vmatrix} 0, 0, \ldots, 0, & 1, & x, \ldots, x^m \\ 1, x_i, \ldots, x_i^m, & a_i x_i, & a_i x_i^2, \ldots, a_i x_i^{m+1}\end{vmatrix}_{(i=1\ldots 2m+1)}$$

THEOREM 18. If $F(x)$ is a non-degenerate solution of problem (e), we have, for any $n > 0$,

$$(3.52)\qquad (-)^k D_n(x/F) > 0,\ (-)^k D^*_n(x\,|\,F) > 0,$$

where either $x < x_1$, $k = 0$, or $x_k < x < x_{k+1}$ ($k = 1, \ldots, n-1$), or $x_n < x$, $k = n$. In particular we have, for any $n > 1$,

$$(3.53)\quad (-)^n\{P_{n-1}(x_n)-a_nQ_{n-1}(x_n)\}>0,\ (-)^{n+1}\{P^*_{n-1}(x_n)-a_nx_nQ^*_{n-1}(x_n)\}>0.$$

If $F(x)$ is a rational solution of problem (e), the expressions D_n and D^*_n satisfy (3.52), the following cases excepted:

$$(3.54)\begin{cases}(\alpha)\ \tau_N>0,\ \tau_1<1;\ \text{then } D_n\equiv 0 \text{ for } n\geqq 2N \text{ and } D^*_n\equiv 0 \text{ for } n\geqq 2N+1;\\ (\beta)\ \tau_N=0,\ \tau_1<1;\ \text{then } D_n\equiv 0 \text{ for } n\geqq 2N-1 \text{ and } D^*_n\equiv 0 \text{ for } n\geqq 2N;\\ (\gamma)\ \tau_N>0,\ \tau_1=1;\ \text{then } D_n\equiv 0 \text{ for } n\geqq 2N \text{ and } D^*_n\equiv 0 \text{ for } n\geqq 2N-1;\\ (\delta)\ \tau_N=0,\ \tau_1=1;\ \text{then } D_n\equiv 0 \text{ for } n\geqq 2N-1 \text{ and } D^*_n\equiv 0 \text{ for } n\geqq 2N-2.\end{cases}$$

A rational solution $F(x)$ is unique; hence a degenerate problem (e) is always determined.

PROOF. If $F(x)$ is not rational we have $N(\chi) = \infty$, hence the inequality (3.41) can be applied to the expressions $\Delta(x, x_1, \ldots, x_n)$ and $\Delta^*(x, x_1, \ldots, x_n)$ as soon as the values $x, x_1, \ldots, x_n$ are so re-arranged as to form an increasing sequence, which can be effected by a permutation of the rows of the determinants Δ and Δ^*. In this way (3.52), and (3.53) as a special case, can be obtained.

The inequalities (3.41) can also be applied when F is rational, and hence (3.52) generally holds, except if we have to do with one of the cases (3.36) . . . (3.39), which yields (3.54).

Now, by (1.03), the expressions $D_n(x_\nu\,|\,F)$ and $D^*_n(x_\nu\,|\,F)$ are independent of the choice of the solution F. Thus it follows from (3.52) and (3.54) that problem (e) cannot have both a rational and a non-rational solution. Moreover, two rational solutions F_1 and F_2 would satisfy the equations $F_1(x_n) = F_2(x_n)$ ($n=1, 2,\ldots$), hence they would be identical.

§ 4. Discussion of the Polynomials $P_n, \ldots, Q^*_n$. Degenerate Solutions.

A further analysis of problem (e) requires a more detailed discussion of the polynomials $P_n, \ldots, Q^*_n$, defined by (3.44) . . .(3.51).

We begin by supposing that the values x_n are different from one another and different from zero, and that the values a_n are quite arbitrary. By elementary properties of determinants we have

$$(4.01) \qquad P_1(x) = a_1,\ Q_1(x) = 1,\ P_1^*(x) = a_1x_1,\ Q_1^*(x) = 1,$$

$$(4.02) \quad P_n(x_k) = a_kQ_n(x_k),\ P_n^*(x_k) = a_kx_kQ_n^*(x_k)\ (k = 1, 2, \ldots, n),$$

and

$$(4.03) \qquad \begin{cases} P_n^*(0) = (-)^n x_1 \ldots x_n\{P_{n-1}(x_n) - a_nQ_{n-1}(x_n)\}, \\ Q_n(0) = (-)^{n+1}\{P_{n-1}^*(x_n) - a_nx_nQ_{n-1}^*(x_n)\} \qquad (n > 1). \end{cases}$$

THEOREM 19. The following recurrence formulae hold for $n > 1$:

$$(4.04) \qquad Q_{n-1}(0)P_n(x) = Q_n(0)P_{n-1}(x) + \frac{P_n^*(0)}{x_1 \ldots x_n} P_{n-1}^*(x),$$

$$(4.05) \qquad Q_{n-1}(0)Q_n(x) = Q_n(0)Q_{n-1}(x) + \frac{P_n^*(0)}{x_1 \ldots x_n} xQ_{n-1}^*(x),$$

$$(4.06) \quad \frac{P_{n-1}^*(0)}{x_1 \ldots x_{n-1}} P_n^*(x) = Q_n(0)xP_{n-1}(x) + \frac{P_n^*(0)}{x_1 \ldots x_{n-1}} P_{n-1}^*(x),$$

$$(4.07) \quad \frac{P_{n-1}^*(0)}{x_1 \ldots x_{n-1}} Q_n^*(x) = Q_n(0)Q_{n-1}(x) + \frac{P_n^*(0)}{x_1 \ldots x_{n-1}} Q_{n-1}^*(x).$$

Hence the expressions $P_n(x), \ldots Q_n^*(x)$ are uniquely determined by (4.01), (4.03), ... (4.07) if and only if

$$(4.08) \qquad P_k^*(0) \neq 0,\ Q_k(0) \neq 0 \quad (k = 1, 2, \ldots, n-1).$$

PROOF. Let the rows and the columns of an n-rowed determinant A be successively denoted by the numbers $1, 2, \ldots n$. Suppose $n > 1$, and let, for $p < n$,

$$1 \leqq \mu_1 < \mu_2 < \ldots < \mu_p \leqq n,\ 1 \leqq \nu_1 < \nu_2 < \ldots < \nu_p \leqq n.$$

Let $A_{\mu_1 \ldots \mu_p}^{(\nu_1 \ldots \nu_p)}$ be the subdeterminant obtained from A by leaving out the rows $\mu_1, \ldots \mu_p$ and the columns $\nu_1, \ldots \nu_p$, and let $A_{1 \ldots n}^{1 \ldots n} = 1$. We then have the well-known identity

$$(4.09) \qquad AA_{\mu_1\mu_2}^{(\nu_1\nu_2)} = A_{\mu_1}^{(\nu_1)} A_{\mu_2}^{(\nu_2)} - A_{\mu_1}^{(\nu_2)} A_{\mu_2}^{(\nu_1)}.$$

If $\mu_1 = 1$, $\mu_2 = 2m + 1$, $\nu_1 = m + 1$, $\nu_2 = 2m + 1$, and

$$A = \begin{vmatrix} 1, x, \ldots, x^{m-1}, 0, 0, \ldots, 0 \\ 1, x_i, \ldots, x_i^{m-1}, a_i, a_ix_i, \ldots, a_ix_i^m \end{vmatrix}_{(i=1\ldots 2m)},$$

this yields

$$A \left| 1, \ldots, x_i^{m-1}, a_i x_i, \ldots, a_i x_i^{m-1} \right|_{(i=1\ldots 2m-1)}$$

$$= \left| 1, \ldots, x_i^{m-1}, a_i x_i, \ldots, a_i x_i^{m} \right|_{(i=1\ldots 2m)} \begin{vmatrix} 1, x, & \ldots, x^{m-1}, & 0, \ldots, & 0 \\ 1, x_i, & \ldots, x_i^{m-1}, & a_i, \ldots, & a_i x_i^{m-1} \end{vmatrix}_{(i=1\ldots 2m-1)}$$

$$- \left| 1, \ldots, x_i^{m-1}, a_i, \ldots, a_i x_i^{m-1} \right|_{(i=1\ldots 2m)} \begin{vmatrix} 1, x, & \ldots, x^{m-1}, & 0, & \ldots, & 0 \\ 1, x_i, & \ldots, x_i^{m-1}, & a_i x_i, & \ldots, & a_i x_i^{m} \end{vmatrix}_{(i=1\ldots 2m-1)}$$

hence, by (3.44) . . . (3.51),

$$Q_{2m-1}(0) P_{2m}(x) = Q_{2m}(0) P_{2m-1}(x) + \frac{P^*_{2m}(0)}{x_1 \ldots x_{2m}} P^*_{2m-1}(x).$$

Next, if we put $\mu_1 = 1$, $\mu_2 = 2m + 2$, $\nu_1 = m + 1$, $\nu_2 = m+2$, and

$$A = \begin{vmatrix} 1, x, \ldots, x^m, 0, & 0, & \ldots, & 0, \\ 1, x_i, \ldots, x_i^m, a_i, & a_i x_i, & \ldots, & a_i x_i^m \end{vmatrix}_{(i=1\ldots 2m+1}, $$

we obtain, by (4.09),

$$Q_{2m}(0) P_{2m+1}(x) = Q_{2m+1}(0) P_{2m}(x) + \frac{P^*_{2m+1}(0)}{x_1 \ldots x_{2m+1}} P^*_{2m}(x),$$

hence (4.04) is right. In the same way (4.05) can be obtained, both for even and odd values of n, by putting $\mu_1 = 1$, $\mu_2 = 2m + 1$, $\nu_1 = m + 1$, $\nu_2 = 2m + 1$ and

$$A = \begin{vmatrix} 0, 0, \ldots, & 0, & 1, & x, & \ldots, & x^m \\ 1, x_i, \ldots, & x_i^{m-1}, & a_i, & a_i x_i, & \ldots, & a_i x_i^m \end{vmatrix}_{(i=1\ldots 2m)},$$

resp. by putting $\mu_1 = 1$, $\mu_2 = 2m + 2$, $\nu_1 = m + 1$, $\nu_2 = m + 2$ and

$$A = \begin{vmatrix} 0, 0, \ldots, & 0, & 1, & x, & \ldots, & x^m \\ 1, x_i, \ldots, & x_i^{m}, & a_i, & a_i x_i, & \ldots, & a_i x_i^m \end{vmatrix}_{(i=1\ldots 2m+1)}.$$

The equalities (4.06) and (4.07) too can be obtained as particular cases of (4.09). However, they can also be deduced from the recurrence formulae for P_n and Q_n by means of a transformation, which will also be useful afterwards. Let φ be a function of $a_1, \ldots a_n$ (which may also depend of $x_1, \ldots x_n$), and put

$$(4.10) \quad \begin{cases} T_n \varphi(a_1, \ldots, a_n) = a_1 \ldots a_n \varphi(a_1^{-1}, \ldots, a_n^{-1}), \\ U_n \varphi(a_1, \ldots, a_n) = \varphi(a_1 x_1, \ldots, a_n x_n). \end{cases}$$

We then have

$$(4.11) \quad \begin{cases} U_n T_n U_n T_n \varphi = x_1 \ldots x_n \varphi, \\ T_n(\varphi_1 \varphi_2) = a_1^{-1} \ldots a_n^{-1} T_n(\varphi_1) T_n(\varphi_2), \ U_n(\varphi_1 \varphi_2) = U_n(\varphi_1) U_n(\varphi_2). \end{cases}$$

Moreover, if φ is independent of a_n, we have

$$(4.12) \qquad T_n \varphi = a_n T_{n-1} \varphi, \ U_n \varphi = U_{n-1} \varphi.$$

Now we have, as an immediate consequence of (3.44) . . . (3.51), and both for even and odd values of n:

$$(4.13) \qquad U_n T_n P_n(x) = Q_n^*(x),\ U_n T_n Q_n(x) = P_n^*(x),$$

and hence, by (4.11),

$$(4.14) \quad U_n T_n P_n^*(x) = x_1 \dots x_n Q_n(x),\ U_n T_n Q_n^*(x) = x_1 \dots x_n P_n(x).$$

By effecting the transformation $U_n T_n$ on the equalities (4.04) and (4.05), we thus obtain the recurrence formulae for P_n^* and Q_n^*.

The condition (4.08) is evident.

THEOREM 20. The following equalities hold for $n > 1$ (and (4.15) for $n = 1$):

$$(4.15) \quad P_n^*(x)Q_n(x) - xP_n(x)Q_n^*(x) = P_n^*(0)Q_n(0) \prod_{k=1}^{n} \left(1 - \frac{x}{x_k}\right),$$

$$(4.16) \quad Q_{n-1}(x)P_n(x) - P_{n-1}(x)Q_n(x) = \frac{P_{n-1}^*(0)P_n^*(0)}{x_1 \dots x_n} \prod_{k=1}^{n-1} \left(1 - \frac{x}{x_k}\right),$$

$$(4.17) \quad Q_{n-1}(x)P_n^*(x) - xP_{n-1}(x)Q_n^*(x) = Q_{n-1}(0)P_n^*(0) \prod_{k=1}^{n-1} \left(1 - \frac{x}{x_k}\right),$$

$$(4.18) \quad P_{n-1}^*(x)Q_n(x) - xQ_{n-1}^*(x)P_n(x) = P_{n-1}^*(0)Q_n(0) \prod_{k=1}^{n-1} \left(1 - \frac{x}{x_k}\right),$$

$$(4.19) \quad P_{n-1}^*(x)Q_n^*(x) - Q_{n-1}^*(x)P_n^*(x) = x_1 \dots x_{n-1} Q_{n-1}(0)Q_n(0) \prod_{k=1}^{n-1} \left(1 - \frac{x}{x_k}\right).$$

PROOF. First (4.15) will be proved by induction. When $n = 1$ (4.15) holds; let (4.15) hold when n is replaced by $n - 1$. We then have, by (4.04) . . . (4.07),

$$P_{n-1}^*(0)Q_{n-1}(0)\{P_n^*(x)Q_n(x) - xP_n(x)Q_n^*(x)\}$$

$$= P_n^*(0)Q_n(0)\left(1 - \frac{x}{x_n}\right)\{P_{n-1}^*(x)Q_{n-1}(x) - xP_{n-1}(x)Q_{n-1}^*(x)\}$$

$$= P_{n-1}^*(0)Q_{n-1}(0)P_n^*(0)Q_n(0) \prod_{k=1}^{n} \left(1 - \frac{x}{x_k}\right),$$

hence, if $P_{n-1}^*(0)Q_{n-1}(0) \neq 0$, (4.15) holds for the given value n. Thus (4.15) holds for any n that satisfies (4.08). Since (4.15) is an identity between polynomials, which is true for arbitrary values of $x_1, \dots x_n, a_1, \dots a_n$ that satisfy (4.08), the equality also holds if (4.08) is not satisfied.

Moreover we have, by (3.44) . . . (3.49), (4.15),

$$Q_{n-1}(0)\{Q_{n-1}(x)P_n(x) - P_{n-1}(x)Q_n(x)\}$$
$$= \frac{P_n^*(0)}{x_1 \dots x_n}\{P_{n-1}^*(x)Q_{n-1}(x) - xP_{n-1}(x)Q_{n-1}^*(x)\}$$
$$= Q_{n-1}(0)\frac{P_{n-1}^*(0)P_n^*(0)}{x_1 \dots x_n}\prod_{k=1}^{n-1}\left(1 - \frac{x}{x_k}\right),$$

hence (4.16) is true if $Q_{n-1}(0) \neq 0$, and consequently if $Q_{n-1}(0)=0$. The equalities (4.17) ... (4.19) can be obtained in the same way.

THEOREM 21. If there is a least value n for which $P_n^*(0)Q_n(0) = 0$, the expressions $P_n^*(0)$ and $Q_n(0)$ cannot both be zero.

PROOF. The theorem is true when $n = 1$; so let $n > 1$. First suppose $P_n^*(0) = 0$. By (4.03) we then have

$$P_{n-1}(x_n) = a_n Q_{n-1}(x_n), \tag{4.20}$$

since the values x_i are all supposed to be different from zero. Now, according to our assumptions, $P_{n-1}^*(0)Q_{n-1}(0) \neq 0$. Hence we have, by (4.15),

$$P_{n-1}^*(x_n)Q_{n-1}(x_n) - x_nP_{n-1}(x_n)Q_{n-1}^*(x_n) = P_{n-1}^*(0)Q_{n-1}(0)\prod_{k=1}^{n-1}\left(1 - \frac{x_n}{x_k}\right) \neq 0,$$

or, by (4.20),

$$Q_{n-1}(x_n)\{P_{n-1}^*(x_n) - a_nx_nQ_{n-1}^*(x_n)\} \neq 0,$$

hence $Q_n(0) \neq 0$ by (4.03). If $Q_n(0) = 0$ we obtain $P_n^*(0) \neq 0$ in the same way.

THEOREM 22. If, in addition to the assumptions made on the sequence $\{x_n\}_1^\infty$ in the beginning of this section, the values $x_1, \dots x_n$ and a_1 are positive, and if

$$P_k^*(0) > 0,\ Q_k(0) > 0 \qquad (k = 2, 3, \dots n), \tag{4.21}$$

the following properties hold:

(a) The polynomials $P_n(x), \dots Q_n^*(x)$ are positive for $x \geqq 0$; the values $a_2, \dots a_n$ are also positive.

(b) The degrees of these polynomials are determined, for $n = 2m$ resp. $n = 2m + 1$, by

$$\begin{cases} [P_{2m}]=m-1, & [Q_{2m}]=m, & [P_{2m}^*]=m, & [Q_{2m}^*]=m-1, \\ [P_{2m+1}]=m, & [Q_{2m+1}]=m, & [P_{2m+1}^*]=m, & [Q_{2m+1}^*] = m. \end{cases} \tag{4.22}$$

(c) The zeros of these polynomials are simple and negative. In what follows the zeros of P_n, Q_n, P_n^* and Q_n^* will be denoted by $(\alpha_{n,i})$, $(\beta_{n,i})$, $(\alpha_{n,i}^*)$ and $(\beta_{n,i}^*)$ respectively, and, if we take $i = 1, 2, \dots$, the absolute values of the zeros will be supposed to form an increasing sequence.

(d) The zeros of P_n as well as those of Q_n^* are separated both by the zeros of P_n^* and by those of Q_n; conversely, the zeros of P_n^* as well as those of Q_n are separated by the zeros of P_n and of Q_n^*. Compared to P_n and Q_n^*, the polynomials P_n^* and Q_n have the zeros with the least absolute values:

$$(4.23)\qquad \begin{cases} \alpha_{n,i} < \alpha_{n,i}^* < \alpha_{n,i-1}, & \alpha_{n,i} < \beta_{n,i} < \alpha_{n,i-1}, \\ \beta_{n,i}^* < \alpha_{n,i}^* < \beta_{n,i-1}^*, & \beta_{n,i}^* < \beta_{n,i} < \beta_{n,i-1}^*. \end{cases}$$

PROOF. By (4.01) the properties (a) and (b) hold when $n = 1$. By induction they hold for any n, which is evident by (4.04), ... (4.07), (4.21) and (4.02).

