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The classes of partially ordered groups

by

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The Hague

§ 1. In 1907 H. Hahn published a paper: Über die nicht-archimedischen Größensysteme [1]). It is a study of commutative simply ordered groups, especially concerning the non-archimedean groups.

Hahn uses the additive notation for the group operation, and he defines the group G to be Archimedean, if the Archimedean postulate (A) is satisfied:

(A) For each pair of positive elements a and b of G ($a > 0$, $b > 0$), there exists a natural multiple $n \cdot a$ of a with the property $n \cdot a > b$, and conversely there is a natural multiple $m \cdot b$ of b with the property $m \cdot b > a$.

If the postulate (A) is not satisfied for all pairs of positive elements, we call the ordering of G non-archimedean.

Suppose G is a commutative simply ordered group, a and b positive elements, then there are only four mutually exclusive possibilities:

I. For each natural multiple $n \cdot a$ of a there exists a natural multiple $m \cdot b$ of b, so that $m \cdot b > n \cdot a$, and conversely for each multiple [2]) $m' \cdot b$ of b there exists a multiple $n' \cdot a$ of a, so that $n' \cdot a > m' \cdot b$.

II. For each multiple $n \cdot a$ of a there exists a multiple $m \cdot b$ of b with $m \cdot b > n \cdot a$, but not conversely.

III. For each multiple $m' \cdot b$ of b there exists a multiple $n' \cdot a$ of a with $n' \cdot a > m' \cdot b$, but not conversely.

IV. Not for every multiple $n \cdot a$ of a does there exist a multiple $m \cdot b$ of b with $m \cdot b > n \cdot a$, nor for every multiple $m' \cdot b$ of b does there exist a multiple $n' \cdot a$ of a with $n' \cdot a > m' \cdot b$.

In case I we call a and b of the same rank, written $a \backsim b$. In case II we call a of a lower rank than b, written $a \prec b$ or $b \succ a$.

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[1]) Sitzungsberichte der Akademie der Wissenschaften, Math. Naturw. Kl. Band 116, 1907, Wien.

[2]) In the following "multiple" will stand for "natural multiple".

Therefore in case III, $b \prec a$ or $a \succ b$. If $a \prec b$, it follows immediately that $n \cdot a < b$ for all natural n.

In the case of simply ordered groups the possibility IV cannot occur. For $a < 0$, $b > 0$ (resp. $a < 0$, $b < 0$) the relation between a and b is defined in the same way as for $-a$ and b (resp. $-a$ and $-b$).

Because of the fact that equality of rank is an equivalence-relation, it is possible to divide G into classes, each class consisting of those and only those elements having the same rank as a given one; therefore two classes either coincide or they are disjunct. If G is non-archimedean ordered, then G has at least two classes A and B different from the zero class (consisting only of the identity). If A and B are two different classes of G and if for $a \in A$, $b \in B$ the relation $a \prec b$ holds, then it is easily proved that this relation is valid for each pair of elements $a' \in A$, $b' \in B$.

Therefore Hahn defines the relation $A \prec B$ for the classes A and B by $a \prec b$ for $a \in A$, $b \in B$. For two different classes A and B of G there exists one and only one of the order relations $A \prec B$ and $B \prec A$. Moreover $A \prec B$ and $B \prec C$ implies $A \prec C$.

The classes of a commutative simply ordered group G form a simply ordered set Λ, the class-set of G, while the ordertype of Λ is called the class-type of G. Conversely Hahn proves: if Λ is a simply ordered set, then there exists always a commutative simply ordered group G such that the class-type of G is equal the ordertype of Λ.

§ 2. We shall try to find a similar partition into classes for partially ordered groups. Though we have later on to restrict ourselves to commutative lattice-ordered groups, for the present we omit this restriction.

Definition: A partially ordered group is a set G satisfying the following conditions:

a) G is a group with the additive notation for the group-operation.

b) G is a partially ordered set.

c) $a \leqq b$ implies $c + a + d \leqq c + b + d$ for each pair c and d of G.

G is called a directed group, if G is a partially ordered group with the property that for each pair a, $b \in G$ there exists an element $c \in G$ with $c \geqq a$, $c \geqq b$.

G is called a lattice-ordered group if G is a lattice instead of a partially ordered set. Then each pair of elements a and b of G have a join $a \cup b$ and a meet $a \cap b$.

Let G be a partially ordered group and $G^{\pm}$ the set of all elements a, comparable with $0 (a \gtreqless 0)$. If a and b are two positive elements, we have for a and b the four possibilities I, II, III and IV of § 1. Likewise we define a and b to be of the same rank ($a \backsim b$) only if the case I occurs.