If $n = 1$ the statements (c) and (d) are meaningless. If $n = 2$, (c) is true by (a) and (b), and (d) is also true, since there is a single zero of P_2^* and of Q_2, and no zero of P_2 or Q_2^*. Let us take $n > 2$ and assume that (c) and (d) hold for $n - 1$. By hypothesis, there will be at least one zero $\alpha_{n-1,1}^*$ of P_{n-1}^*, and we have, according to (4.23),

$$\operatorname{sgn} P_{n-1}(\alpha_{n-1,i}^*) = (-)^{i+1},$$

hence, by (4.04) and (4.06),

$$(4.24)\qquad \begin{aligned} \operatorname{sgn} P_n(\alpha_{n-1,i}^*) &= (-)^{i+1}, \\ \operatorname{sgn} P_n^*(\alpha_{n-1,i}^*) &= (-)^{i} \end{aligned} \quad (i=1, 2, \ldots, [P_{n-1}^*]).$$

If $n = 2m$, we have $P_{2m}(0) > 0$, $P_{2m}^*(0) > 0$ by (a), hence $P_{2m}(x)$ changes sign in at least $m - 2$ points, and $P_{2m}^*(x)$ in at least $m - 1$ points of the interval $(\alpha_{2m-1,m-1}^*, 0)$. If $n = 2m + 1$, we obtain in the same way that P_{2m+1} changes sign in at least $m - 1$ points, and P_{2m+1}^* in at least m points of $(\alpha_{2m,m}^*, 0)$. Since P_n and P_n^* are positive if $x > 0$, the coefficient of the highest power of x of these polynomials must be positive. This yields, for $x < 0$ and $|x|$ sufficiently large, and for even resp. odd values of n,

$$(4.25)\qquad \begin{cases} \operatorname{sgn} P_{2m}(x) = (-)^{m-1}, & \operatorname{sgn} P_{2m}^*(x) = (-)^m, \\ \operatorname{sgn} P_{2m+1}(x) = (-)^m, & \operatorname{sgn} P_{2m+1}^*(x) = (-)^m. \end{cases}$$

Comparing this result with (4.24) we obtain that P_{2m} and P_{2m}^* change sign at least once in $(-\infty, \alpha_{2m-1,m-1}^*)$, and that P_{2m+1} changes sign at least once in $(-\infty, \alpha_{2m,m}^*)$. Hence, if we denote the number of negative zeros of a polynomial f by $\nu(f)$, we have, for even resp. for odd values of n,

$$\nu(P_{2m}) \geqq m - 1, \ \nu(P_{2m}^*) \geqq m, \ \nu(P_{2m+1}) \geqq m, \ \nu(P_{2m+1}^*) \geqq m,$$

and, by (4.22), all zeros of P_n and P_n^* will be simple and negative.

Now the recurrence formulae (4.04) ... (4.07) are invariant, by (4.13) and (4.14), for the transformation U_nT_n. Moreover, by (4.10), (4.13) and (4.14), we have for $k \leqq n$:

$$U_nT_nP_k^*(x) = a_{k+1}\dots a_nx_1\dots x_nQ_k(x),$$
$$U_nT_nQ_k(x) = a_{k+1}\dots a_nx_{k+1}\dots x_nP^*(x),$$

and since the values a_i and x_i are all positive, the set of inequalities (4.21) is also invariant for the transformation U_nT_n. Hence the zeros of the polynomials

$$U_nT_nP_n(x) = Q_n^*(x),\ U_nT_nP_n^*(x) = x_1\dots x_nQ_n(x)$$

are simple and negative, which completes the proof of (c) for the value n.

Next we prove that the zeros of P_n and P_n^* separate one another, and that $(\alpha_{n,1}^*, 0)$ contains no zeros of P_n. In the particular case $n = 3$, P_3 as wel as P_3^* have a single zero, and, by (4.24), we have

$$\operatorname{sgn} P_3(\alpha_{2,1}^*) = 1,\ \operatorname{sgn} P_3^*(\alpha_{2,1}^*) = -1,$$

hence our statement is true. If $n > 3$ there is at least one zero of P_{n-1}, and we have, in virtue of our hypothesis,

$$\operatorname{sgn} P_{n-1}^*(\alpha_{n-1,i}) = (-)^i,$$

hence, by (4.04) and (4.06),

$$\operatorname{sgn} P_n(\alpha_{n-1,i}) = (-)^i,\ \ \operatorname{sgn} P_n^*(\alpha_{n-1,i}) = (-)^i \qquad (i=1, 2, \dots, [P_{n-1}]). \tag{4.26}$$

If $n = 2m$, both P_{2m} and P_{2m}^* will change sign at least $m-1$ times in the interval $(\alpha_{2m-1,\, m-1}, 0)$; if $n = 2m+1$, both P_{2m+1} and P_{2m+1}^* will do so at least $m-1$ times in $(\alpha_{2m,\, m-1}, 0)$. Comparing (4.26) for $n = 2m$ and $i = m-1$ with (4.25), we obtain that P_{2m}^* changes sign at least once in $(-\infty, \alpha_{2m-1,\, m-1})$; the same holds for P_{2m+1} and P_{2m+1}^* with respect to the interval $(-\infty, \alpha_{2m,\, m-1})$. Now, by (4.22), we can infer that both the zeros of P_n and of P_n^* are separated by those of P_{n-1}, hence

$$\alpha_{n-1,i} < \alpha_{n,i} < \alpha_{n-1,i-1},\ \alpha_{n-1,i} < \alpha_{n,i}^* < \alpha_{n-1,i-1}. \tag{4.27}$$

Moreover, it follows from (4.24) and (4.25) that both the zeros of P_n and of P_n^* are separated by those of P_{n-1}^*, hence

$$\alpha_{n-1,i+1}^* < \alpha_{n,i} < \alpha_{n-1,i}^*\ \alpha_{n-1,i}^* < \alpha_{n,i}^* < \alpha_{n-1,i-1}^*,$$

and by (4.27)

$$\alpha_{n,i} < \alpha_{n-1,i}^* < \alpha_{n,i}^* < \alpha_{n-1,i-1} < \alpha_{n,i-1},$$

which proves the statement. Applying the transformation U_nT_n

to the polynomials P_n and P_n^*, we get the result that the zeros of Q_n and Q_n^* also separate one another, and that Q_n has the zero with the smallest absolute value.

From (4.15) we get, by putting $x = \alpha_{n,i}$,

$$P_n^*(\alpha_{n,i})Q_n(\alpha_{n,i}) > 0 \quad (i = 1, 2, \ldots, [P_n]),$$

and since we have just shown

$$\operatorname{sgn} P_n^*(\alpha_{n,i}) = (-)^i,$$

we also have

$$\operatorname{sgn} Q_n(\alpha_{n,i}) = (-)^i.$$

Since Q_n has at most one zero more than P_n, the zeros of P_n and Q_n separate one another, and evidently Q_n has the zero with the least absolute value. Applying the transformation $U_n T_n$ we get the corresponding property for the zeros of P_n^* and Q_n^*, which completes the proof of (d) for the value n.

From now on it will again be supposed that (1.01) and (1.10) hold, which implies, by (4.01),

$$(4.28) \qquad P_1(x) = Q_1(x) = P_1^*(x) = Q_1^*(x) = 1.$$

Let us put

$$(4.29) \qquad R_n(x) = \frac{P_n(x)}{Q_n(x)}, \quad R_n^*(x) = \frac{P_n^*(x)}{xQ_n^*(x)} \quad (n = 1, 2, \ldots),$$

hence, by (4.28),

$$(4.30) \qquad R_1(x) = 1, \quad R_1^*(x) = \frac{1}{x},$$

and, by (4.02),

$$(4.31) \qquad R_n(x_k) = R_n^*(x_k) = a_k \quad (k = 1, 2, \ldots n),$$

in all cases where these expressions are not indeterminate.

THEOREM 23. The following statements are consequences of (4.21):

(a) R_n and R_n^* are positive for $x > 0$.

(b) Putting $x_0 = 0$ we have

$$(4.32) \qquad (-)^k \{R_n^*(x) - R_n(x)\} > 0$$

for $x_k < x < x_{k+1}$ $(k = 0, 1, \ldots, n-1)$, or $x_n < x$, $k = n$, while

$$(4.33) \quad (-)^k\{R_{n-1}^*(x) - R_n^*(x)\} > 0, \quad (-)^k\{R_n(x) - R_{n-1}(x)\} > 0,$$

for $x_k < x < x_{k+1}$ $(k = 0, 1, \ldots, n-2)$, or $x_{n-1} < x$, $k = n-1$.

(c) R_n and R_n^* belong to the class $\{F\}$.

(d) R_n and R_n^* are increasing functions of a_n for $x > x_{n-1}$.

PROOF.

Ad (*a*). According to Theorem (22a), the polynomials $P_n, \dots Q_n^*$ are positive when $x > 0$. Hence R_n and R_n^* are positive, by (4.29).

Ad (*b*). We get (4.32) and (4.33) as immediate consequences of (4.15), (4.16), (4.19) and (4.21).

Ad (*c*). According to Theorem 22(c), the zeros $\beta_{n,i}$ and $\beta_{n,i}^*$ of Q_n and Q_n^* are simple and negative; according to Theorem 22(d), P_n and Q_n have no zeros in common, nor have P_n^* and Q_n^*. Hence the poles of R_n and R_n^* are of the first order, and, except for a pole of R_n^* in the origin, they coincide with the zeros of Q_n resp. of Q_n^*. We thus obtain

$$R_n(z) = \sum_{(i)} \frac{A_{n,i}}{z - \beta_{n,i}} + G_n(z), \quad R_n^*(z) = \frac{A_{n,0}^*}{z} + \sum_{(i)} \frac{A_{n,i}^*}{z - \beta_{n,i}^*} + G_n^*(z),$$

where G_n and G_n^* are polynomials, and where the residues $A_{n,i}$, $A_{n,i}^*$ must be positive, according to Theorem 22(d). If the degrees of $P_n, \dots, Q_n^*$ are taken into account, it is clear that G_n and G_n^* are constants. Let them be denoted by u_n and u_n^*. This yields

$$(4.34) \quad R_n(z) = \sum_{(i)} \frac{A_{n,i}}{z - \beta_{n,i}} + u_n, \quad R_n^*(z) = \frac{A_{n,0}^*}{z} + \sum_{(i)} \frac{A_{n,i}^*}{z - \beta_{n,i}^*} + u_n^*,$$

hence

$$(4.35) \qquad u_n = \lim_{x \to \infty} R_n(x), \; u_n^* = \lim_{x \to \infty} R_n^*(x).$$

According to (a), these limits are non-negative, so now it follows from (4.34) that R_n and R_n^* can be represented by a Stieltjes integral of the form (1.05). Finally we have, by (4.31) and (1.10),

$$R_n(1) = R_n^*(1) = 1,$$

hence R_n and R_n^* belong to the class $\{F\}$.

Ad (*d*). By (4.03), (4.04) and (4.05) we have

$$(-)^n Q_{n-1}(0) \frac{\partial P_n(x)}{\partial a_n} = x_n Q_{n-1}^*(x_n) P_{n-1}(x) - Q_{n-1}(x_n) P_{n-1}^*(x),$$

$$(-)^n Q_{n-1}(0) \frac{\partial Q_n(x)}{\partial a_n} = x_n Q_{n-1}^*(x_n) Q_{n-1}(x) - x Q_{n-1}(x_n) Q_{n-1}^*(x),$$

hence, by (4.16) and (4.18),

$$(-)^n Q_{n-1}(0)Q_n^2(x)\frac{\partial R_n(x)}{\partial a_n} = (-)^n Q_{n-1}(0)\left\{Q_n(x)\frac{\partial P_n(x)}{\partial a_n} - P_n(x)\frac{\partial Q_n(x)}{\partial a_n}\right\}$$

$$= Q_n(x)\{x_n Q_{n-1}^*(x_n)P_{n-1}(x) - Q_{n-1}(x_n)P_{n-1}^*(x)\}$$

$$- P_n(x)\{x_n Q_{n-1}^*(x_n)Q_{n-1}(x) - xQ_{n-1}(x_n)Q_{n-1}^*(x)\}$$

$$= x_n Q_{n-1}^*(x_n)\{P_{n-1}(x)Q_n(x) - Q_{n-1}(x)P_n(x)\}$$

$$+ Q_{n-1}(x_n)\{xQ_{n-1}^*(x)P_n(x) - P_{n-1}^*(x)Q_n(x)\}$$

$$= - P_{n-1}^*(0)\left\{\frac{P_n^*(0)}{x_1 \ldots x_{n-1}} Q_{n-1}^*(x_n) + Q_n(0)Q_{n-1}(x_n)\right\}\prod_{k=1}^{n-1}\left(1 - \frac{x}{x_k}\right),$$

or

$$(4.36)\quad Q_n^2(x)\frac{\partial R_n(x)}{\partial a_n} = (-)^{n-1}P_{n-1}^*(0)Q_n(x_n)\prod_{k=1}^{n-1}\left(1 - \frac{x}{x_k}\right).$$

In the same way we obtain

$$(437)\qquad xQ_n^{*2}(x)\frac{\partial R_n^*(x)}{\partial a_n} = (-)^{n-1}x_1 \ldots x_n Q_{n-1}(0)Q_n^*(x_n)\prod_{k=1}^{n-1}\left(1 - \frac{x}{x_k}\right),$$

which proves the statement.

Theorem 24. The set of inequalities (4.21) is equivalent to the system

$$(4.38)\quad a_k = \vartheta_k R_{k-1}(x_k) + \vartheta_k^* R_{k-1}^*(x_k),\ \vartheta_k + \vartheta_k^* = 1,\ 0 < \vartheta_k < 1 \qquad (k = 2, 3, \ldots, n).$$

Proof. First let (4.21) hold. According to theorem 22(a) the polynomials $P_k, \ldots, Q_k^*$ are positive for $k = 2, 3, \ldots n$ and $x \geqq 0$. We thus have, by (4.03),

$$\operatorname{sgn}\{R_{k-1}(x_k) - a_k\} = (-)^k,\quad \operatorname{sgn}\{R_{k-1}^*(x_k) - a_k\} = (-)^{k+1},$$

hence a_k is included in the strict sense between $R_{k-1}(x_k)$ and R_{k-1}^* (x_k), which yields (4.38).

The converse can be proved by induction. If (4.38) holds, we have, for $k = 2$,

$$a_2 = \vartheta_2 + \frac{\vartheta_2^*}{x_2},$$

and, since $x_2 > 1$,

$$a_2 < 1 < a_2 x_2.$$

Moreover we have, by (4.03) and (4.28),

$$P_2^*(0) = x_1 x_2(1 - a_2),\ Q_2(0) = -1 + a_2 x_2,$$

hence (4.21) holds for $k = 2$. Let (4.21) hold for $k \leqq \nu - 1$, where $\nu \leqq n$. By Theorem 23(a) the functions $R_{\nu-1}$ (x) and $R_{\nu-1}^*$ (x) are positive for $x > 0$, while, by (4.32),

$$(-)^{\nu-1}\{R^*_{\nu-1}(x_\nu) - R_{\nu-1}(x_\nu)\} > 0,$$

and, by (4.38),

$$a_\nu = \vartheta_\nu R_{\nu-1}(x_\nu) + \vartheta^*_\nu R^*_{\nu-1}(x_\nu),\ \vartheta_\nu + \vartheta^*_\nu = 1,\ 0 < \vartheta_\nu < 1.$$

Hence we have

$$(-)^{\nu+1}\{R^*_{\nu-1}(x_\nu) - a_\nu\} > 0,\ (-)^\nu\{R_{\nu-1}(x_\nu) - a_\nu\} > 0,$$

and, by (4.03),

$$P^*_\nu(0) > 0,\ Q_\nu(0) > 0,$$

which completes the proof.

The necessary conditions for the existence of a solution of problem (e), which have been obtained in the preceding section, can now be expressed in a somewhat different way, and the explicit solution of a degenerate problem can be given.

THEOREM 25. If problem (e) has a solution, all values

$$P^*_n(0),\ Q_n(0) \qquad (n = 2, 3, \ldots)$$

are positive, except for the following cases, where the problem is degenerate:

α) $\tau_N > 0,\ \tau_1 < 1$; then $P^*_n(0) = 0$ for $n \geqq 2N + 1$, $Q_n(0) = 0$ for $n \geqq 2N + 2$;
β) $\tau_N = 0,\ \tau_1 < 1$; then $P^*_n(0) = 0$ for $n \geqq 2N$, $Q_n(0) = 0$ for $n \geqq 2N + 1$;
γ) $\tau_N > 0,\ \tau_1 = 1$; then $P^*_n(0) = 0$ for $n \geqq 2N + 1$, $Q_n(0) = 0$ for $n \geqq 2N$;
δ) $\tau_N = 0,\ \tau_1 = 1$; then $P^*_n(0) = 0$ for $n \geqq 2N$, $Q_n(0) = 0$ for $n \geqq 2N - 1$.

Here the values N, τ_1 and τ_N have the same meaning as in Theorem 17.

PROOF. By (3.43) and (4.03) we have, for any solution F of problem (e),

$$P^*_n(0) = (-)^{n-1} x_1 \ldots x_n\, D_{n-1}(x|_n F),\ Q_n(0) = (-)^{n-1} D^*_{n-1}(x_n|F).$$

Hence, by Theorem 18, the values $P^*_n(0)$ and $Q_n(0)$ are positive, except for the cases (α) ... (δ).

According to Theorem 16, any rational function of the class $\{F\}$ can be represented by an expression of the form

$$(4.39)\quad F(x) = u_0 + \sum_{i=1}^{\nu-1} \frac{u_i}{1 - t_i + t_i x} + \frac{u_\nu}{x} \quad (u_0 \geqq 0;\ 0 < t_i < 1,\ u_i > 0 \text{ for } i = 1, \ldots, \nu-1;\ u_\nu \geqq 0).$$

If this representation contains n positive parameters u, t, I shall denote $F(x)$ by $r_n(x)$ if the origin is a regular point, and by $r^*_n(x)$ if it is a pole. This agrees with the notation of the functions $R_n(x)$ and $R_n^*(x)$: if these belong to $\{F\}$, they depend, by (4.22), of exactly n positive parameters.

THEOREM 26. If a rational function $r_n(x)$ of the class $\{F\}$ satisfies the equalities

$$r_n(x_k) = a_k \qquad (k = 1, 2, \ldots, n),$$

(4.38) holds for $k = 2, 3, \ldots n$, and we have identically $r_n(x) = R_n(x)$.

PROOF. First let $n = 2m$. Since $r_n(x)$ satisfies (1.03) for $k = 1, 2, \ldots, n$, we can apply the preceding theorem, where

$$N = m,\ \tau_1 < 1,\ \tau_N > 0.$$

Hence (4.21) holds, and, by Theorem 24, the inequalities (4.38) hold also. Moreover, by (3.43) and Theorem 18, we have

$$D_{2m}(x|r_{2m}) = -P_{2m}(x) + Q_{2m}(x)r_{2m}(x) \equiv 0,$$

which proves the identity of r_{2m} and R_{2m}. If $n = 2m + 1$, we have in the same way

$$N = m + 1,\ \tau_1 < 1,\ \tau_N = 0,$$

which yields (4.21) and (4.38), whereas

$$D_{2m+1}(x|r_{2m+1}) = -P_{2m+1}(x) + Q_{2m+1}(x)r_{2m+1}(x) \equiv 0,$$

which proves that r_{2m+1} and R_{2m+1} are identical.

THEOREM 27. Let a rational function $r_n^*(x)$ of the class $\{F\}$ satisfy the equalities

$$r_n^*(x_k) = a_k \qquad (k = 1, 2, \ldots n).$$

Then (4.38) holds for $k = 2, 3, \ldots, n$, and $r_n^*(x) \equiv R_n^*(x)$.

PROOF. It is similar to the preceding one.

THEOREM 28. If problem (e) has a non-degenerate solution, all values ϑ_n and ϑ_n^*, defined by

$$(4.40)\quad a_n = \vartheta_n R_{n-1}(x_n) + \vartheta_n^* R_{n-1}^*(x_n), \vartheta_n + \vartheta_n^* = 1 \qquad (n = 2, 3, \ldots),$$

are positive. In the case of a degenerate problem we have

$$\vartheta_k > 0,\ \vartheta_k^* > 0 \quad (k = 2, 3, \ldots, n-1), \quad \vartheta_n\vartheta_n^* = 0,$$

for a definite value of n. According as $\vartheta_n = 0$ or $\vartheta_n^* = 0$, the solution of the problem is $R_{n-1}^*(x)$ or $R_{n-1}(x)$.

PROOF. If the problem is solvable and non-degenerate, all values $P_n^*(0)$, $Q_n(0)$ are positive, by Theorem 25. Hence all values ϑ_n, ϑ_n^* are positive, by Theorem 24.

If the problem is degenerate, we have, for a definite value of n,

$$P_k^*(0) > 0,\ Q_k(0) > 0\ (k = 2, 3, \ldots, n-1),\ P_n^*(0)Q_n(0) = 0.$$

In particular we have, by Theorem 25, $P_n^*(0) = 0$ if $\tau_1 < 1$, and $Q_n(0) = 0$ if $\tau_1 = 1$. Hence the solution of the problem can be represented by $r_\nu(x)$ in the first case and by $r_\nu^*(x)$ in the second one, where ν has still to be determined. In both cases we have, by Theorem 24,

$$0 < \vartheta_k < 1 \qquad (k = 2, 3, \ldots, n-1),$$

hence, by Theorem 23,

$$R_{n-1}(x_n) \neq R_{n-1}^*(x_n).$$

Thus we can always write

$$a_n = \vartheta_n R_{n-1}(x_n) + \vartheta_n^* R_{n-1}^*(x_n), \; \vartheta_n + \vartheta_n^* = 1.$$

Now by (4.03) we have

$$(4.41) \begin{cases} P_n^*(0) = (-)^n \vartheta_n^* x_1 \ldots x_n Q_{n-1}(x_n)\{R_{n-1}(x_n) - R_{n-1}^*(x_n)\}, \\ Q_n(0) = (-)^n \vartheta_n x_n Q_{n-1}^*(x_n)\{R_{n-1}(x_n) - R_{n-1}^*(x_n)\}, \end{cases}$$

hence $\vartheta_n^* = 0$ or $\vartheta_n = 0$ according as $P_n^*(0) = 0$ or $Q_n(0) = 0$. In other words: the degenerate problem (e) has either a solution $r_\nu(x)$ when $\vartheta_n^* = 0$, or a solution $r_\nu^*(x)$ when $\vartheta_n = 0$. In both cases we have $\nu < n$, by Theorems 26 and 27, and, by the same theorems, the solution is equal to $R_\nu(x)$ resp. to $R_\nu^*(x)$. On the contrary, the equations $a_{n-1} = R_\nu(x_{n-1})$ resp. $a_{n-1} = R_\nu^*(x_{n-1})$ are incompatible, by Theorem 23, with $0 < \vartheta_k < 1$ $(k = 2, 3, \ldots, n-1)$ and $\nu < n-1$. Hence $\nu = n-1$, which proves the theorem.

THEOREM 29. In order that problem (e) be degenerate it is necessary and sufficient that

$$0 < \vartheta_k < 1 \qquad (k = 2, 3, \ldots, n-1),$$

and either

$$a_{n+\nu} = R_{n-1}(x_{n+\nu}) \qquad (\nu = 0, 1, 2, \ldots)$$

or

$$a_{n+\nu} = R_{n-1}^*(x_{n+\nu}) \qquad (\nu = 0, 1, 2, \ldots)$$

for a definite number $n > 1$. In the first case the solution of the problem is $R_{n-1}(x)$, while $\vartheta_n^* = 0$; in the second case it is $R_{n-1}^*(x)$, while $\vartheta_n = 0$.

PROOF. By Theorem 28 the conditions are necessary; by (4.31) and Theorems 23 and 24 they are sufficient.

From now on I shall leave degenerate problems out of consideration; so henceforth a solution of problem (e) will always be supposed to be non-rational.

§ 5. Existence of a Solution: sufficient Conditions.

In Theorem 28 the conditions

$$(5.01)\quad a_n = \vartheta_n R_{n-1}(x_n) + \vartheta_n^* R_{n-1}^*(x_n),\ \vartheta_n + \vartheta_n^* = 1,\ 0 < \vartheta_n < 1 \qquad (n = 2, 3, \ldots)$$

have been shown to be necessary for the existence of a solution of problem (e). We shall now prove that these conditions are also sufficient. So in this section it will always be supposed that (5.01) holds; hence (4.41) is also true for any $n > 1$.

Theorem 30. For any $x > 0$ the sequences $\{R_n^*(x)\}_{n=1}^{\infty}$ and $\{R_n^*(x)\}_{n=1}^{\infty}$ are monotonic and bounded; hence the limits

$$(5.02)\qquad R(x) = \lim_{n\to\infty} R_n(x),\ R^*(x) = \lim_{n\to\infty} R_n^*(x)$$

exist (for the present only for $x > 0$). Moreover we have

$$(5.03)\qquad R(x_k) = R^*(x_k) = a_k \qquad (k = 1, 2, \ldots)$$

and

$$(5.04)\quad \begin{cases} R_n^*(x) > R^*(x) \geqq R(x) > R_n(x) \text{ for } x_{2l} < x < x_{2l+1},\ n \geqq 2l, \\ R_n^*(x) < R^*(x) \leqq R(x) < R_n(x) \text{ for } x_{2l+1} < x < x_{2l+2},\ n \geqq 2l+1, \end{cases}$$

A solution $F(x)$ of problem (e) satisfies the inequalities

$$(5.05)\quad (-)^k\{R^*(x) - F(x)\} \geqq 0,\ (-)^k\{F(x) - R(x)\} \geqq 0 \qquad (x_k < x < x_{k+1},\ k = 0, 1, 2, \ldots),$$

hence any solution is included (in the wide sense) between R and R^* (for $x > 0$).

Proof. According to (5.01) and Theorem 24 the conditions of Theorem 23 hold for any n. We thus have

$$R_n^*(x) > R_{n+1}^*(x) > R_{n+1}(x) > R_n(x) \text{ for } x_{2l} < x < x_{2l+1},\ n \geqq 2l,$$
$$R_n^*(x) < R_{n+1}^*(x) < R_{n+1}(x) < R_n(x) \text{ for } x_{2l+1} < x < x_{2l+2},\ n \geqq 2l+1,$$

hence the sequences $\{R_n(x)\}_{n=1}^{\infty}$ and $\{R_n^*(x)\}_{n=1}^{\infty}$ are monotonic and bounded, which proves (5.02). In particular, when x is different from $x_1, x_2, \ldots$, there is always one sequence increasing and the other one decreasing.

Moreover, (5.03) holds by (4.31), and (5.04) by (4.33). Finally (5.05) holds by (3.43), (3.52) and (4.29).

Theorem 31. For all complex values of z, with the possible exception of a set of values on the half line $z \leqq 0$, the functions $R(z)$ and $R^*(z)$ are holomorphic, while

$$(5.06)\qquad R(z) = \lim_{n\to\infty} R_n(z),\ R^*(z) = \lim_{n\to\infty} R^*(z)$$

holds uniformly in any domain $G(\varepsilon, \varrho)$, such as it has been defined in Lemma 1.

PROOF. By Theorems 1 and 23, the functions R_n and R_n^* are uniformly bounded in any domain $G(\varepsilon, \varrho)$. Hence the theorem is an immediate consequence of (5.02) and of the Porter-Vitali theorem.

By (5.03) the functions R and R^* have the required value a_k when $x = x_k$, for every $k \geqq 1$. Hence, if we show that R and R^* belong to $\{F\}$, we are sure that the conditions (5.01) are sufficient for the existence of a solution of problem (e), since there will be at least one solution indeed (R and R^* may be identical). This will be done by means of Theorem 33. The real difficulty of this theorem, however, is how to prove (5.16). The formula (5.16) can be obtained in a very elegant way by means of a theorem of Helly [1], which has also served to overcome a similar difficulty in the theory of continued fractions [3]). Yet, the theorem of Helly is based on Zermelo's axiom of choice, and the well-known objections can be raised against it. For this reason I shall proceed in another, though more complicated way.

THEOREM 32. Let χ be the corresponding function of F, and

$$c(s) = \int_0^1 t^s d\chi(t). \tag{5.07}$$

We then have

$$c(s) = \frac{1}{2\pi i}\int_{\alpha-i\infty}^{\alpha+i\infty} \frac{F(z)dz}{(1-z)^{s+1}} \quad (0<\alpha<1,\ \sigma=\operatorname{Re} s>0), \tag{5.08}$$

if we put

$$|\arg(1-z)| < \frac{\pi}{2}$$

along the path of integration.

PROOF. By the theory of residues we have

$$\int_{\alpha-i\infty}^{\alpha+i\infty} \frac{dz}{(1-z)^{s+1}(1-t+tz)} = 2\pi i t^s \qquad (0 \leqq t \leqq 1), \tag{5.09}$$

for, if we shift the path of integration to the left, we only pass the pole $z = 1 - t^{-1}$. Moreover we have, along the path of integration in (5.08),

$$|1-t+tz| \geqq \alpha \qquad (0 \leqq t \leqq 1),$$

[3]) See e.g. J. GROMMER [1].

and

$$(5.10)\quad \begin{aligned}|(1-z)^{-s-1}|&=|\exp[-(\sigma+1+i\tau)\{\log|1-z|+i\arg(1-z)\}]|\\&=\exp\{-(\sigma+1)\log|1-z|+\tau\arg(1-z)\}\leqq|1-z|^{-\sigma-1}e^{\frac{1}{2}\pi|\tau|},\end{aligned}$$

if we put $s = \sigma + i\tau$. Hence

$$\left|\int_{\alpha+i\beta}^{\alpha+i\infty}\frac{dz}{(1-z)^{s+1}(1-t+tz)}\right| \leqq \alpha^{-1}e^{\frac{1}{2}\pi|\tau|}\int_{\beta}^{\infty}|1-z|^{-\sigma-1}dy$$

$$< \alpha^{-1}e^{\frac{1}{2}\pi|\tau|}\int_{\beta}^{\infty}y^{-\sigma-1}dy = \alpha_1\beta^{-\sigma},$$

where the value α_1 is independent of t. By (5.09) we thus have

$$t^s = \frac{1}{2\pi i}\int_{\alpha-i\beta}^{\alpha+i\beta}\frac{dz}{(1-z)^{s+1}(1-t+tz)} + O(\beta^{-\sigma}),$$

uniformly for $0 \leqq t \leqq 1$. By (5.07) we obtain

$$c(s) = \frac{1}{2\pi i}\int_0^1\left\{\int_{\alpha-i\beta}^{\alpha+i\beta}\frac{dz}{(1-z)^{s+1}(1-t+tz)} + O(\beta^{-\sigma})\right\}d\chi(t)$$

$$= \frac{1}{2\pi i}\int_{\alpha-i\beta}^{\alpha+i\beta}\frac{dz}{(1-z)^{s+1}}\int_0^1\frac{d\chi(t)}{1-t+tz} + O(\beta^{-\sigma}),$$

which yields (5.08).

Lemma 2. If

$$\alpha_\nu \geqq 0,\ \beta_\nu \geqq 0\ \ (\nu = 1, 2, \ldots n),$$

$$\sum_1^n \alpha_\nu = A \leqq 1,\quad \sum_1^n \beta_\nu = B \leqq 1,\quad \max_{1\leqq\nu\leqq n}\ \alpha_\nu\beta_\nu = \varepsilon,$$

we have

$$\sum_1^n \alpha_\nu\beta_\nu \leqq \sqrt{\varepsilon}.$$

Proof. The lemma is true when $n = 1$, so let $n > 1$.

(a) Since $\alpha_\nu \leqq 1$, $\beta_\nu \leqq 1$ $(\nu = 1, 2, \ldots n)$, we have $0 \leqq \varepsilon \leqq 1$. When $\varepsilon = 0$ or $\varepsilon = 1$ the lemma is trivial. Hence we can take $0 < \varepsilon < 1$, which implies that at least one product $\alpha_\nu\beta_\nu$ is different from zero, and that A and B are positive.

(b) Putting

$$\alpha'_\nu = \beta'_\nu = \sqrt{\alpha_\nu\beta_\nu} \qquad (\nu = 1, 2, \ldots n),$$

we have

$$\sum_1^n\alpha'_\nu = \sum_1^n\beta'_\nu = \sum_1^n\sqrt{\alpha_\nu\beta_\nu} \leqq \sqrt{AB} \leqq 1,\ \max_{1\leqq\nu\leqq n}\alpha'_\nu\beta'_\nu = \varepsilon,\ \sum_1^n\alpha'_\nu\beta'_\nu = \sum_1^n\alpha_\nu\beta_\nu,$$

hence we may confine ourselves to the case $\alpha_\nu = \beta_\nu$ ($\nu=1, 2,\dots n$), which implies $A = B$.

(c) Let the restrictions of (a) and (b) hold. Putting

$$\alpha'_\nu = A^{-1}\alpha_\nu \qquad (\nu = 1, 2, \dots n),$$

we have

$$\sum_1^n \alpha'_\nu = 1, \quad \max_{1\leqq\nu\leqq n} \alpha'^2_\nu = A^{-2}\varepsilon.$$

Now

$$\sum_1^n \alpha'^2_\nu \leqq A^{-1}\sqrt{\varepsilon}$$

would yield

$$\sum_1^n \alpha^2_\nu = A^2 \sum_1^n \alpha'^2_\nu \leqq A\sqrt{\varepsilon} \leqq \sqrt{\varepsilon},$$

so if the lemma were true for the set $(\alpha'_1, \dots \alpha'_n)$, it would also be true for the set $(\alpha_1, \dots \alpha_n)$. Hence we may confine ourselves to the case $A = 1$.

(d) So now we can put

$$\alpha_\nu \geqq 0 \quad (\nu = 1, 2, \dots n); \qquad \sum_1^n \alpha_\nu = 1; \qquad \max_{1\leqq\nu\leqq n} \alpha^2_\nu = \varepsilon.$$

This yields

$$\max_{1\leqq\nu\leqq n} \alpha_\nu = \sqrt{\varepsilon},$$

hence

$$\sum_1^n \alpha^2_\nu \leqq \sum_1^n \alpha_\nu \max_{1\leqq\nu\leqq n} \alpha_\nu = \sqrt{\varepsilon},$$

which proves the lemma.

It can easily be shown that in the preceding lemma the equality

$$\sum_1^n \alpha_\nu\beta_\nu = \sqrt{\varepsilon}$$

can hold only if $\varepsilon = n^{-2}$ ($n = 1, 2, \dots$).

Lemma 3. Let $\varphi(u)$ be of limited variation in $\langle 0, 1\rangle$. Let

(5.11) $$\varphi(0) = \varphi(1) = 0, \max_{0\leqq t\leqq 1} \left|\int_0^t \varphi(u)du\right| = \varepsilon,$$

and let the total variation of φ be limited by

(5.12) $$\int_0^1 |\, d\varphi(u)\,| \leqq 2.$$

We then have

(5.13) $$\int_0^1 |\,\varphi(u)\,|\, du \leqq \sqrt{2\varepsilon}.$$

PROOF. Without loss of generality we may suppose that the range $\langle 0, 1\rangle$ can so be divided into a finite number of subintervals $i_1, i_2, \dots i_n$, that either $\varphi \geqq 0$ or $\varphi \leqq 0$ within each i_ν, while φ takes a different sign in every pair of consecutive intervals $i_\nu, i_{\nu+1}$. For, as easily can be shown, any function $\Phi(u)$ of limited variation in $\langle 0, 1\rangle$ is the limit of a sequence of functions $\varphi_k(u)$ with the property just mentioned.

Let β_ν be the length of i_ν, and put

$$\left| \int_{i_\nu} \varphi(u)du \right| = \alpha_\nu\beta_\nu \qquad (\nu = 1, 2, \dots n).$$

Hence $\alpha_\nu\beta_\nu \leqq 2\varepsilon$, by (5.11). Moreover we have $\Sigma\,\beta_\nu = 1$. Obviously α_ν is equal to the maximum value of $|\,\varphi\,|$ in i_ν at the utmost, and, since $\varphi(0) = \varphi(1) = 0$, the expression $2\Sigma\alpha_\nu$ is equal to the total variation of φ in $\langle 0, 1\rangle$ at most, hence, by (5.12),

$$\sum_{\nu=1}^{n} \alpha_\nu \leqq 1.$$

Now it follows from Lemma 2:

$$\int_0^1 |\,\varphi(u)\,|\,du = \sum_{\nu=1}^{n} \alpha_\nu\beta_\nu \leqq \sqrt{2\varepsilon}.$$

By (5.11) and (5.12) we have $\varepsilon \leqq 1$. Now put

$$\varphi(u) = \begin{cases} \vartheta_1\sqrt{2\varepsilon} \text{ when } 0 < u < \vartheta_1\sqrt{2\varepsilon}, \\ -\vartheta_2\sqrt{2\varepsilon} \text{ when } \vartheta_1\sqrt{2\varepsilon} < u < (\vartheta_1 + \vartheta_2)\sqrt{2\varepsilon}, \\ (-)^k\sqrt{2\varepsilon} \text{ when } (\vartheta_1+\vartheta_2+k)\sqrt{2\varepsilon}<u<(\vartheta_1+\vartheta_2+k+1)\sqrt{2\varepsilon}, \end{cases}$$

$$\left(k = 0, 1, \dots, \left[\frac{1}{\sqrt{2\varepsilon}} - \frac{1}{\sqrt{2}}\right] - 1\right),$$

where ϑ_1 and ϑ_2 are defined by

$$\vartheta_1 + \vartheta_2 = \frac{1}{\sqrt{2\varepsilon}} - \left[\frac{1}{\sqrt{2\varepsilon}} - \frac{1}{\sqrt{2}}\right],\ \vartheta_2^2 - \vartheta_1^2 = \tfrac{1}{2},$$

and where φ is defined, in the remaining points of the segment (0,1), by

$$\varphi(u) = \tfrac{1}{2}\{\varphi(u + 0) + \varphi(u - 0)\}.$$

This example shows that the coefficient $\sqrt{2}$ in (5.13) cannot be replaced by a smaller one. For, the integral

$$\int_0^1 |\,\varphi(u)\,|\,du$$

attains its maximum value

$$\sqrt{2\varepsilon} + 2\varepsilon(\vartheta_1^2 + \vartheta_2^2 - \vartheta_1 - \vartheta_2)$$

when φ is the function just mentioned, and here the expression $\vartheta_1^2 + \vartheta_2^2 - \vartheta_1 - \vartheta_2$ is bounded and negative (hence equality is only possible in (5.13) when $\varepsilon = 0$). I leave out the proof which is rather long.

Of course Lemma 3 can be given in a less restricted form, where the total variation of φ has an arbitrary positive value.

THEOREM 33. Let $\xi_k > 0$ for $k = 1, 2, \ldots$, and let $\xi_k \to \xi > \frac{1}{2}$ for $k \to \infty$. Let the functions F_n all belong to the class $\{F\}$, and let the limits

$$\lim_{n\to\infty} F_n(\xi_k) = A_k \tag{5.14}$$

exist for $k = 1, 2, \ldots$ We then have

$$\lim_{n\to\infty} F_n(z) = F(z), \tag{5.15}$$

uniformly in any domain $G(\varepsilon, \varrho)$ as defined in Lemma 1. The limit F belongs to $\{F\}$, while the corresponding function χ satisfies

$$\lim_{n\to\infty} \int_0^1 |\chi(t) - \chi_n(t)|\, dt = 0. \tag{5.16}$$

(The theorem also holds when ξ_k and ξ are arbitrary complex numbers, different from the values $z \leqq 0$, but we need not use this generalization, which requires less elementary estimations in part (a) of the proof).

PROOF.

(a) By Theorem 1, the functions F_n are uniformly bounded within a given domain $G(\varepsilon, \varrho)$. Moreover, if $\varrho < \xi$, which can always be supposed, the sequence $\{F_n(z)\}_{n=1}^{\infty}$ converges in an infinite set of G. By the Porter-Vitali theorem (5.15) holds uniformly in G.

(b) Putting

$$c_n(s) = \int_0^1 t^s d\chi_n(t) \qquad (n = 1, 2, \ldots, \sigma \geqq 0), \tag{5.17}$$

we have, by (2.04),

$$c_n(k) = \frac{(-)^k}{k!} F_n^{(k)}(1) \qquad (k = 0, 1, 2, \ldots).$$

By the uniform convergence of the sequence $\{F_n(z)\}_{n=1}^{\infty}$ we have

$$F^{(k)}(1) = \lim_{n\to\infty} F_n^{(k)}(1),$$

hence the limits

$$(5.18)\quad c(k) = \lim_{n\to\infty} c_n(k) = \frac{(-)^k}{k!} F^{(k)}(1) \qquad (k = 0, 1, 2, \ldots)$$

exist. According to the theory of the Hausdorff moment problem, the sequences $\{c_n(k)\}_{k=0}^{\infty}$ are completely monotonic, i.e. the inequalities

$$\sum_{\nu=0}^{N} (-)^{\nu} \binom{N}{\nu} c_n(\nu + k) \geqq 0$$

hold for any $N \geqq 0$, $k \geqq 0$ and $n \geqq 1$. Hence, by (5.18), the sequence $\{c(k)\}_{k=0}^{\infty}$ is also completely monotonic, which implies that the moment problem

$$c(k) = \int_0^1 t^k d\chi(t) \qquad (k = 0, 1, \ldots)$$

has a uniquely determined non-decreasing solution χ. Since $c_n(0) = 1$ for any $n \geqq 1$, we have also $c(0) = 1$ and $\chi(1) = 1$. Now it follows from Theorem 4 and (5.18) that $F(z)$ belongs to the class $\{F\}$.