If there is a natural number m_0, so that $n \cdot a < m_0 \cdot b$ for all natural n, we shall call a of a lower rank than $b (a \prec b$ or $b \succ a)$. If the positive elements a and b are such that neither $a \backsim b$, $a \prec b$, nor $b \prec a$, we call a and b of incomparable rank. For $a < 0$, $b > 0$ (resp. $a < 0$, $b < 0$) the relation between a and b is defined in the same way as for $-a$ and b (resp. $-a$ and $-b$). It is easily proved, that for any two elements a and b of G at most one of the relations $a \backsim b$, $a \prec b$, or $b \succ a$ holds. If none of these relations is satisfied, then a and b are of incomparable rank. Thus we obtain: For each pair of elements a and b of $G^{\pm}$ there exists exactly one of the four possibilities: $a \backsim b$, $a \prec b$, $a \succ b$, or a and b of incomparable rank. If $a \in G^{\pm}$ $(a \neq 0)$ we define $0 \prec a$ for each $a \in G^{\pm}$. We prove the following statement:

If $a \prec b$, $a \backsim a'$, $b \backsim b'$, then we have $a' \prec b'$. For the sake of convenience we suppose $a > 0$, $b > 0$ and moreover $m \cdot a < n_0 \cdot b$ for all natural m.

For each multiple $m' \cdot a'$ of a' there is a multiple $m \cdot a$ of a with

$$m \cdot a > m' \cdot a'$$

and for each multiple $r \cdot b$ of b there is a multiple $r' \cdot b'$ with

$$r' \cdot b' > r \cdot b.$$

For every natural c we have

$$c \cdot r \cdot b < c \cdot r' \cdot b';$$

we choose c in such a manner, that $c \cdot r \geqq n_0$. Thus

$$m \cdot a < c \cdot r' \cdot b'$$

for all natural m we have: For all $m' \cdot a'$ we can find a multiple $m \cdot a$ with

$$m' \cdot a' < m \cdot a,$$

therefore $m' \cdot a' < c \cdot r' \cdot b'$ for all m' and so we have $a' \prec b'$.

If a and b are of incomparable rank and $a \backsim a'$, $b \backsim b'$, then

a' and b' are of incomparable rank too; in fact, should a' and b' be of comparable rank, it follows from the preceding result, that a and b should be of comparable rank. The relation "equality of rank" enables us to divide the set $G^{\pm}$ into classes. A class A consists of those and only those elements which are of the same rank. The zero class O is the class consisting of the identity of G. It follows that two classes A and B either coincide or are disjunct. Just as for the simply ordered groups it is possible to define an order relation $A \succ B$ for the two classes A and B, if and only if $a \succ b$ for $a \in A$, $b \in B$. Two such classes A and B are called incomparable if two elements $a \in A$ and $b \in B$ are of incomparable rank. Therefore each pair of different classes A and B defines one and only one of the three relations $A \succ B$, $B \succ A$, or A and B are incomparable. Moreover $A \succ B$, $B \succ C$ implies $A \succ C$. The classes of a partially ordered group G form a partially ordered set Λ, called the class-set of G. Λ possesses a least element O, the zero class. The Hasse-diagram of Λ is called the class-diagram of G.

§ 3. Examples.

1. The class-set Λ of a simply ordered group G is a chain.

2. Let G be the group of the pairs $(m; n)$, m and n integers with the operation: $(m_1; n_1) + (m_2; n_2) = (m_1 + m_2; n_1 + n_2)$ while the ordering is defined by $(m_1; n_1) \leqq (m_2; n_2)$ if and only if $m_1 \leqq m_2$, $n_1 \leqq n_2$ (cardinal-ordering of the group of pairs). G has four different classes: the zero class O, the class A of elements $(0; n)$ (n integer and $\neq 0$), the class B of elements $(n; 0)$ with $n \neq 0$, and the class C of the elements $(m; n)$ with $m > 0$, $n > 0$, or $m < 0$, $n < 0$. Each pair of the classes A, B, and C is incomparable since the elements $a = (0; 1)$, $b = (1; 0)$ and $c = (1; 1)$ are incomparable. The class-diagram of G is given in fig. 1.

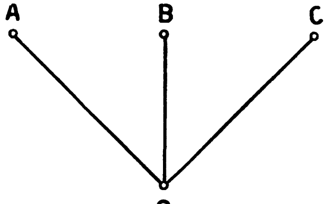

Fig. 1.

3. G is the group of the triples $(m, n; p)$, in which m, n and p are integers such that

$$(m_1, n_1; p_1) + (m_2, n_2; p_2) = (m_1 + m_2, n_1 + n_2; p_1 + p_2).$$

The ordering is defined as follows: the pairs $\alpha = (m, n)$ of the first two components are cardinally ordered (as in ex. 2); on the other hand the pairs $(\alpha; p)$, in which (m, n) is replaced by α, are ordinally ordered (e.g. lexicographically ordered). Contrary to the examples 1. and 2. this group is not a lattice-ordered group since the elements (0,1; 0) and (1,0; 0) have no join. Let A be the class containing the element (0,0; 1), B the class containing (0,1; 0), C the class containing (1,0; 0) and D the class containing (1,1; 0). There exist no other classes, hence the class-diagram has a form like that in fig. 2.