(c) The expression (5.07) is bounded in the half plane $\sigma \geqq 0$, since $|\, c(s) \,| \leqq c(\sigma) \leqq 1$; it is also holomorphic in the half plane $\sigma > 0$. The same holds for the expressions $c_n(s)$. Moreover we can prove

$$(5.19)\qquad \lim_{n\to\infty} c_n(s) = c(s),$$

uniformly in any rectangle $0 < \sigma_0 \leqq \sigma \leqq \sigma_1$, $|\, \tau \,| \leqq \tau_0$. For, let $0 < \alpha < 1$. Since the sequence $\{F_n(z)\}_1^{\infty}$ converges uniformly on the line $\mathrm{Re}\, z = \alpha$, we can assign to any $\varepsilon > 0$ a $N(\varepsilon)$ such that

$$|\, F_n(z) - F(z) \,| < \varepsilon$$

for $n > N$ and $z = \alpha + iy$. Hence, by (5.08) and (5.10)

$$\left| c_n(s) - c(s) \right| = \frac{1}{2\pi} \left| \int_{\alpha - i\infty}^{\alpha + i\infty} \frac{F_n(z) - F(z)}{(1 - z)^{s+1}} dz \right|$$

$$\leqq \frac{\varepsilon}{2\pi} e^{\frac{1}{2}\pi |\tau|} \int_{-\infty}^{\infty} \frac{dy}{|\, 1 - z \,|^{\sigma+1}}$$

$$< \frac{\varepsilon}{2\pi} e^{\frac{1}{2}\pi\tau_0} \left\{ \int_{|1-z|<1} \frac{dy}{|\, 1 - z|^{\sigma_1 + 1}} + \int_{|1-z|>1} \frac{dy}{|\, 1 - z \,|^{\sigma_0 + 1}} \right\},$$

which proves (5.19).

(d) Now (5.16) remains to be proved. According to Burkill [1], the transformation of Mellin can be applied to the integral (5.07), which yields

$$(5.20)\quad 1-\chi(t)=\int_t^1 d\chi(u)=\frac{1}{2\pi i}\int_{\sigma-i\infty}^{\sigma+i\infty}\frac{c(s)t^{-s}}{s}ds \quad (\sigma>0,\ 0<t<1).$$

If we replace $d\chi(u)$ in this formula by $ud\chi(u)$, and next replace s by $s-1$, we obtain

$$(5.21)\qquad \int_t^1 ud\chi(u)=\frac{1}{2\pi i}\int_{\sigma-i\infty}^{\sigma+i\infty}\frac{c(s)t^{1-s}}{s-1}ds \quad (\sigma>1,\ 0<t<1).$$

Since

$$\int_t^1 \chi(u)du = 1-t\chi(t)-\int_t^1 ud\chi(u)$$

we have, by applying (5.20) and (5.21),

$$\int_t^1 \chi(u)du = 1-t-\frac{1}{2\pi i}\int_{\sigma-i\infty}^{\sigma+i\infty}\frac{c(s)t^{1-s}}{s(s-1)}ds,$$

hence, by (5.17),

$$\int_t^1 \{\chi(u)-\chi_n(u)\}du=-\frac{1}{2\pi i}\int_{\varrho-i\infty}^{\varrho+i\infty}\frac{\{c(s)-c_n(s)\}t^{1-s}}{s(s-1)}ds \quad (\sigma>1,\ 0<t\leqq 1).$$

Now the integrand in the right hand member is holomorphic for $\sigma>0$, with the exception of the pole $s=1$. Since $c(s)-c_n(s)$ is bounded for $\sigma\geqq 0$, this integrand is $O(\tau^{-2})$ as $|\tau|\to\infty$, uniformly in a strip $\frac{1}{2}\leqq\sigma\leqq 2$. By a change of the path of integration we thus get

$$\int_t^1 \{\chi(u)-\chi_n(u)\}du=-\frac{1}{2\pi i}\int_{\frac{1}{2}-i\infty}^{\frac{1}{2}+i\infty}\frac{\{c(s)-c_n(s)t^{1-s}}{s(s-1)}ds-\frac{1}{2\pi i}\int_{(1+)}$$

The integral round about the point $s=1$ can be evaluated, which yields

$$(5.22)\quad \int_t^1 \{\chi(u)-\chi_n(u)\}du=-\frac{1}{2\pi i}\int_{\frac{1}{2}-i\infty}^{\frac{1}{2}+i\infty}\frac{\{c(s)-c_n(s)\}t^{1-s}}{s(s-1)}ds-c(1)+c_n(1) \quad (0<t\leqq 1).$$

According to (c) we have, for any $\varepsilon>0$ and $T>0$,

$$|c(s)-c_n(s)|<\varepsilon$$

for $\frac{1}{2} \leqq \sigma \leqq 1$, $|\tau| \leqq T$ and $n > N(\varepsilon, T)$. Hence

$$\left| \int_t^1 \{\chi(u) - \chi_n(u)\} du \right| \leqq \frac{1}{\pi} \left| \int_0^{\frac{1}{2}+iT} \frac{\{c(s) - c_n(s)\} t^{1-s}}{s(s-1)} ds \right|$$
$$+ \frac{1}{\pi} \left| \int_{\frac{1}{2}+iT}^{\frac{1}{2}+i\infty} \right| + |c(1) - c_n(1)| \leqq O(\varepsilon\sqrt{t}) + O(T^{-1}\sqrt{t}) + \varepsilon$$

for $\varepsilon \to 0$, $T \to \infty$ and $0 < t \leqq 1$. Putting $T = \varepsilon^{-1}$ we thus obtain

$$\left| \int_t^1 \{\chi(u) - \chi_n(u)\} du \right| = O(\varepsilon) \tag{5.23}$$

for $0 < t \leqq 1$ and $n > N(\varepsilon, \varepsilon^{-1})$. Since the left hand member is continuous for $t = 0$, (5.23) also holds for $0 \leqq t \leqq 1$, which yields

$$\max_{0 \leqq t \leqq 1} \left| \int_t^1 \{\chi(u) - \chi_n(u)\} du \right| \leqq A\varepsilon,$$

where A is a positive constant, and $n > N(\varepsilon, \varepsilon^{-1})$. Now

$$\chi(0) - \chi_n(0) = \chi(1) - \chi_n(1) = 0,$$

while the total variation of $\chi(t) - \chi_n(t)$ in $\langle 0, 1 \rangle$ is equal to 2 at the utmost. Hence, by Lemma 3,

$$\int_0^1 |\chi(t) - \chi_n(t)| \, dt \leqq \sqrt{2A\varepsilon},$$

so (5.16) is true.

Using the theorem of Helly, we would have obtained

$$\lim_{n \to \infty} \chi_n(t) = \chi(t)$$

for every t where χ is continuous. It can easily be shown that this result is equivalent to (5.16), but I prefer the latter statement from the standpoint of intuistionistic mathematics.

Theorem 34. The functions R and R^*, defined by (5.02), belong to the class $\{F\}$.

Proof. All functions R_n and R_n^* belong to $\{F\}$. Hence the theorem is right, by (5.02) and Theorem 33.

Theorem 35. The conditions (5.01) are sufficient for the existence of a solution of problem (e).

Proof. By Theorem 34 the functions R and R^* are solutions of problem (e), hence there is at least one solution.

§ 6. Questions of Uniqueness.

In this section it will always be supposed that problem (e) is solvable, i.e. that (5.01) holds. If the solutions R and R^* are identical, the problem is determined, by (5.05); if not, the indeterminacy of the problem is a tautology. Hence the question whether a problem (e) is determined or not comes to the question whether $R(x) - R^*(x)$ is identically zero.

I begin by replacing the polynomials $P_n, \dots Q_n^*$ of § 3 by other ones, which satisfy recurrence formulae that are somewhat simpler. Let

$$\eta_n = \frac{P_{n-1}^*(0)P_n^*(0)}{x_1 \dots x_n Q_{n-1}(0) Q_n(0)} \tag{6.01}$$

for $n > 1$, hence, by (4.41),

$$\eta_n = \frac{P_{n-1}^*(0)Q_{n-1}(x_n)}{x_n Q_{n-1}(0) Q_{n-1}^*(x_n)} \cdot \frac{\vartheta_n^*}{\vartheta_n}. \tag{6.02}$$

The first factor in the right hand member is positive, and independent of ϑ_n; hence $\eta_n > 0$. The converse also holds: to any set $\{\eta_n\}_2^\infty$ of positive values there is a corresponding set of values ϑ_n that satisfy $0 < \vartheta_n < 1$. Putting

$$\varphi_n(x) = \frac{P_n(x)}{Q_n(0)},\ \psi_n(x) = \frac{Q_n(x)}{Q_n(0)},\ \varphi_n^*(x) = \frac{P_n^*(x)}{P_n^*(0)},\ \psi_n^*(x) = \frac{Q_n^*(x)}{P_n^*(0)}, \tag{6.03}$$

we have, by (4.28),

$$\varphi_1(x) = \psi_1(x) = \varphi_1^*(x) = \psi_1^*(x) = 1. \tag{6.04}$$

Moreover, the recurrence formula

$$\begin{cases} \varphi_n(x) = \varphi_{n-1}(x) + \eta_n \varphi_{n-1}^*(x), & \varphi_n^*(x) = \varphi_{n-1}^*(x) + (x_n\eta_n)^{-1} x \varphi_{n-1}(x), \\ \psi_n(x) = \psi_{n-1}(x) + \eta_n x \psi_{n-1}^*(x), & \psi_n^*(x) = \psi_{n-1}^*(x) + (x_n\eta_n)^{-1} \psi_{n-1}(x), \end{cases} \tag{6.05}$$

can be obtained from (4.04) . . . (4.07) for any $n > 1$. By (4.15) we have

$$\varphi_n^*(x)\psi_n(x) - x\varphi_n(x)\psi_n^*(x) = \prod_1^n \left(1 - \frac{x}{x_k}\right), \tag{6.06}$$

while (4.29) yields

$$R_n(x) = \frac{\varphi_n(x)}{\psi_n(x)},\quad R_n^*(x) = \frac{\varphi_n^*(x)}{x\psi_n^*(x)}, \tag{6.07}$$

hence

$$R_n^*(x) - R_n(x) = \frac{\prod_1^n \left(1 - \frac{x}{x_k}\right)}{x\psi_n(x)\psi_n^*(x)}.$$

By Theorem 31 we now have

$$R^*(z) - R(z) = \lim_{n\to\infty} \frac{\prod_1^n \left(1 - \frac{z}{x_k}\right)}{z\psi_n(z)\psi_n^*(z)} \tag{6.08}$$

for all values of z, except the values $z \leqq 0$.

Lemma 4. Let $A_1 = B_1 = 1$; let $a_n > 0$, $b_n > 0$ for $n=2, 3, \ldots$; let

$$A_n = A_{n-1} + a_n B_{n-1},\ B_n = B_{n-1} + b_n A_{n-1}.$$

We then have

$$A_n \geqq 1 + \sum_2^n a_k,\ B_n \geqq 1 + \sum_2^n b_k \qquad (n = 1, 2, \ldots).$$

(A confusion with the values $a_n = F(x_n)$ might be excluded).

Proof. The lemma is true for $n = 1$. Assuming the lemma holds for $n - 1$, we have

$$A_n \geqq 1 + \sum_2^{n-1} a_k + a_n(1 + \sum_2^{n-1} b_k) \geqq 1 + \sum_2^n a_k,$$

$$B_n \geqq 1 + \sum_2^{n-1} b_k + b_k(1 + \sum_2^{n-1} a_k) \geqq 1 + \sum_2^n b_k.$$

Lemma 5. Let the assumptions of the former lemma hold; let

$$\sum_2^\infty (a_k + b_k) = \infty.$$

We then have $A_nB_n \to \infty$ as $n \to \infty$.

Proof. According to Lemma 4 we have

$$A_nB_n \geqq (1 + \sum_2^n a_k)(1 + \sum_2^n b_k) > \sum_2^n (a_k + b_k),$$

hence A_nB_n increases indefinitely as $n \to \infty$.

Theorem 36. If the series

$$\sum_2^\infty \left(\eta_n + \frac{1}{x_n\eta_n}\right) \tag{6.09}$$

diverges, problem (e) is determined.

Proof. According to our assumption, at least one of the series

$$\sum_2^\infty \eta_n,\ \sum_2^\infty \frac{1}{x_n\eta_n},$$

is divergent, hence

$$\sum_{2}^{\infty}\left(x\eta_n + \frac{1}{x_n\eta_n}\right) = \infty$$

for a given positive value of x. Putting

$$A_n = \psi_n(x),\quad B_n = \psi_n^*(x),\quad a_n = x\eta_n,\quad b_n = (x_n\eta_n)^{-1},$$

we have, by Lemma 5,

$$\psi_n(x)\psi_n^*(x) \to \infty \qquad (n \to \infty).$$

Moreover, if $n \to \infty$, the product

$$\prod_{1}^{n}\left(1 - \frac{x}{x_k}\right)$$

tends to a finite limit, which is either zero or different from zero. In both cases the right hand member in (6.08) will be zero for $z = x > 0$.

THEOREM 37. If

$$\sum_{1}^{\infty} \frac{1}{\sqrt{x_n}} = \infty, \tag{6.10}$$

the problem (e) is determined.

PROOF. Since

$$\eta_n + \frac{1}{x_n\eta_n} = \frac{1}{\sqrt{x_n}}\left(\eta_n\sqrt{x_n} + \frac{1}{\eta_n\sqrt{x_n}}\right) \geqq \frac{2}{\sqrt{x_n}},$$

(6.10) implies the series (6.09) to be divergent.

Since problem (e) is determined if (6.10) holds, it will so much the more be determined if

$$\sum_{1}^{\infty} \frac{1}{x_n} = \infty. \tag{1.04}$$

This is a special case of a well-known result of Hausdorff and Feller, which has already been mentioned in the introduction.

LEMMA 6. Let $A_1 = B_1 = 1$; let a_n and b_n be arbitrary complex numbers; let

$$A_n = A_{n-1} + a_nB_{n-1},\quad B_n = B_{n-1} + b_nA_{n-1}$$

for $n = 2, 3, \ldots$ We then have

$$|A_n| \leqq \prod_{2}^{n}(1 + |a_k| + |b_k|),\quad |B_n| \leqq \prod_{2}^{n}(1 + |a_k + |b_k|) \qquad (n = 1, 2, \ldots).$$

PROOF. If $n = 1$ the lemma is true. Assuming the lemma is true for $n - 1$, we have

$$|A_n| \leqq |A_{n-1}| + |a_n B_{n-1}| \leqq (1+|a_n|) \prod_2^{n-1} (1+|a_k|+|b_k|) \leqq \prod_2^{n} (1+|a_k|+|b_k|),$$

and similarly for $|B_n|$.

LEMMA 7. Let the assumptions of the former lemma hold. We then have

$$|A_m - A_n| \leqq \prod_2^m (1+|a_k|+|b_k|) - \prod_2^n (1+|a_k|+|b_k|),$$

$$|B_m - B_n| \leqq \prod_2^m (1+|a_k|+|b_k|) - \prod_2^n (1+|a_k|+|b_k|)$$

for $m \geqq n \geqq 1$.

PROOF. If $m = n$ the lemma holds; suppose it holds for $(m-1, n)$, where $m-1 \geqq n$. According to the former lemma we have

$$|A_m - A_n| = |A_{m-1} + a_m B_{m-1} - A_n| \leqq |A_{m-1} - A_n| + |a_m B_{m-1}|$$

$$\leqq \prod_2^{m-1} (1+|a_k|+|b_k|) - \prod_2^n (1+|a_k|+|b_k|) + |a_m| \prod_2^{m-1} (1+|a_k|+|b_k|)$$

$$\leqq \prod_2^m (1+|a_k|+|b_k|) - \prod_2^n (1+|a_k|+|b_k|),$$

and similarly for $|B_m - B_n|$.

THEOREM 38. If the series (6.09) converges, the sequences $\{\varphi_n(z)\}_{n=1}^{\infty}, \ldots \{\psi_n^*(z)\}_{n=1}^{\infty}$ are uniformly convergent within any circle $|z| \leqq \varrho$. The limits

$$(6.11)\quad \varphi(z) = \lim_{n\to\infty} \varphi_n(z),\ \varphi^*(z) = \lim_{n\to\infty} \varphi_n^*(z),\ \psi(z) = \lim_{n\to\infty} \psi_n(z),\ \psi^*(z) = \lim_{n\to\infty} \psi_n^*(z),$$

are entire transcendental functions of z.

PROOF. Putting

$$A_n = \varphi_n(z),\ B_n = \varphi_n^*(z),\ a_n = \eta_n,\ b_n = (x_n \eta_n)^{-1} z,$$

we have, by Lemma 7,

$$|\varphi_m(z) - \varphi_n(z)| \leqq \prod_2^n \left(1 + \eta_k + \frac{|z|}{x_k \eta_k}\right) \left\{ \prod_{n+1}^m \left(1 + \eta_k + \frac{|z|}{x_k \eta_k}\right) - 1 \right\},$$

$$|\varphi_m^*(z) - \varphi_n^*(z)| \leqq \prod_2^n \left(1 + \eta_k + \frac{|z|}{x_k \eta_k}\right) \left\{ \prod_{n+1}^m \left(1 + \eta_k + \frac{|z|}{x_k \eta_k}\right) - 1 \right\},$$

where $m > n$. For $|z| \leqq \varrho$ the first factor in the right hand member of these inequalities is uniformly bounded, since

$$\sum_2^{\infty} \left(\eta_n + \frac{\varrho}{x_n \eta_n}\right) < \infty.$$

For the same reason the second factor tends to zero as $n \to \infty$, which proves the uniform convergence of the sequences $\{\varphi_n\}$ and $\{\varphi_n^*\}$. Hence φ and φ^* are entire functions. The statements concerning $\{\psi_n\}$ and $\{\psi_n^*\}$ can be obtained in the same way.

By (6.05) the functions $\varphi(x), \ldots, \psi^*(x)$ are increasing more rapidly than any polynomial of arbitrarily given degree, when $x \to \infty$; hence these functions are transcendental.

THEOREM 39. If the series (6.09) converges, the functions R and R^* are meromorphic and not identical.

PROOF. If the series (6.09) is convergent, we have, by Theorems 31 and 38,

$$R(z) = \frac{\varphi(z)}{\psi(z)}, \quad R^*(z) = \frac{\varphi^*(z)}{z\psi^*(z)}, \tag{6.12}$$

for all z except the values $z \leqq 0$. By analytic continuation (6.12) holds for any z, so R and R^* must be meromorphic. Moreover we have, by (6.08),

$$R^*(z) - R(z) = \frac{\prod_1^\infty \left(1 - \frac{z}{x_n}\right)}{z\psi(z)\,\psi^*(z)}. \tag{6.13}$$

Since

$$\sum_2^\infty \frac{1}{x_n} < \sum_2^\infty \frac{1}{\sqrt{x_n}} \leqq \tfrac{1}{2}\sum_2^\infty \left(\eta_n + \frac{1}{x_n \eta_n}\right) < \infty,$$

the product

$$\prod_1^\infty \left(1 - \frac{z}{x_n}\right)$$

is not identically zero, hence R and R^* are not identical.

Summarizing the results of Theorems 36 and 39 we can now say:

THEOREM 40. In order that problem (e) be determined, a necessary and sufficient condition is

$$\sum_2^\infty \left(\eta_n + \frac{1}{x_n \eta_n}\right) = \infty.$$

Some remarks on the indeterminate case may be inserted here.

Evidently there is no criterion which, analogous to Theorem 37, only depends on the values x_n, and which implies the indeterminacy of problem (e). For, whatever the values x_n may be, we can always take

$$\sum_2^\infty \eta_n = \infty,$$

which implies a determined problem, according to Theorem 40.

By Theorem 40 it is also evident that both the determined and the indeterminate case of problem (e) can occur. Hence, if we change our problem by requiring that solutions only have to be completely monotonic, and thus leave the condition (1.05) out, this new problem can still more be indeterminate. I do not know whether an example of this case is known, but at any rate an explicit example of an indeterminate problem (e) will be given in the next section.

THEOREM 41. The zeros of $\varphi(z), \ldots \psi^*(z)$ are simple and negative. The zeros of φ, as well as those of ψ^*, are separated both by the zeros of φ^* and by those of ψ, and conversely. Compared to φ and ψ^*, the functions φ^* and ψ have the zeros with the least absolute values. The functions $\varphi, \ldots \psi^*$ are of genus zero.

PROOF. By (6.03) the zeros of $\varphi_n, \ldots \psi_n^*$ coincide with those of $P_n, \ldots Q_n^*$; they will be denoted as in Theorem 22. According to the proof of this theorem we have

$$(6.14) \qquad \begin{cases} \alpha_{n,i} < \alpha_{n+1,i} < 0, & \alpha_{n,i}^* < \alpha_{n+1,i}^* < 0, \\ \beta_{n,i} < \beta_{n+1,i} < 0, & \beta_{n,i}^* < \beta_{n+1,i}^* < 0. \end{cases}$$

Hence the limits

$$(6.15) \qquad \begin{cases} \alpha_i = \lim\limits_{n\to\infty} \alpha_{n,i}, & \beta_i = \lim\limits_{n\to\infty} \beta_{n,i}, \\ \alpha_i^* = \lim\limits_{n\to\infty} \alpha_{n,i}^*, & \beta_i^* = \lim\limits_{n\to\infty} \beta_{n,i}^* \end{cases} \qquad (i=1, 2, \ldots)$$

exist; they are all real and $\leqq 0$. Moreover it is evident, by (6.14), that the values $\alpha_i, \ldots \beta_i^*$ and the cluster points of the sets $\{\alpha_{n,i}\}, \ldots \{\beta_{n,i}^*\}$ are identical. Hence, by a well-known theorem of Hurwitz [1], the values $\alpha_i, \ldots \beta_i^*$ coincide with all zeros of $\varphi, \ldots \psi^*$.

Next we have, by (4.23) and (6.15),

$$(6.16) \quad \alpha_i \leqq \alpha_i^* \leqq \alpha_{i-1}, \ \alpha_i \leqq \beta_i \leqq \alpha_{i-1}, \ \beta_i^* \leqq \alpha_i^* \leqq \beta_{i-1}^*, \ \beta_i^* \leqq \beta_i \leqq \beta_{i-1}^*.$$

Now, by (6.12) and (6.13),

$$(6.17) \qquad \varphi^*(z)\psi(z) - z\varphi(z)\psi^*(z) = \prod_1^\infty \left(1 - \frac{z}{x_n}\right),$$

and since the right hand member is different from zero for $z \leqq 0$, the expressions $\varphi^*(z)\psi(z)$ and $z\varphi(z)\psi^*(z)$ cannot be zero at the same time when $z \leqq 0$. Hence (6.16) implies

$$(6.18) \quad \alpha_i < \alpha_i^* < \alpha_{i-1}, \ \alpha_i < \beta_i < \alpha_{i-1}, \ \beta_i^* < \alpha_i^* < \beta_{i-1}^*, \ \beta_i^* < \beta_i < \beta_{i-1}^*,$$

which proves that the zeros of $\varphi, \ldots \psi^*$ are simple. Moreover

(6.18) proves that the statements about the separation of the zeros hold, and that φ^* and ψ have the zeros with the least absolute value. Putting $z = 0$ we have, by (6.17),

$$\varphi^*(0)\psi(0) > 0,$$

hence $\alpha_1^* < 0$ and $\beta_1 < 0$, hence all zeros of $\varphi, \ldots \psi^*$ are negative.

We still have to discuss the genus of $\varphi, \ldots \psi^*$. Take $x > 0$. By (6.14) we have $|\alpha_{n,i}| > |\alpha_{n+1,i}|$, hence

$$1 - \frac{x}{\alpha_{n,i}} = 1 + \frac{x}{|\alpha_{n,i}|} < 1 + \frac{x}{|\alpha_{n+1,i}|} = 1 - \frac{x}{\alpha_{n+1,i}},$$

which implies that the product

$$\prod_{i=1}^{j} \left(1 - \frac{x}{\alpha_{n,i}}\right)$$

is an increasing function both of n and j. Since

$$\frac{\varphi(x)}{\varphi(0)} = \lim_{n\to\infty} \frac{\varphi_n(x)}{\varphi_n(0)} = \lim_{n\to\infty} \lim_{j\to\infty} \prod_{i=1}^{j} \left(1 - \frac{x}{\alpha_{n,i}}\right),$$

we thus obtain

$$\frac{\varphi(x)}{\varphi(0)} = \lim_{j\to\infty} \lim_{n\to\infty} \prod_{i=1}^{j} \left(1 - \frac{x}{\alpha_{n,i}}\right) = \lim_{j\to\infty} \prod_{1}^{j} \left(1 - \frac{x}{\alpha_i}\right) = \prod_{1}^{\infty} \left(1 - \frac{x}{\alpha_i}\right),$$

where the infinite product in the right hand member is convergent. Hence

$$\sum_{1}^{\infty} \frac{1}{|\alpha_i|} < \infty,$$

and the function φ must be of genus zero. For similar reasons ψ, φ^* and ψ^* are of genus zero.

We thus have, for any finite value of z,

$$(6.19) \quad \begin{cases} \varphi(z) = \varphi(0) \prod_{1}^{\infty} \left(1 - \frac{z}{\alpha_i}\right), & \psi(z) = \psi(0) \prod_{1}^{\infty} \left(1 - \frac{z}{\beta_i}\right), \\ \varphi^*(z) = \varphi^*(0) \prod_{1}^{\infty} \left(1 - \frac{z}{\alpha_i^*}\right), & \psi^*(z) = \psi^*(0) \prod_{1}^{\infty} \left(1 - \frac{z}{\beta_i^*}\right), \end{cases}$$

where, by (6.04) and (6.05),

$$(6.20) \quad \varphi(0)=1+\sum_{2}^{\infty} \eta_n, \; \psi(0)=1, \; \varphi^*(0)=1, \; \psi^*(0)=1+\sum_{2}^{\infty}(x_n\eta_n)^{-1}$$

An indeterminate problem (e) remains indeterminate, if one of the conditions $F(x_n) = a_n$ is left out. This leads to the question what becomes of an indeterminate problem, if a condition $F(\xi)=\alpha$, where ξ is positive and different from all values x_n, is added. I begin with

LEMMA 8. Let the polynomials $\Phi_n(x)$, $\Phi_n^*(x)$, $\Psi_n(x)$ and $\Psi_n^*(x)$ satisfy

$$\Phi_1(x) = \Phi_1^*(x) = \Psi_1(x) = \Psi_1^*(x) = 1$$

and

$$\Phi_n = \Phi_{n-1} + \lambda_n \Phi_{n-1}^*, \qquad \Phi_n^* = \Phi_{n-1}^* + \mu_n x \Phi_{n-1},$$
$$\Psi_n = \Psi_{n-1} + \lambda_n x \Psi_{n-1}^*, \qquad \Psi_n^* = \Psi_{n-1}^* + \mu_n \Psi_{n-1},$$

for $n > 1$, where λ_n and μ_n are arbitrary positive values. We then have:

(a) the zeros of these polynomials are simple and negative;

(b) the zeros of Φ_n as well as those of Ψ_n^* are separated both by the zeros of Φ_n^* and by those of Ψ_n; conversely, the zeros of Φ_n^* and Ψ_n are separated by those of Φ_n and Ψ_n^*; compared to the zeros of Φ_n and Ψ_n^*, the polynomials Φ_n^* and Ψ_n have the zeros with the least absolute values.

PROOF. The proof is nearly the same as that of Theorem 22, which is only a particular case of the present lemma.

THEOREM 42. Let the condition $F(\xi) = \alpha$ be added to an indeterminate problem of type (e), where ξ is positive and different from all values x_n. If α is included (in the strict sense) between $R(\xi)$ and $R^*(\xi)$, the new problem will still be indeterminate. If α is equal to $R(\xi)$ or to $R^*(\xi)$, the new problem has a unique solution $R(x)$ resp. $R^*(x)$.

PROOF. Since α is included in the strict sense between $R(\xi)$ and $R^*(\xi)$, it will still more be included between $R_n(\xi)$ and $R_n^*(\xi)$, by Theorem 30. Hence the value

$$(6.21) \quad \zeta_n = -\frac{\varphi_n(\xi) - \alpha\psi_n(\xi)}{\varphi_n^*(\xi) - \alpha\xi\psi_n^*(\xi)} = -\frac{\psi_n(\xi)}{\xi\psi_n^*(\xi)} \cdot \frac{R_n(\xi) - \alpha}{R_n^*(\xi) - \alpha}$$

must be finite and positive. Now we can introduce a set of polynomials $\bar{\varphi}_n(x), \ldots \bar{\psi}_n^*(x)$ by putting

$$(6.22) \quad \begin{cases} \bar{\varphi}_n = \varphi_n + \zeta_n \varphi_n^*, & \bar{\psi}_n = \psi_n + \zeta_n x \psi_n^*, \\ \bar{\varphi}_n^* = \varphi_n^* + (\xi\zeta_n)^{-1} x \varphi_n, & \bar{\psi}_n^* = \psi_n^* + (\xi\zeta_n)^{-1}\psi_n. \end{cases}$$

By (4.31) and (6.07) we have

$$\frac{\bar{\varphi}_n(x_k)}{\bar{\psi}_n(x_k)} = \frac{\bar{\varphi}_n^*(x_k)}{x_k\bar{\psi}_n^*(x_k)} = a_k \qquad (k = 1, 2, \ldots n),$$

while, by (6.21),

(6.23) $$\frac{\bar{\varphi}_n(\xi)}{\bar{\psi}_n(\xi)} = \frac{\bar{\varphi}_n^*(\xi)}{\xi\bar{\psi}_n^*(\xi)} = \alpha.$$

Hence, if we put

$$\overline{R}_n(x) = \frac{\bar{\varphi}_n(x)}{\bar{\psi}_n(x)}, \quad \overline{R}_n^*(x) = \frac{\bar{\varphi}_n^*(x)}{x\bar{\psi}_n^*(x)},$$

the functions $\overline{R}_n$ and $\overline{R}_n^*$ take the values required for the arguments $x_1, \ldots x_n$ and ξ. By (6.22) and Lemma 8 these functions belong to the class $\{F\}$. Now, if $n \to \infty$, ζ_n tends to a positive value ζ, since both the numerator and the denominator in (6.21) tend to finite values different from zero. Thus the limits

$$\bar{\varphi}(z) = \lim_{n\to\infty} \bar{\varphi}_n(z) = \varphi(z) + \zeta\varphi^*(z), \ldots,$$
$$\bar{\psi}^*(z) = \lim_{n\to\infty} \bar{\psi}_n^*(z) = \psi^*(z) + (\xi\zeta)^{-1}\psi(z)$$

exist for all values of z. Putting

$$\overline{R}(z) = \frac{\bar{\varphi}(z)}{\bar{\psi}(z)}, \quad \overline{R}^*(z) = \frac{\bar{\varphi}^*(z)}{z\bar{\psi}^*(z)},$$

we have $\overline{R}_n(x) \to \overline{R}(x)$, $\overline{R}_n^*(x) \to \overline{R}^*(x)$ for any $x > 0$ and $n \to \infty$. Hence, by Theorem 33, $\overline{R}$ and $\overline{R}^*$ belong to $\{F\}$. The functions $\overline{R}$ and $\overline{R}^*$ are solutions of the original problem (e); by (6.23) they also satisfy the condition

$$\overline{R}(\xi) = \overline{R}^*(\xi) = \alpha.$$

Finally, $\overline{R}$ and $\overline{R}^*$ cannot be identical, which is immediately evident by (6.06) and (6.22). Hence the first part of the theorem is true.

As to the second part, we may suppose $\alpha = R(\xi)$ and $x_{2\iota} < \xi < x_{2\iota+1}$ in order to fix the ideas; the other cases can be treated in the same way. Now, if we add the condition $F(\xi) = \alpha$, the problem remains solvable, since $F = R$ is a solution. By Theorem 30 any solution of the new problem will be included (in the wide sense) between two such solutions $\overline{R}(x)$ and $\overline{R}^*(x)$, whether these be identical or not. Hence, by Theorem 30,

$$\overline{R}^*(x) \geqq R(x) \geqq \overline{R}(x), \qquad (x_{2\iota} < x < \xi),$$
$$\overline{R}^*(x) \leqq R(x) \leqq \overline{R}(x) \qquad (\xi < x < x_{2\iota+1}).$$

Now $\overline{R}$ and $\overline{R}^*$ are solutions of the original problem too. Hence

$$\overline{R}(x) \geqq R(x), \qquad \overline{R}^*(x) \geqq R(x) \quad (x_{2\iota} < x < x_{2\iota+1}),$$

which yields

$$\overline{R}(x) = R(x) \qquad (x_{2\iota} < x < \xi),$$
$$\overline{R}^*(x) = R(x) \qquad (\xi < x < x_{2\iota+1}).$$

Since R, $\overline{R}$ and $\overline{R}^*$ are holomorphic for $x > 0$, these functions are identical, which proves the statement.

Theorem 42 contains a slight improvement of Theorem 30. For we now have:

THEOREM 43. Any solution $F(x)$ of an indeterminate problem of type (e), which is not identical with R or R^*, is included (in the strict sense) between $R(x)$ and $R^*(x)$, for any positive x different from the values x_n.

The results of Theorem 42 can be extended without difficulty to the case where a finite number of conditions $F(\xi_\nu) = \alpha_\nu$ ($\nu=1, 2, \ldots N$) is added. Yet, the fact that there are special cases, where an indeterminate problem becomes determined if a single condition is added, only leaves room for generalizations of Theorem 42 that are rather cumbrous. Perhaps it is useful to introduce a notion here which seems to be new. If a determined problem of type (e) can be made indeterminate by leaving out a certain set of N conditions $F(x_\nu) = a_\nu$ ($\nu = \nu_1, \nu_2, \ldots \nu_N$), but not by leaving out less than N conditions, the number N will be called the *degree of definiteness* of the problem.

THEOREM 44. Let a problem of degree 1 become indeterminate when the condition $F(x_\mu) = a_\mu$ is left out. The problem will also become indeterminate when any condition $F(x_\nu) = a_\nu$ is left out.

PROOF. In what follows any problem of type (e) will be denoted by $P(x)$, where (x) is the set of the values x for which $F(x)$ is given. Moreover, it will always be supposed that the sequence $\{x_n\}_1^\infty$ belongs to (x); for the sake of concision the x_n will be dropped in the notation. Thus, in the case of $P(\xi_1, \xi_2)$ the values of F will be prescribed for $x = x_1, x_2, \ldots, \xi_1, \xi_2$, whereas in the case of P these values are given for $x = x_1, x_2, \ldots$ only. It will also be supposed that the arguments $x_1, x_2, \ldots, \xi_1, \xi_2, \ldots$ are all different and positive. Finally we shall put $F(\xi_i) = \alpha_i$ whenever the value $F(\xi_i)$ is given.

Now let $P(\xi_1, \xi_2)$ be determined, and let $P(\xi_1)$ be indeterminate. In order to prove the theorem it will suffice to show that $P(\xi_2)$ is indeterminate. Since $P(\xi_1)$ is indeterminate, P will be still more so. Hence all solutions of P will be included (in the wide sense) between two non-identical solutions $R(x)$ and $R^*(x)$ of P, and in the same way all solutions of $P(\xi_1)$ are included between $\overline{R}(x)$ and $\overline{R}^*(x)$. According to Theorem 42, α_1 must be included in the strict sense between $R(\xi_1)$ and $R^*(\xi_1)$. Hence $\overline{R}$ and $\overline{R}^*$ cannot be identical with R and R^*, so it follows from Theorem 43 that $\overline{R}(\xi_2)$ and $\overline{R}^*(\xi_2)$ are included in the strict sense between $R(\xi_2)$ and $R^*(\xi_2)$. Now $P(\xi_1, \xi_2)$ is determined, while $P(\xi_1)$ is indeterminate; hence, by Theorem 42, α_2 must be equal either to $\overline{R}(\xi_2)$ or to $\overline{R}^*(\xi_2)$, so α_2 is also included in the strict sense between $R(\xi_2)$ and $R^*(\xi_2)$. Now $P(\xi_2)$ must be indeterminate, again by Theorem 42, since P is indeterminate.

THEOREM 45. A problem of degree N becomes indeterminate if N arbitrarily chosen conditions $F(x_\nu) = a_\nu$ are left out.

PROOF. According to the former theorem the statement is true when $N = 1$; let it hold for any degree $< N$. Using our previous notation, we can suppose, without loss of generality, that $P(\xi_1, \ldots \xi_{2N})$ is of degree N, and that $P(\xi_1, \ldots \xi_N)$ is indeterminate.

Let $1 \leqq k \leqq N$. According to our assumptions and to the definition of degree, $P(\xi_1, \ldots, \xi_N, \xi_{N+k+1}, \ldots, \xi_{2N})$ is of degree $N - k$, hence $P(\xi_1, \ldots, \xi_k, \xi_{N+k+1}, \ldots, \xi_{2N})$ is indeterminate (according to our hypothesis). Hence it remains to show that $P(\xi_{N+1}, \ldots, \xi_{2N})$ is indeterminate.

Now $P(\xi_1, \ldots \xi_{2N-1})$ is of degree $N - 1$ (according to the definition of degree), hence $P(\xi_1, \xi_{N+1}, \ldots \xi_{2N-1})$ is indeterminate (according to our hypothesis), while $P(\xi_1, \xi_{N+1}, \ldots \xi_{2N})$ is determined (according to the definition of degree), hence $P(\xi_{N+1}, \ldots \xi_{2N})$ is indeterminate (according to Theorem 44).

Now it is natural to extend the notion of degree of definiteness to all problems of type (e). According to this generalized notion of degree, indeterminate problems are of zero degree, while determined problems are either of (finite) positive or of infinite degree. An introduction of negative degrees is not to the purpose, since a further classification of the indeterminate problems of type (e) seems to be impossible. Any indeterminate problem can be made determined by the addition of an arbitrary number, or even of an infinite number, of appropriate conditions $F(\xi_k) = \alpha_k$, which is immediately evident by Theorem 42.

In the case of a positive degree N, the number $N - 1$ can be interpreted, according to Theorem 45, as the number of superfluous equations of the system $F(x_n) = a_n$ $(n = 1, 2, \ldots)$, which determines F. One might connect the notion of degree with the theory of infinite matrices, but this would give rise to questions that are beyond the scope of this paper.

THEOREM 46. The solution of a problem of type (e) of finite positive degree is meromorphic.

PROOF. Let N be the degree of the problem considered. First leave $N-1$ conditions $F(x_n) = a_n$ out. By Theorem 45, this gives rise to a determined problem of unity degree, which has the same solution F as the original problem. Now this new problem becomes indeterminate if one more condition is left out. This can only happen, by Theorem 42, if F is equal to one of the functions R or R^* that correspond to the final indeterminate problem. Since the latter are meromorphic, F must be meromorphic too.

Some questions concerning the notion of infinite degree must still be viewed here.

THEOREM 47. If a problem of type (e) is of infinite degree, a denumbrable set of conditions

$$F(x_{i_\nu}) = a_{i_\nu} \qquad (\nu = 1, 2, \ldots)$$

can be left out, and still the problem remains determined.

PROOF. Let, in the original problem P, $\{x_n\}_1^\infty$ be the sequence of abscissae x for which the values $F(x_n) = a_n$ are given. Since P is supposed to be determined, the inequality

$$| R_n(x) - R_n^*(x) | < \varepsilon$$

will hold for any $\varepsilon > 0$, $x \geqq 1$ and $n \geqq N(\varepsilon)$, if $N(\varepsilon)$ is an appropriate function of ε. Put $\varepsilon = 2^{-1}$ and $i_1 = N(2^{-1}) + 1$, and leave the condition $F(x_{i_1}) = a_{i_1}$ out, which gives rise to a new problem P_1. Instead of the sequences $\{R_n\}_1^\infty$ and $\{R_n^*\}_1^\infty$ of functions that limit the solution of P, there will be two other sequences $\{R_{n,1}\}_1^\infty$ and $\{R_{n,1}^*\}_1^\infty$ that correspond to P_1. Evidently we have

$$R_{n,1} = R_n,\ R_{n,1}^* = R_n^* \qquad (n = 1, 2, \ldots, i_1 - 1),$$
$$| R_{n,1}(x) - R_{n,1}^*(x) | < 2^{-1} \quad (x \geqq 1,\ n \geqq i_1 - 1).$$

According to our assumption, P_1 is determined. Hence the inequality

$$| R_{n,1}(x) - R_{n,1}^*(x) | < \varepsilon$$

will hold for $\varepsilon > 0$, $x \geqq 1$ and $n \geqq N_1(\varepsilon)$, if $N_1(\varepsilon)$ is appropriately chosen. Put $\varepsilon = 2^{-2}$ and $i_2 = N_1(2^{-2}) + 1$, and leave the condition $F(x_{i_2}) = a_{i_2}$ out. This gives rise to a problem P_2 with the corresponding sequences $\{R_{n,2}\}_1^\infty$ and $\{R_{n,2}^*\}_1^\infty$. We now have

$$R_{n,2} = R_{n,1},\ R_{n,2}^* = R_{n,1}^* \quad (n = 1, 2, \ldots, i_2 - 1),$$
$$| R_{n,2}(x) - R_{n,2}^*(x) | < 2^{-2} \quad (x \geqq 1,\ n \geqq i_2 - 1).$$

This process can be carried on. After having left out k conditions

(6.24) $$F(x_{i_\nu}) = a_{i_\nu},$$

where $\nu = 1, 2, \ldots k$ and

$$i_\nu = N_{\nu-1}(2^{-\nu}) + 1,$$

we obtain a problem P_k with the corresponding sequences $\{R_{n,k}\}_{n=1}^{\infty}$ and $\{R^*_{n,k}\}_{n=1}^{\infty}$, which satisfy

$$R_{n,k} = R_{n,k-1}, \; R^*_{n,k} = R^*_{n,k-1} \quad (n = 1, 2, \ldots, i_k - 1),$$
$$| R_{n,k}(x) - R^*_{n,k}(x) | < 2^{-k} \quad (x \geqq 1, \; n \geqq i_k - 1).$$

Let P_∞ be the problem that comes into being if we leave out the denumbrable set of conditions (6.24), where now $\nu = 1, 2, \ldots$ Evidently the solutions of P_∞ are included between the elements of equal order of the sequences $\{\overline{R}_n\}_1^\infty$, $\{\overline{R}^*_n\}_1^\infty$, where

$$\overline{R}_n = R_{n,k}, \; \overline{R}^*_n = \overline{R}^*_{n,k} \quad (n = 1, 2, \ldots, i_k - 1)$$

for $k = 1, 2, \ldots$. Hence P_∞ is determined, which proves the theorem.

Next, the question must be put what becomes of a problem of infinite degree, when a denumbrable set of conditions $F(x_n) = a_n$ is left out. The degree of the new problem may be zero, positive or infinite again. Evidently two cases can be distinguished here: either a problem of infinite degree may be transformed into another problem of positive degree by leaving out an appropriate denumbrable set of conditions, or this may be impossible. The problems of the first kind will be said to belong to the class A, while those of the second kind will belong to the class B. Neither of these two classes is empty; for, all problems that have a non-meromorphic solution belong to B, according to Theorem 46, whereas the problems that arise from an indeterminate problem by the addition of the conditions $R(\xi_\nu) = \alpha_\nu$ resp. $R^*(\xi_\nu) = \alpha_\nu$, where $\nu = 1, 2, \ldots$, all belong to the class A. One might conjecture that the set of problems of infinitive degree, which have a meromorphic solution, is identical with the class A. However, I have not as yet solved this very interesting question.

It has already been said in the introduction that completely monotonic functions can be represented by Newton series. If

(1.04) $$\sum_1^\infty \frac{1}{x_n} = \infty$$

holds, any function f, which is completely monotonic in (x_0, ∞), where $x_0 < x_1$, and which satisfies the conditions $f(x_n) = a_n$ $(n = 1, 2, \ldots)$, can be represented by the expression

$$(6.25)\qquad \sum_{n=1}^{\infty} [a_1, a_2, \ldots a_n](x - x_1)(x - x_2) \ldots (x - x_{n-1}),$$

where the divided differences $[a_1, \ldots a_n]$ are defined by

$$(6.26)\qquad [a_n] = a_n,\ [a_{i_1} \ldots a_{i_n}] = \frac{[a_{i_2} \ldots a_{i_n}] - [a_{i_1} \ldots a_{i_{n-1}}]}{x_{i_n} - x_{i_1}}.$$

Since the functions F are completely monotonic, (6.25) will also represent $F(x)$ if (1.04) holds, which implies that we have to do with a determined case of problem (e). The question can be put what becomes of the series (6.25) if we give up the condition (1.04). An answer is given by

Theorem 48. Let the problem $F(x_n) = a_n$ $(n = 1, 2, \ldots)$ be solvable. The series

$$(6.27)\qquad \sum_{n=1}^{\infty} [a_1 \ldots a_n](z - x_1)(z - x_2) \ldots (z - x_{n-1})$$

will then be convergent in the half plane Re $z > 0$. In order that the series represents a function of the class $\{F\}$ it is necessary and sufficient that (1.04) hold (in which case the series represents the unique solution F).

Proof. Let

$$(1.05)\qquad F(z) = \int_0^1 \frac{d\chi(t)}{1 - t + tz}$$

be a solution of the problem considered. We can exclude the case where F is a constant; hence there will be at least one value of t, different from zero, where χ increases. For convenience put

$$t^{-1} - 1 = \lambda.$$

Moreover, let

$$(6.28)\qquad F(z) = \sum_{k=1}^{n} [a_1 \ldots a_k](z - x_1)(z - x_2) \ldots (z - x_{k-1}) + \varrho_n(z).$$

Now we obtain from (1.05) and (6.26), by induction,

$$[a_1 \ldots a_n] = (-)^{n-1} \int_0^1 \frac{d\chi(t)}{t \prod_{k=1}^{n} (\lambda + x_k)},$$

hence we have, by (6.28),

$$(6.29)\quad \varrho_n(z) = \prod_1^n \left(1 - \frac{z}{x_k}\right) \cdot \int_0^1 \frac{d\chi(t)}{(1 - t + tz)\prod_{k=1}^n \left(1 + \frac{\lambda}{x_k}\right)},$$

and here the integral in the right hand member converges in any domain $G(\varepsilon, \varrho)$ as $n \to \infty$, as can be easily seen (for the definition of G see Lemma 1).

Now we must distinguish between the case where

$$(6.30)\qquad \sum_1^\infty \frac{1}{x_n}$$

diverges and the case where this series is convergent.

(a) If (6.30) diverges, the product

$$\prod_1^n \left(1 - \frac{z}{x_k}\right)$$

is divergent as $n \to \infty$, when Re $z < 0$. When Re $z > 0$, this product converges and is equal to zero, hence

$$\varrho(z) = \lim_{n\to\infty} \varrho_n(z) = 0.$$

The series (6.27) is then convergent and, by (6.28), it represents the unique solution F.

(b) If (6.30) is convergent, we have

$$(6.31)\quad \varrho(z) = \lim_{n\to\infty} \varrho_n(z) = \prod_1^\infty \left(1 - \frac{z}{x_k}\right) \cdot \int_0^1 \frac{d\chi(t)}{(1 - t + tz)\prod_{k=1}^\infty \left(1 + \frac{\lambda}{x_k}\right)}$$

for any z, if we exclude the values $z \leqq 0$ (since (6.30) holds in any domain G). By (6.28), the series (6.27) will be convergent again, even in any domain G. Yet, as $\varrho(z)$ cannot be identically zero now, the series (6.27) cannot represent the solution F. Now $F(z)$ might be *any* solution of problem (e), hence (6.27) represents no solution whatever of this problem.

We can replace (6.28) by

$$\sum_{k=1}^\infty [a_1 \ldots a_k](z - x_1)(z - x_2) \ldots (z - x_{k-1}) = F(z) - \varrho(z).$$

Since the Newton series is everywhere convergent in case (b), the singularities of F and ϱ on the half line $z \leqq 0$ must neutralize each other in the right hand member.

§ 7. Applications.

In this section different qucstions will be treated that are in some way connected with our subject. I begin by giving some examples of functions that belong to $\{F\}$.

If $0 < \alpha < 1$, the function $z^{-\alpha}$ provides such an example. For we have, when $|z-1| < 1$,

$$z^{-\alpha}=\{1+(z-1)\}^{-\alpha}=\sum_{k=0}^{\infty}(-)^k\frac{\alpha(\alpha+1)\dots(\alpha+k-1)}{k!}(z-1)^k,$$

while

$$\frac{\alpha(\alpha+1)\dots(\alpha+k-1)}{k!}=\frac{\Gamma(k+\alpha)}{\Gamma(k+1)\Gamma(\alpha)}=\{\Gamma(\alpha)\Gamma(1-\alpha)\}^{-1}\frac{\Gamma(k+\alpha)\Gamma(1-\alpha)}{(k+1)}$$
$$=\frac{\sin\pi\alpha}{\pi}\int_0^1 t^{k+\alpha-1}(1-t)^{-\alpha}dt,$$

hence, by Theorem 4,

$$z^{-\alpha}=\frac{\sin\pi\alpha}{\pi}\int_0^1\frac{t^{\alpha-1}(1-t)^{-\alpha}}{1-t+tz}\,dt \qquad (0<\alpha<1). \tag{7.01}$$

Evidently this is a special case of well-known formulae in the theory of the Gamma-function. Integrating (7.01) we obtain

$$\int_0^1 z^{-\alpha}d\alpha=\frac{z-1}{z\log z}=\int_0^1\frac{\chi'(t)dt,}{1-t+tz}, \tag{7.02}$$

where

$$\chi'(t)=\frac{1}{\pi}\int_0^1 t^{\alpha-1}(1-t)^{-\alpha}\sin\pi\alpha\,d\alpha=\frac{1}{t(1-t)\,(\pi^2+\varrho^2)}\left(\varrho=\log\frac{t}{1-t}\right).$$

Hence the function (7.02) also belongs to $\{F\}$. According to Theorem 4 it can be expanded in a series

$$\frac{z-1}{z\log z}=\sum_{k=0}^{\infty}(-)^k c_k(z-1)^k,$$

valid for $|z-1| < 1$. Hence we have

$$\sum_{n=0}^{\infty}(-)^n(z-1)^n=\sum_{j=0}^{\infty}\frac{(-)^j(z-1)^j}{j+1}\sum_{k=0}^{\infty}(-)^k c_k(z-1)^k,$$

or

$$\sum_{k=0}^{n}\frac{c_k}{n-k+1}=1,$$

which yields

$$c_0 = 1,\ c_1 = \tfrac{1}{2},\ c_2 = \tfrac{5}{12},\ c_3 = \tfrac{3}{8},\ c_4 = \tfrac{251}{720}, \dots$$

Since the values c_k are moments of Hausdorff, the sequence $\{c_k\}_0^\infty$ must be completely monotonic.

Taking $\alpha = \frac{1}{2}$ we obtain from (7.01)

$$\frac{1}{\sqrt{z}} = \frac{1}{\pi}\int_0^1 \frac{du}{(1-u+uz)\sqrt{u(1-u)}},$$

hence

$$\frac{1}{\sqrt{1-t^2+zt^2}} = \frac{1}{\pi}\int_0^1 \frac{du}{(1-ut^2+ut^2z)\sqrt{u(1-u)}} = \frac{1}{\pi}\int_0^{t^2} \frac{dv}{(1-v+vz)\sqrt{v(t^2-v)}},$$

where $0 \leqq t \leqq 1$ and where the square root in the left hand member is to be taken positive when $z > 0$. This being so, the integral

$$F_1(z) = \frac{2}{\pi}\int_0^1 \frac{dt}{\sqrt{(1-t^2)(1-t^2+zt^2)}} \tag{7.03}$$

can be written in the form

$$F_1(z) = \frac{2}{\pi^2}\int_0^1 \frac{dt}{\sqrt{1-t^2}}\int_0^{t^2} \frac{dv}{(1-v+vz)\sqrt{v(t^2-v)}} = \int_0^1 \frac{\chi_1'(v)dv}{1-v+vz},$$

where

$$\chi_1'(v) = \frac{2}{\pi^2\sqrt{v}}\int_{\sqrt{v}}^1 \frac{dt}{\sqrt{(1-t^2)(t^2-v)}} = \frac{2}{\pi^2\sqrt{v}}\int_0^1 \frac{d\tau}{\sqrt{(1-\tau^2)(1-v'\tau^2)}} \quad (v'=1-v).$$

Hence F_1 belongs to $\{F\}$ since $F_1(1) = 1$. A more elegant way to obtain this result is as follows. If $|z-1| < 1$, F_1 can be expressed by a Taylor series

$$F_1(z) = \sum_{k=0}^{\infty} (-)^k c_k (z-1)^k,$$

where

$$c_k = \left\{\frac{1.3\dots(2k-1)}{2.4\dots(2k)}\right\}^2.$$

Now we have

$$\sqrt{c_k} = \frac{1.3\dots(2k-1)}{2.4\dots(2k)} = \frac{2}{\pi}\int_0^1 \frac{t^{2k}dt}{\sqrt{1-t^2}},$$

so the sequence $\{\sqrt{c_k}\}_0^\infty$ is a set of moments of Hausdorff. The same must hold for the sequence $\{c_k\}_0^\infty$, hence F_1 belongs to $\{F\}$ by Theorem 4.

According to Theorem 40, a problem of type (e) will be determined if and only if

$$\sum_{n=2}^{\infty}\left(\lambda_n+\frac{1}{\lambda_n}\right)\frac{1}{\sqrt{x_n}}=\infty, \tag{7.04}$$

where

$$\lambda_n=\eta_n\sqrt{x_n}\qquad (n=2,3,\ldots). \tag{7.05}$$

The case $\lambda_n=1$ $(n=2,3,\ldots)$ is easy. By (6.05) we have

$$\varphi_n(x)=\varphi_{n-1}(x)+\frac{\varphi^*_{n-1}(x)}{\sqrt{x_n}},\ \varphi^*_n(x)=\varphi^*_{n-1}(x)+\frac{x\varphi_{n-1}(x)}{\sqrt{x_n}},$$

$$\psi_n(x)=\psi_{n-1}(x)+\frac{x\psi^*_{n-1}(x)}{\sqrt{x_n}},\ \psi^*_n(x)=\psi^*_{n-1}(x)+\frac{\psi_{n-1}(x)}{\sqrt{x_n}},$$

hence, by (6.04),

$$\varphi^*_n(x)=\psi_n(x),\ \psi^*_n(x)=\varphi_n(x)\quad (n=1,2,\ldots),$$

which yields

$$\psi_n(x)+\sqrt{x}\varphi_n(x)=\left(1+\sqrt{\frac{x}{x_n}}\right)\{\psi_{n-1}(x)+\sqrt{x}\varphi_{n-1}(x)\}=\prod_1^n\left(1+\sqrt{\frac{x}{x_k}}\right),$$

$$\psi_n(x)-\sqrt{x}\varphi_n(x)=\left(1-\sqrt{\frac{x}{x_n}}\right)\{\psi_{n-1}(x)-\sqrt{x}\varphi_{n-1}(x)\}=\prod_1^n\left(1-\sqrt{\frac{x}{x_k}}\right),$$

and

$$\varphi_n(x)=\frac{1}{2\sqrt{x}}\left\{\prod_1^n\left(1+\sqrt{\frac{x}{x_k}}\right)-\prod_1^n\left(1-\sqrt{\frac{x}{x_k}}\right)\right\},$$

$$\psi_n(x)=\tfrac{1}{2}\left\{\prod_1^n\left(1+\sqrt{\frac{x}{x_k}}\right)+\prod_1^n\left(1-\sqrt{\frac{x}{x_k}}\right)\right\},$$

so the functions R_n and R^*_n can be expressed by the formulae

$$(7.06)\quad \begin{cases} R_n(x)=\dfrac{\varphi_n(x)}{\psi_n(x)}=\dfrac{1}{\sqrt{x}}\cdot\dfrac{\prod_1^n\left(1+\sqrt{\frac{x}{x_k}}\right)-\prod_1^n\left(1-\sqrt{\frac{x}{x_k}}\right)}{\prod_1^n\left(1+\sqrt{\frac{x}{x_k}}\right)+\prod_1^n\left(1-\sqrt{\frac{x}{x_k}}\right)}, \\ R^*_n(x)=\dfrac{\varphi^*_n(x)}{x\psi^*_n(x)}=\dfrac{1}{\sqrt{x}}\cdot\dfrac{\prod_1^n\left(1+\sqrt{\frac{x}{x_k}}\right)+\prod_1^n\left(1-\sqrt{\frac{x}{x_k}}\right)}{\prod_1^n\left(1+\sqrt{\frac{x}{x_k}}\right)-\prod_1^n\left(1-\sqrt{\frac{x}{x_k}}\right)}. \end{cases}$$

By (7.04) the problem will be determined if and only if

$$\sum_{2}^{\infty} \frac{1}{\sqrt{x_n}} = \infty.$$

In this case we obviously have

$$R(x) = R^*(x) = \frac{1}{\sqrt{x}}.$$

On the other hand we have in the indeterminate case, by (7.06),

$$R(x) = \frac{1}{\sqrt{x}} \cdot \frac{\prod_{1}^{\infty}\left(1+\sqrt{\frac{x}{x_k}}\right) - \prod_{1}^{\infty}\left(1-\sqrt{\frac{x}{x_k}}\right)}{\prod_{1}^{\infty}\left(1+\sqrt{\frac{x}{x_k}}\right) + \prod_{1}^{\infty}\left(1-\sqrt{\frac{x}{x_k}}\right)},$$

$$R^*(x) = \frac{1}{\sqrt{x}} \cdot \frac{\prod_{1}^{\infty}\left(1+\sqrt{\frac{x}{x_k}}\right) + \prod_{1}^{\infty}\left(1-\sqrt{\frac{x}{x_k}}\right)}{\prod_{1}^{\infty}\left(1+\sqrt{\frac{x}{x_k}}\right) - \prod_{1}^{\infty}\left(1-\sqrt{\frac{x}{x_k}}\right)},$$

and these solutions can also be characterized by

$$R(x)R^*(x) = x^{-1},$$

$$R^*(x) - R(x) = \frac{4}{\sqrt{x}} \cdot \frac{\prod_{1}^{\infty}\left(1-\frac{x}{x_k}\right)}{\prod_{1}^{\infty}\left(1+\sqrt{\frac{x}{x_k}}\right)^2 - \prod_{1}^{\infty}\left(1-\sqrt{\frac{x}{x_k}}\right)^2}, \quad R(x) > 0 \quad (x > 0).$$

Now let us consider the particular case

$$x_n = n^4 \qquad (n = 1, 2, \ldots),$$

which corresponds to an indeterminate problem. Putting

$$s = \pi\sqrt[4]{z} \qquad (z = x + iy,\ s = \sigma + it), \tag{7.07}$$

where $s > 0$ will be taken when $z > 0$, we obtain

$$\prod_{1}^{\infty}\left(1-\sqrt{\frac{x}{x_k}}\right) = \prod_{k=1}^{\infty}\left(1-\frac{\sigma^2}{\pi^2 k^2}\right) = \frac{\sin\sigma}{\sigma},$$

$$\prod_{1}^{\infty}\left(1+\sqrt{\frac{x}{x_k}}\right) = \prod_{k=1}^{\infty}\left(1+\frac{\sigma^2}{\pi^2 k^2}\right) = \frac{e^{\sigma}-e^{-\sigma}}{2\sigma},$$

hence

$$R(x) = \frac{\pi^2}{\sigma^2} \cdot \frac{e^{\sigma}-e^{-\sigma}-2\sin\sigma}{e^{\sigma}-e^{-\sigma}+2\sin\sigma}, \quad R^*(x) = \frac{\pi^2}{\sigma^2} \cdot \frac{e^{\sigma}-e^{-\sigma}+2\sin\sigma}{e^{\sigma}-e^{-\sigma}-2\sin\sigma},$$

or

$$R(x) = \pi^2 \sum_{k=0}^{\infty} \frac{(\pi^4 x)^k}{(4k+3)!} : \sum_{k=0}^{\infty} \frac{(\pi^4 x)^k}{(4k+1)!},$$

$$R^*(x) = \sum_{k=0}^{\infty} \frac{(\pi^4 x)^k}{(4k+1)!} : \pi^2 x \sum_{k=0}^{\infty} \frac{(\pi^4 x)^k}{(4k+3)!}.$$

It is immediately evident, by the last formulae, that $R(z)$ and $R^*(z)$ are meromorphic functions. R is holomorphic in the origin, whereas R^* has a pole of the first order there with a residue $6\pi^{-2}$. The other poles of R and R^* are identical with the zeros of the functions

$$\sum_{k=0}^{\infty} \frac{(\pi^4 z)^k}{(4k+1)!}, \quad \sum_{k=0}^{\infty} \frac{(\pi^4 z)^k}{(4k+3)!}.$$

According to Theorem 34 these must be simple and negative. By (7.07), these zeros correspond to the roots of the equations

$$e^s - e^{-s} + 2 \sin s = 0, \quad e^s - e^{-s} - 2 \sin s = 0,$$

and, in order to compute the first, we thus can put $s = (1+i)\varrho$, where $\varrho > 0$, which yields

$$(7.08) \quad e^{2\sigma} = \frac{\cos \varrho - \sin \varrho}{\cos \varrho + \sin \varrho} = \cot\left(\varrho + \frac{\pi}{4}\right), \quad e^{2\varrho} = \frac{\cos \varrho + \sin \varrho}{\cos \varrho - \sin \varrho} = \operatorname{tg}\left(\varrho + \frac{\pi}{4}\right).$$

If the increasing sequences of the positive roots of the equations (7.08) are denoted by $\{\varrho_n\}_1^\infty$ and $\{\varrho_n^*\}_1^\infty$, we evidently have

$$\varrho_n = n\pi - \frac{\pi}{4} + o(1), \quad \varrho_n^* = n\pi + \frac{\pi}{4} + o(1),$$

as $n \to \infty$. Of course more accurate asymptotic formulae can be given.

If F_1, F_2 belong to $\{F\}$, other functions with the same property can be obtained by a transformation. Some of these transformations are trivial, e.g.:

$$\begin{aligned} F(z) &= F_1(z+\alpha) : F_1(1+\alpha) && (\alpha > 0), \\ F(z) &= F_1(\alpha z) : F_1(\alpha) && (\alpha > 0), \\ F(z) &= \alpha F_1(z) + (1-\alpha) F_2(z) && (0 < \alpha < 1). \end{aligned}$$

Less trivial is

THEOREM 49. Let F_1 belong to $\{F\}$. Let

$$(7.09) \quad F(z) = z^{-1} F_1(z^{-1}).$$

Then F belongs to $\{F\}$.

PROOF. Let

$$F_1(z) = \int_0^1 \frac{d\chi_1(t)}{1 - t + tz}.$$

Putting $u = 1 - t$, $\chi(u) = 1 - \chi_1(t)$, we get

$$z^{-1}F_1(z^{-1}) = \int_0^1 \frac{d\chi(u)}{1 - u + uz},$$

which proves the theorem.

THEOREM 50. Let

$$F(z)F_1(z^{-1}) = 1. \tag{7.10}$$

When F_1 belongs to $\{F\}$, the same will hold for F.

PROOF. First let $F_1(z)$ be a rational function, which, if represented by (4.39), contains n positive parameters. According to the symbolism of § 4, F_1 can be denoted either by $r_n(z)$, when F_1 is regular in the origin, or by $r_n^*(z)$, when the origin is a pole. Let $\{x_k\}_1^n$ be an increasing sequence of arbitrary positive values; put $F_1(x_k) = a_k$ $(k = 1, 2, \ldots n)$. According to Theorems 26 and 27 the function F_1 will then be identical with $R_n(z)$ resp. with $R_n^*(z)$.

By (7.10) we thus obtain the expressions

$$\frac{Q_n(z^{-1})}{P_n(z^{-1})}, \quad \frac{Q_n^*(z^{-1})}{zP_n^*(z^{-1})}$$

for F; it must be shown that these belong to $\{F\}$. Let us consider the expressions

$$\left\{\begin{array}{ll} p_{2m}(z) = z^{m-1}Q_{2m}^*(z^{-1}), & q_{2m}(z) = z^m P_{2m}^*(z^{-1}), \\ p_{2m}^*(z) = z^m Q_{2m}(z^{-1}), & q_{2m}^*(z) = z^{m-1}P_{2m}(z^{-1}), \\ p_{2m+1}(z) = z^m Q_{2m+1}(z^{-1}), & q_{2m+1}(z) = z^m P_{2m+1}(z^{-1}), \\ p_{2m+1}^*(z) = z^m Q_{2m+1}^*(z^{-1}), & q_{2m+1}^*(z) = z^m P_{2m+1}^*(z^{-1}), \end{array}\right. \tag{7.11}$$

which have the following properties.

(a) $p_n, \ldots q_n^*$ are polynomials of z, which are positive for $z = 0$. This is an immediate consequence of Theorem 22(a) and(b).

(b) The degrees of $p_n, \ldots q_n^*$ are equal to those of $P_n, \ldots Q_n^*$, since $P_n(0), \ldots Q_n^*(0)$ are positive and since

$$[P_{2m}] = [Q_{2m}^*], \; [P_{2m}^*] = [Q_{2m}], \; [P_{2m+1}] = [Q_{2m+1}], \; [P_{2m+1}^*] = [Q_{2m+1}^*].$$

(c) The zeros of $p_n, \ldots q_n^*$ are simple and negative, since these are the reciprocals of the zeros of $P_n, \ldots Q_n^*$ (in another arrangement).

(d) The zeros of p_n as well as those of q_n^* are separated both by the zeros of p_n^* and q_n, and conversely. Compared to p_n and q_n^*, the polynomials p_n^* and q_n have the zeros with the least absolute values. For, by Theorem 22(b) and (d), the following couples of polynomials have zeros that separate one another:

$$(P_{2m}\, \underline{P_{2m}^*}), \quad (P_{2m}, \underline{Q_{2m}}), \quad (Q_{2m}^*, \underline{P_{2m}^*}), \quad (Q_{2m}^*, \underline{Q_{2m}}),$$
$$(\underline{P_{2m+1}}, P_{2m+1}^*), \ (\underline{P_{2m+1}}, Q_{2m+1}), \ (\underline{Q_{2m+1}^*}, P_{2m+1}^*), \ (\underline{Q_{2m+1}^*}, Q_{2m+1}).$$

In each of these cases the polynomial that has the zero with the greatest absolute value has been underlined. We thus have, by (7.11), the following set of corresponding couples whose zeros separate one another:

$$(q_{2m}^*, \underline{q_{2m}}), \quad (q_{2m}^*, \underline{p_{2m}^*}), \quad (p_{2m}, \underline{q_{2m}}), \quad (p_{2m}, \underline{p_{2m}^*}),$$
$$(\underline{q_{2m+1}}, q_{2m+1}^*), \ (\underline{q_{2m+1}}, p_{2m+1}), \ (\underline{p_{2m+1}^*}, q_{2m+1}^*), \ (\underline{p_{2m+1}^*}, q_{2m+1}),$$

where the underlined polynomials now have the zero with the least absolute value, which proves the statement (d).

This being so, the expressions

$$\frac{Q_{2m}(z^{-1})}{P_{2m}(z^{-1})} = \frac{p_{2m}^*(z)}{zq_{2m}^*(z)}, \quad \frac{Q_{2m}^*(z^{-1})}{zP_{2m}^*(z^{-1})} = \frac{p_{2m}(z)}{q_{2m}(z)},$$
$$\frac{Q_{2m+1}(z^{-1})}{P_{2m+1}(z^{-1})} = \frac{p_{2m+1}(z)}{q_{2m+1}(z)}, \quad \frac{Q_{2m+1}^*(z^{-1})}{zP_{2m+1}^*(z^{-1})} = \frac{p_{2m+1}^*(z)}{zq_{2m+1}^*(z)},$$

must belong to $\{F\}$, for similar reasons as have been used in the proof of Theorem 23(c). Hence the theorem is true when F_1 is rational.

In order to prove the theorem when F_1 is non-rational, we can take $x_n = n$ and $a_n = F_1(n)$. The corresponding problem (e) will be solvable, since it has the solution F_1. Moreover it is determined, since

$$\sum_1^\infty \frac{1}{n} = \infty.$$

Hence we have at any rate

$$F_1(z) = \lim_{n\to\infty} R_n(z) \tag{7.12}$$

for $z > 0$. Let

$$S_n(z) = 1 : R_n(z^{-1}).$$

As it has just been shown, $S_n(z)$ belongs to $\{F\}$ for any n. Moreover, by (7.10) and (7.12), the limit

$$\lim_{n\to\infty} S_n(z) = F(z)$$

exists for $z > 0$. Hence F belongs to $\{F\}$, by Theorem 33.

The function

$$F(z) = \int_0^1 \frac{dt}{1 - t + tz} = \frac{\log z}{z - 1}$$

belongs to $\{F\}$. Hence

$$1 : F(z^{-1}) = \frac{z - 1}{z \log z}$$

also belongs to $\{F\}$, in accordance with (7.02).

By Theorems 49 and 50 we also have the transformation

$$\{F_1(z)\}^{-1} = zF(z), \tag{7.13}$$

which provides other couples of functions belonging to $\{F\}$. For the rest, (7.13) is substantially equivalent to a theorem of Kaluza [1]. Now F can be expanded, by Theorem 4, in a Taylor series

$$F(z) = \sum_{n=0}^{\infty} (-)^n c_n (z - 1)^n,$$

valid for $|z - 1| < 1$. Hence we have, by (1.13), (2.06) and (7.13),

$$\frac{1}{F_1(z)} = \{1 + (z - 1)\} \sum_{n=0}^{\infty} (-)^n c_n (z - 1)^n = 1 - \sum_{n=1}^{\infty} (-)^n (c_{n-1} - c_n)(z - 1)^n.$$

Putting

$$\frac{1}{F_1(z)} = 1 - \sum_{n=1}^{\infty} A_n (1 - z)^n \qquad (|z - 1| < 1), \tag{7.14}$$

we thus have, by (2.06),

$$A_n = c_{n-1} - c_n = \int_0^1 t^{n-1}(1 - t) d\chi(t).$$

Hence the sequence $\{A_n\}_1^{\infty}$ is a set of moments of Hausdorff, which implies that it is completely monotonic.

As an example, we can take for F_1 the elliptic integral (7.03), which belongs to $\{F\}$. Hence, if we replace z in (7.14) by $1 - z$, we get the development

$$1 : \frac{2}{\pi} \int_0^1 \frac{dt}{\sqrt{(1 - t^2)(1 - zt^2)}} = 1 - \sum_1^{\infty} A_n z^n \qquad (|z| < 1),$$

where the sequence of coefficients A_n must be completely monotonic. This is a generalization of a result of van Veen [1], who proved that these coefficients are positive and decreasing.

If, in (7.10), we require $F = F_1$, we obtain the functional equation

$$F(z)F(z^{-1}) = 1, \tag{7.15}$$

which is satisfied by

$$F(z) = z^{-\alpha} \qquad (0 \leqq \alpha \leqq 1).$$

It would be interesting to know whether there are other solutions F of (7.15), but as yet I have not solved this problem.

No use has been made, in the preceding investigation, of continued fractions. Of course we could have done so, since the expressions $R_n(z)$ and $R_n^*(z)$ are approximants of odd resp. of even order of the continued fraction

$$\frac{1|}{|1} + \frac{z-1|}{|1} + \frac{1|}{|\eta_2} + \frac{1-\dfrac{z}{x_2}\Big|}{\Big|(x_2\eta_2)^{-1}z} + \frac{1|}{|\eta_3} + \frac{1-\dfrac{z}{x_3}\Big|}{\Big|(x_3\eta_3)^{-1}z} + \cdots, \tag{7.16}$$

which is an immediate consequence of (6.04), (6.05) and (6.07). If the corresponding problem (e) has a unique solution F, (7.16) obviously converges for any z, different from the values $z \leqq 0$; there will also be convergence within the open intervals of the half line $z \leqq 0$, where F is holomorphic.

When the corresponding problem (e) is indeterminate, (7.16) diverges, except for the values $z = x_n$ $(n = 1, 2, \ldots)$. However, by contraction of (7.16) we can obtain the continued fractions

$$\frac{1|}{|1} - \frac{\eta_2(1-z)|}{|1+\eta_2} - \frac{\dfrac{\eta_3}{\eta_2}\left(1-\dfrac{z}{x_2}\right)\Big|}{\Big|1+\dfrac{\eta_3}{\eta_2}} - \frac{\dfrac{\eta_4}{\eta_3}\left(1-\dfrac{z}{x_3}\right)\Big|}{\Big|1+\dfrac{\eta_4}{\eta_3}} - \cdots, \tag{7.17}$$

$$\frac{1|}{|z} + \frac{\dfrac{z}{x_2\eta_2}(1-z)\Big|}{\Big|1+\dfrac{z}{x_2\eta_2}} - \frac{\dfrac{x_2\eta_2}{x_3\eta_3}\left(1-\dfrac{z}{x_2}\right)\Big|}{\Big|1+\dfrac{x_2\eta_2}{x_3\eta_3}} - \frac{\dfrac{x_3\eta_3}{x_4\eta_4}\left(1-\dfrac{z}{x_3}\right)\Big|}{\Big|1+\dfrac{x_3\eta_3}{x_4\eta_4}} - \cdots, \tag{7.18}$$

which have the sequences of approximants $\{R_n(z)\}_{n=1}^{\infty}$ and $\{R_n^*(z)\}_{n=1}^{\infty}$ respectively, and thus converge for every z, the poles of R resp. of R^* excepted. Evidently (7.17) and (7.18) will diverge, as a rule, for all values $z \leqq 0$ when the problem (e) is determined. These continued fractions can also be obtained as an immediate consequence of the recurrence formulae

$$7.19 \quad \begin{cases} \varphi_n(z) = \left(1 + \dfrac{\eta_n}{\eta_{n-1}}\right)\varphi_{n-1}(z) - \dfrac{\eta_n}{\eta_{n-1}}\left(1 - \dfrac{z}{x_{n-1}}\right)\varphi_{n-2}(z), \\[2ex] \psi_n(z) = \left(1 + \dfrac{\eta_n}{\eta_{n-1}}\right)\psi_{n-1}(z) - \dfrac{\eta_n}{\eta_{n-1}}\left(1 - \dfrac{z}{x_{n-1}}\right)\psi_{n-2}(z), \\[2ex] \varphi_n^*(z) = \left(1 + \dfrac{x_{n-1}\eta_{n-1}}{x_n\eta_n}\right)\varphi_{n-1}^*(z) - \dfrac{x_{n-1}\eta_{n-1}}{x_n\eta_n}\left(1 - \dfrac{z}{x_{n-1}}\right)\varphi_{n-2}^*(z), \\[2ex] \psi_n^*(z) = \left(1 + \dfrac{x_{n-1}\eta_{n-1}}{x_n\eta_n}\right)\psi_{n-1}^*(z) - \dfrac{x_{n-1}\eta_{n-1}}{x_n\eta_n}\left(1 - \dfrac{z}{x_{n-1}}\right)\psi_{n-2}^*(z), \end{cases}$$

which in their turn follow from (6.05). Properly speaking, (7.17) and (7.18) are only another way of writing the system (7.19).

Conversely, for any increasing sequence $\{x_n\}_1^\infty$, where $x_1 = 1$ and $x_n \to \infty$ as $n \to \infty$, and for any positive sequence $\{\eta_n\}_2^\infty$, the continued fractions (7.17) and (7.18) represent a function F, and so does (7.16) if it is convergent. Since there are always determined problems (e) that have an arbitrarily given function F as a unique solution, the following theorem holds:

THEOREM 51. Any function $F(z)$ can be represented for any z, save perhaps for the values $z \leqq 0$, and even in an infinity of ways, by continued fractions of the types (7.16), . . . (7.18).

If $F(z)$ is given along a line $\mathrm{Re}\, z = \alpha$ $(0 < \alpha < 1)$, we can state an explicit formula for χ, thus solving the problem of the inversion of (1.05) in a stricter sense than it had be done in Theorem 8 and (2.17). For, combining (5.08) and (5.20) we have

$$(7.20) \quad \chi(t) = 1 + \frac{1}{4\pi^2}\int_{\sigma-i\infty}^{\sigma+i\infty} \frac{t^{-s}}{s}\, ds \int_{\alpha-i\infty}^{\alpha+i\infty} \frac{F(z)}{(1-z)^{s+1}}\, dz \quad (\sigma>0,\ 0<\alpha<1),$$

for $0 < t < 1$. I have not succeeded in finding an expression for χ that contained only one integration.

Next, some remarks concerning the limit

$$(7.21) \qquad u = \lim_{x \to +\infty} F(x)$$

may be added. Apart from the trivial case $F(x) \equiv 1$ we evidently have $0 \leqq u < 1$, and any value u contained in $\langle 0, 1)$ actually occurs. Besides, all solutions F of a problem of type (e) have the same limit value u, since

$$u = \lim_{n\to+\infty} a_n.$$

For any value of u there are corresponding problems (e) that are determined. On the other hand, we have shown by an example that an indeterminate problem can correspond to $u = 0$. Now it is clear by the transformation

$$F(x) = \frac{F_1(x) + \alpha}{1 + \alpha} \qquad (\alpha > 0)$$

that at least one indeterminate problem (e) corresponds to any given u. Hence the question whether a problem of type (e) has a unique solution, has nothing to do with the value of u.

While stating problem (e) in the beginning of this paper, we have used the sequences $\{x_n\}_1^\infty$ and $\{a_n\}_1^\infty$. Now it is possible, by the transformations (6.02) and (7.05), to express the same problem in terms of the sequences $\{x_n\}_1^\infty$ and $\{\lambda_n\}_2^\infty$, and we can ask what becomes of the condition $u = 0$ in this new formulation. The answer is contained in the following theorem:

THEOREM 52. Let

$$\pi_{2k} = \lambda_{2k}, \ \pi_{2k+1} = \lambda_{2k+1}^{-1}. \tag{7.22}$$

In order that a problem of type (e) be determined, it is necessary and sufficient that

$$\sum_2^\infty \frac{\pi_n + \pi_n^{-1}}{\sqrt{x_n}} = \infty.$$

In order that its solutions tend to zero as $x \to \infty$, it is necessary and sufficient that

$$\sum_2^\infty (\pi_2\pi_3 \ldots \pi_{n-1})^2 \pi_n \sqrt{x_n} = \infty. \tag{7.23}$$

PROOF. Evidently the first statement is only a transcription of (7.04). In order to prove the second part of the theorem, we need an expression for the limits u_n and u_n^* defined by (4.35). By (4.22) we have

$$u_{2m} = u_{2m+1}^* = 0,$$

so we can confine our attention to the limits u_{2m+1} and u_{2m}^*. Let $\varphi_n, \ldots \psi_n^*$ be the coefficients of the highest powers of x in $\varphi_n(x), \ldots \psi_n^*(x)$. By (4.22) and (6.05) we obtain the equations

$$(7.24)\quad \begin{cases} \varphi_{2m} = \varphi_{2m-1} + \eta_{2m}\varphi^*_{2m-1}, & \varphi^*_{2m} = (x_{2m}\eta_{2m})^{-1}\varphi_{2m-1}, \\ \psi_{2m} = \eta_{2m}\psi^*_{2m-1}, & \psi^*_{2m} = \psi^*_{2m-1} + (x_{2m}\eta_{2m})^{-1}\psi_{2m-1}, \\ \varphi_{2m+1} = \eta_{2m+1}\varphi^*_{2m}, & \varphi^*_{2m+1} = \varphi^*_{2m} + (x_{2m+1}\eta_{2m+1})^{-1}\varphi_{2m}, \\ \psi_{2m+1} = \psi_{2m} + \eta_{2m+1}\psi^*_{2m}, & \psi_{2m+1} = (x_{2m+1}\eta_{2m+1})^{-1}\psi_{2m}. \end{cases}$$

Hence, by (6.07),

$$(7.25)\quad \begin{cases} \dfrac{1}{u_{2m+1}} = \dfrac{\psi_{2m+1}}{\varphi_{2m+1}} = \dfrac{1}{u^*_{2m}} + \dfrac{\psi_{2m}}{\eta_{2m+1}\varphi^*_{2m}} = \dfrac{1}{u^*_{2m}} + \dfrac{x_{2m}\eta^2_{2m}}{\eta_{2m+1}} \cdot \dfrac{\psi^*_{2m-1}}{\varphi_{2m-1}}, \\ \dfrac{1}{u^*_{2m}} = \dfrac{\psi^*_{2m}}{\varphi^*_{2m}} = \dfrac{1}{u_{2m-1}} + x_{2m}\eta_{2m}\dfrac{\psi^*_{2m-1}}{\varphi_{2m-1}}. \end{cases}$$

Since all values in these formulae are positive, we have

$$u_{2m-1} > u^*_{2m} > u_{2m+1} \qquad (m = 1, 2, \ldots),$$

whereas it is obvious that

$$(7.26)\qquad u = \lim_{m\to\infty} u_{2m+1} = \lim_{m\to\infty} u^*_{2m}.$$

Using the notations (7.05) and (7.22), we have, by (7.24),

$$\frac{\psi^*_{2m+1}}{\varphi_{2m+1}} = \frac{x_{2m}\eta^2_{2m}\psi^*_{2m-1}}{x_{2m+1}\eta^2_{2m+1}\varphi_{2m-1}} = \frac{\lambda^2_{2m}\psi^*_{2m-1}}{\lambda^2_{2m+1}\varphi_{2m-1}} = \pi^2_{2m}\pi^2_{2m+1}\frac{\psi^*_{2m-1}}{\varphi_{2m-1}},$$

hence, by (6.04),

$$(7.27)\qquad \frac{\psi^*_{2m+1}}{\varphi_{2m+1}} = (\pi_2\pi_3\ldots\pi_{2m+1})^2.$$

By (7.05), (7.22), (7.25) and (7.27) we have

$$\begin{aligned} \frac{1}{u_{2m+1}} &= \frac{1}{u_{2m-1}} + \left(x_{2m}\eta_{2m} + \frac{x_{2m}\eta^2_{2m}}{\eta_{2m+1}}\right)\frac{\psi^*_{2m-1}}{\varphi_{2m-1}} \\ &= \frac{1}{u_{2m-1}} + (\pi_{2m}\sqrt{x_{2m}} + \pi^2_{2m}\pi_{2m+1}\sqrt{x_{2m+1}})\frac{\psi^*_{2m-1}}{\varphi_{2m-1}} \\ &= \frac{1}{u_{2m-1}} + (\pi_2\pi_3\ldots\pi_{2m-1})^2\pi_{2m}\sqrt{x_{2m}} + (\pi_2\pi_3\ldots\pi_{2m})^2\pi_{2m+1}\sqrt{x_{2m+1}}, \end{aligned}$$

or

$$(7.28)\qquad \frac{1}{u_{2m+1}} = 1 + \sum_{k=2}^{2m+1}(\pi_2\pi_3\ldots\pi_{k-1})^2\pi_k\sqrt{x_k},$$

since $u_1 = 1$. Now, by (7.26) and (7.28), the condition $u = 0$ and the statement (7.23) are obviously equivalent.

THEOREM 53. In order that a function F admits an asymptotic expansion

(7.29) $$F(z) \sim d_0 + \frac{d_1}{z} + \frac{d_2}{z^2} + \ldots,$$

valid in any domain $|\arg z| \leqq \varphi < \pi$ as $|z| \to \infty$, it is necessary and sufficient that

(7.30) $$\int_{+0}^{1} t^{-n} d\chi(t) < \infty \qquad (n = 1, 2, \ldots)$$

and

(7.31) $$d_0 = \chi(+0),\ d_n = (-)^{n-1} \int_{+0}^{1} t^{-n}(1-t)^{n-1} d\chi(t) \quad (n=1, 2, \ldots).$$

Proof. First let the moments (7.30) be finite, and (7.31) hold. Since

(7.32) $$F(z) = \int_{0}^{1} \frac{d\chi(t)}{1-t+tz} = \chi(+0) + \int_{+0}^{1} \frac{d\chi(t)}{1-t+tz},$$

we have

$$F(z) - \sum_{0}^{n} \frac{d_k}{z^k} = F(z) - d_0 - \sum_{1}^{n} \frac{d_k}{z^k}$$
$$= \int_{+0}^{1} \left\{ \frac{1}{1-t+tz} + \sum_{k=1}^{n} (-)^k (tz)^{-k} (1-t)^{k-1} \right\} d\chi(t)$$
$$= (-)^n \int_{+0}^{1} \frac{(1-t)^n}{(tz)^n(1-t+tz)} d\chi(t),$$

hence

(7.33) $$z^n \left\{ F(z) - \sum_{0}^{n} \frac{d_k}{z^k} \right\} = (-)^n \int_{+0}^{1} \frac{(1-t)^n}{t^n(1-t+tz)} d\chi(t).$$

Now, in any domain $|\arg z| \leqq \varphi < \pi$ either $\operatorname{Im} z \to \infty$ or $\operatorname{Re} z \to \infty$ as $|z| \to \infty$; hence the integral in the right hand member is smaller than

(7.34) $$\frac{1}{k|z|} \int_{+0}^{1} \frac{(1-t)^n}{t^{n+1}} d\chi(t),$$

where k only depends on φ. By (7.30) the expression (7.34) is finite; it tends to zero as $|z| \to \infty$, and thus our conditions are sufficient.

Conversely, let (7.29) hold. It will suffice to suppose that (7.29) is valid if z tends to infinity along the half line $x > 0$. First we have, by (7.32),

$$d_0 = \lim_{x\to\infty} F(x) = \chi(+0) + \lim_{x\to\infty} \int_{+0}^{1} \frac{d\chi(t)}{1-t+tx}.$$

Now, for $x > 1$ and for any $\varepsilon > 0$,

$$\int_{+0}^{1} \frac{d\chi(t)}{1-t+tx} = \int_{+0}^{\varepsilon} + \int_{\varepsilon}^{1} \leqq \int_{+0}^{\varepsilon} d\chi(t) + \frac{1}{x}\int_{\varepsilon}^{1} \frac{d\chi(t)}{t}.$$

Since both integrals in the right hand member can be made arbitrarily small by an appropriate choice of ε and x, the left hand integral tends to zero as $x \to \infty$. Hence $d_0 = \chi(+0)$. Next we have, by (7.29),

$$(7.35)\quad d_1 = \lim_{x\to\infty} x\{F(x) - d_0\} = \lim_{x\to\infty} \int_{+0}^{1} \frac{x\,d\chi(t)}{1-t+tx}$$

$$= \lim_{x\to\infty} \int_{+0}^{1} \left\{ \frac{1}{t} - \frac{1-t}{t(1-t+tx)} \right\} d\chi(t).$$

Since the integrand is positive we get, for any $\varepsilon > 0$,

$$\lim_{x\to\infty} \int_{\varepsilon}^{1} \left\{ \frac{1}{t} - \frac{1-t}{t(1-t+tx)} \right\} d\chi(t) = \int_{\varepsilon}^{1} \frac{d\chi(t)}{t} \leqq d_1,$$

hence, as $\varepsilon \to 0$,

$$\int_{+0}^{1} \frac{d\chi(t)}{t} \leqq d_1,$$

and by (7.35)

$$d_1 = \int_{+0}^{1} \frac{d\chi(t)}{t} - \lim_{x\to\infty} \int_{+0}^{1} \frac{1-t}{t(1-t+tx)}\, d\chi(t).$$

Since the limit in the right hand member is non-negative, we obtain

$$d_1 = \int_{+0}^{1} \frac{d\chi(t)}{t}.$$

Now, let $d_0, d_1, \ldots d_n$ be expressed by (7.31). Hence (7.33) holds, and consequently we have

$$d_{n+1} = \lim_{x\to\infty} x^{n+1}\left\{ F(x) - \sum_{0}^{n} \frac{d_k}{x^k} \right\} = (-)^n \lim_{x\to\infty} x \int_{+0}^{1} \frac{(1-t)^n}{t^n(1-t+tx)}\, d\chi(t)$$

or

$$(7.36)\quad (-)^n d_{n+1} = \lim_{x\to\infty} \int_{+0}^{1} \frac{(1-t)^n}{t^n} \left\{ \frac{1}{t} - \frac{1-t}{t(1-t+tx)} \right\} d\chi(t).$$

Since the integrand in the right hand member is positive, we get, for any $\varepsilon > 0$,

$$\lim_{x\to\infty}\int_{\varepsilon}^{1}\frac{(1-t)^n}{t^n}\left\{\frac{1}{t}-\frac{1-t}{t(1-t+tx)}\right\}d\chi(t)=\int_{\varepsilon}^{1}\frac{(1-t)^n}{t^{n+1}}d\chi(t)\leqq(-)^n d_{n+1},$$

hence, as $\varepsilon \to 0$,

$$\int_{+0}^{1} t^{-n-1}(1-t)^n\, d\chi(t) \leqq (-)^n\, d_{n+1},$$

and by (7.36)

$$(-)^n\, d_{n+1} = \int_{+0}^{1} t^{-n-1}(1-t)^n d\chi(t) - \lim_{x\to\infty}\int_{+0}^{1}\frac{(1-t)^{n+1}}{t^{n+1}(1-t+tx)}\, d\chi(t).$$

Since the limit in the right hand member is non-negative, we get

$$(-)^n d_{n+1} = \int_{+0}^{1} t^{-n-1}(1-t)^n\, d\chi(t).$$

Hence the conditions (7.30) and (7.31) are necessary.

Evidently one part of Theorem 53, where the conditions are said to be sufficient, is a transcription of a result of Stieltjes [4]). As to the other part, this is closely related to a theorem of Hamburger [5]) and R. Nevanlinna [2], if and only if

$$\int_{+0}^{1}\frac{d\chi(t)}{t} < \infty,$$

for else the transformation (1.08) does not apply to F.

From (7.29) we obtain, by the transformation (1.08), the asymptotic development of Stieltjes

$$F(z) - \chi(+0) = \int_{0}^{\infty}\frac{d\bar{\chi}(u)}{z+u} \sim \frac{c_0}{z} - \frac{c_1}{z^2} + \frac{c_2}{z^3} - \ldots,$$

where the moments c_n are finite:

$$c_n = (-)^n d_{n+1} = \int_{+0}^{1}(t^{-1}-1)^n\, t^{-1}\, d\chi(t) = \int_{0}^{\infty} u^n\, d\bar{\chi}(u) < \infty.$$

THEOREM 54. Let $\{\xi_i\}_1^\infty$ be an increasing sequence. Let $\xi_1 = 1$ and $\xi_i \to \infty$ as $i \to \infty$. Furthermore, let

$$\alpha_1 = 1;\ 0 < \alpha_i < 1\ (i = 2, 3, \ldots);\ \lim_{i\to\infty}\alpha_i = 0.$$

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[4]) STIELTJES [1], pp. 436, 498.

[5]) HAMBURGER [1], p. 268

There are functions F that satisfy

$$F(\xi_i) > \alpha_i \quad (i = 2, 3, \ldots); \ \lim_{x \to \infty} F(x) = 0.$$

Proof. The proof is based on the construction of a problem of type (e), which is characterized by two sequences $\{x_n\}_1^\infty$ and $\{a_n\}_1^\infty$, defined for the purpose; so in particular we must put $x_1 = a_1 = 1$. The functions $R_n(x)$ and $R_n^*(x)$ will have the same meaning as before, while a_n will be included, in the strict sense and for any $n > 1$, between $R_{n-1}(x_n)$ and $R_{n-1}^*(x_n)$. Thus R_n and R_n^* belong to $\{F\}$ for any n, and our problem (e) is solvable.

In order to avoid the necessity of distinguishing at every turn even and odd values of n, I shall write $(\overline{R}_n, \underline{R}_n)$ instead of (R_n, R_n^*), where $\underline{R}_n(x)$ is the function of the couple (R_n, R_n^*) that tends to zero, while $\overline{R}_n(x)$ is the one that tends to a positive value as $x \to \infty$. If we have, for a certain value of n,

$$\alpha_i < \underline{R}_n(\xi_i) \qquad (i = 2, 3, \ldots),$$

our theorem is obviously true. Henceforth this case will be excluded.

Next, an increasing sequence $\{j(n)\}_{n=1}^\infty$ of indices will be defined, where $j(1) = 1$, while we shall take $x_n = \xi_{j(n)}$. Hence $x_1 = 1$, whereas the sequence $\{x_n\}_1^\infty$ is increasing, and $x_n \to \infty$ as $n \to \infty$, as it is required. Moreover the following conditions, where $n > 1$, can be fulfilled:

$$(7.37) \qquad \begin{cases} \alpha_i < a_n < 2\alpha_i & \text{for } i = j(n), \\ \alpha_i < \min\{\overline{R}_n(\xi_i),\ \underline{R}_n(\xi_i)\} & \text{for } 1 < i < j(n), \\ \alpha_i < \overline{R}_n(\xi_i) & \text{for } i > j(n). \end{cases}$$

Evidently (7.37) can be satisfied if $n = 2$. For, if we assign to x_2 and ϑ_2 any provisional values (provided $x_2 > 1$ and $0 < \vartheta_2 < 1$), and if we put, as before,

$$a_2 = \vartheta_2 R_1(x_2) + \vartheta_2^* R_1^*(x_2), \qquad \vartheta_1 + \vartheta_2 = 1,$$

there can only be a finite set of values α_i such that $\alpha_i > R_2^*(\xi_i) = \overline{R}_2(\xi_i)$. Now, by Theorem 23(d), R_2^* is an increasing function of a_2, and hence of ϑ_2. Thus by an appropriate choice of ϑ_2 (while x_2 remains fixed, though arbitrary), the inequalities

$$(7.38) \qquad \alpha_i < \overline{R}_2(\xi_i) \qquad (i = 2, 3, \ldots)$$

will hold. On the other hand, there must be an infinite set of indices i such that $\alpha_i > R_1^*(\xi_i)$, according to the assumption just made. Hence ϑ_2 can so be chosen that (7.38) holds, while at the same, time there is an index $i = j(2)$ among the latter set that satisfies

$$\alpha_{j(2)} < \overline{R}_2(\xi_{j(2)}) < 2\alpha_{j(2)}.$$

Since $\xi_{j(2)} = x_2$ we have $\overline{R}_2(\xi_{j(2)}) = a_2$, so our statement is true.

Now let (7.37) hold for an arbitrary value of n. The argument is the same as in the case $n = 2$. For any provisional couple of values $(x_{n+1}, \vartheta_{n+1})$ there is a finite set of indices $i > j(n)$ such that $\alpha_i > \overline{R}_{n+1}(\xi_i)$, whereas there is an infinite set of these indices such that $\alpha_i > \underline{R}_n(\xi_i)$. Hence, by an appropriate choice of ϑ_{n+1} (while x_{n+1} remains constant, though it still has an arbitrary value $> x_n$), we can obtain $\alpha_i < \overline{R}_{n+1}(\xi_i)$ for all $i > j(n)$, whereas there is a particular index $i = j(n+1)$ such that $\underline{R}_n(\xi_i) < \alpha_i < \overline{R}_{n+1}(\xi_i) < 2\alpha_i$. In this way (7.37) can be satisfied for $n + 1$, and hence for any value of n.

By (7.37) we have $\alpha_i < R(\xi_i)$ for $1 < i < j(n)$ (since R is included in the strict sense between $\overline{R}_n$ and $\underline{R}_n$), i.e. we have $\alpha_i < R(\xi_i)$ for any value of i. Now $R(x)$ tends to zero as $x \to \infty$, since $\alpha_i \to 0$ as $i \to \infty$, hence the theorem is true.

It was the aim of this paper to get some information about completely monotonic functions that decrease arbitrarily slow as $x \to \infty$. The results, as seen from this point of view, are rather scanty, and no doubt much work on this subject remains to be done.

BIBLIOGRAPHY

J. Barkley Rosser

[1] The complete Monotonicity of certain Functions derived from completely monotone Functions, Duke Math. J. **15** (1948), 313—331.

J. C. Burkill

[1] On Mellin's inversion Formula, Proc. Cambr. Phil. Soc. **23** (1927), 356—360.

W. Feller

[1] Completely monotone Functions and Sequences, Duke Math. J. **5** (1939), 661—674.

J. Grommer

[1] Ganze transzendente Funktionen mit lauter reellen Nullstellen, Diss. Göttingen 1914, p. 28. = J. f. d. r. u. a. Math. **144**, Heft 2.

H. Hamburger

[1] Ueber eine Erweiterung des Stieltjesschen Momentenproblems, Math. Ann. **81** (1920), 235—319.

[2] Id. **82** (1921), 120—164, 168—187.

F. HAUSDORFF

[1] Summationsmethoden und Momentfolgen, Math. Ztsch. **9** (1921), 74—109, 280—299.

[2] Momentprobleme für ein endliches Intervall, Math. Ztsch. **16** (1923), 220—248.

E. HELLY

[1] Ueber lineare Funktionaloperationen, Sitzungsber. Ak. Wien, math-naturw. Kl. **121** (1912), Abt. II^a, 265—297.

D. HILBERT

[1] Grundzüge einer allgemeinen Theorie der linearen Integralgleichungen, Gött. Nachr., math-ph. Kl. (1906), 192.

A. HURWITZ

[1] Ueber die Nullstellen der Besselschen Funktion, Math. Ann. **33** (1889), 246—266 = Werke I, 266—286.

TH. KALUZA

[1] Ueber die Koeffizienten reziproker Potenzreihen, Math. Ztsch. **28** (1928), 161—170.

O. LOKKI

[1] Ueber analytische Funktionen deren Dirichletintegral endlich ist und die in gegebenen Punkten vorgeschriebene Werte annehmen, Diss. Helsinki 1947. = Ann. Ac. Sc. Fennicae, Series A1, 1947, nr. 39.

R. NEVANLINNA

[1] Ueber beschränkte analytische Funktionen, Comm. in honorem Ernesti Leonardi Lindelöf, Helsinki 1929. = Ann. Ac. Sc. Fennicae, Series A, **32** (1929), nr. 7.

[2] Asymptotische Entwicklungen beschränkter Funktionen und das Stieltjessche Momentenproblem, Ann. Ac. Sc. Fennicae, Series A, **18** (1922), 45—48.

TH. J. STIELTJES

[1] Recherches sur les fractions continues, Oevres II, 402—559.

S. C. VAN VEEN

[1] Onderzoek van de coefficiënten A_n in de ontwikkeling van

$$1 : \frac{2}{\pi} \int_0^1 \frac{dt}{\sqrt{(1-t^2)(1-zt^2)}} = 1 - \sum_{n=1}^{\infty} A_n z^n,$$

Nw. Arch. v. Wisk. (2) **15** (1927), 256—262.

H. WALL

[1] Analytic theory of continued Fractions, (1948), p. 314.

A. WINTNER

[1] Spektraltheorie der unendlichen Matrizen, (1929), p. 97.

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