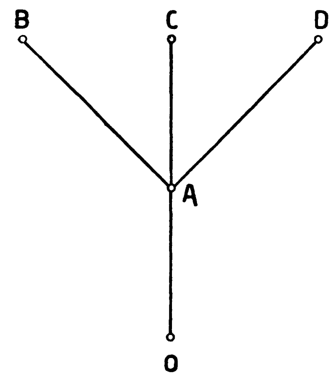

Fig. 2.

These and other examples show that in general the class-set Λ is not a lattice. Moreover a question arises: Do there exist groups with a prescribed class-set Γ? If we restrict ourselves to commutative lattice-ordered groups then it is possible to prove that the answer is negative. Since the class-set of a partially ordered group is not in general a lattice, we have a strong reason to ask whether it is possible to solve the problem of the division of classes of partially ordered groups in such a way, that we are able to find another sort of class-set with — at least — the properties of a lattice. This question can be answered affirmatively.

§ 4. Supposing now that G is a commutative lattice-ordered group we will proceed in the following paragraph to give some definitions and properties of these groups.

$$|a| = a \cup -a; \text{ if } a \neq 0, \text{ we have}$$

$$|a| > 0; \ |0| = 0; \ |a \pm b| \leqq |a| + |b|.$$

Two lattice-ordered groups G and G' are called isomorphic if there is a group-isomorphic relation between G and G' such that $a \leqq b$ implies $a' \leqq b'$ and $a' \leqq b'$ implies $a \leqq b$. It is easily proved that in the case of isomorphism $p \cup q$ (resp. $p \cap q$) corresponds to $p' \cup q'$ (resp. $p' \cap q'$).

A lattice-ordered subgroup H of G is a lattice-ordered group, which is a subgroup of G while the lattice H is a sublattice of G. Now we need the following:

THEOREM 4.1: If G is a commutative lattice-ordered group and n a natural number, then the correspondence $a \rightarrow n \cdot a$ is an isomorphism of G with a lattice-ordered subgroup of G [3]).

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[3]) G. BIRKHOFF, Lattice Theory p. 221; Ex. 3.

Proof: From $a \to n \cdot a$, $b \to n \cdot b$, it follows that $a + b \to n \cdot (a + b)$, and $n \cdot a = n \cdot b$ implies $a = b$. If $a \leqq b$, then also $n \cdot a \leqq n \cdot b$, and conversely $n \cdot a \leqq n \cdot b$ implies $a \leqq b$ (because of the commutative property of the groupoperation). It follows that $a \cup b \longleftrightarrow n \cdot (a \cup b)$, but also $a \cup b \longleftrightarrow n \cdot a \cup n \cdot b$; therefore $n \cdot (a \cup b) = n \cdot a \cup n \cdot b$, and in the same way $n \cdot (a \cap b) = n \cdot a \cap n \cdot b$.

By an L-ideal of the lattice-ordered group G is meant a normal subgroup of G which contains with any a, also all x with $|x| \leqq |a|$ [4]). G and O are L-ideals of G, and are called improper L-ideals, whereas all other L-ideals of G are called proper L-ideals. If N is an L-ideal of G, then N contains with a and b also $a + b$, $a \cup b$, $a \cap b$, and all x with the property $a \cap b \leqq x \leqq a \cup b$. Now let a be some element of G. The set $I(a)$ of elements $x \in G$ which satisfy the relation $|x| \leqq n \cdot |a|$ for some natural n is an L-ideal. Because, if $|b| \leqq m \cdot |a|$, $|c| \leqq n \cdot |a|$, then $|b \pm c| \leqq |b| + |c| \leqq (m + n) \cdot |a|$; and if $b \in I(a)$ and $|x| \leqq |b|$, then $|x| \leqq m \cdot |a|$; hence $I(a)$ is an L-ideal. Moreover $I(a)$ is the smallest L-ideal which contains a. In fact, an L-ideal containing a contains also $n \cdot a$ (for all natural n) and therefore all b with $|b| \leqq |n \cdot a| = n \cdot |a|$. In addition it is obvious, that $I(a) = I(-a) = I(|a|)$.

All L-ideals $I(a)$ of G will be called I-ideals.

For subsequent use we now give a theorem first proved by Birkhoff [5]): A commutative lattice-ordered group G has two proper disjunct L-ideals (e.g. two proper L-ideals with intersection O) unless G is simply ordered. The proof of this theorem is based on the consideration that G contains an element a incomparable with O unless G is simply ordered. To prove the theorem Birkhoff constructs two disjunct L-ideals S and S', of which S' contains the element $a^+ = a \cup 0$ but not $a^- = a \cap 0$, while S contains a^- but not a^+. This enables us to prove the following.

Theorem 4.2: A commutative lattice-ordered group G is simply ordered if and only if the I-ideals of G form a chain.

Proof: Suppose that G is simply ordered and that $I(a)$ and $I(b)$ are two I-ideals, $a \neq 0$, $b \neq 0$. $I(a) = I(-a)$, therefore we suppose $a > 0$, $b > 0$ and $a < b$. Then $I(a) \subseteq I(b)$, because $x \in I(a)$ implies $|x| \leqq n . a$ for some natural n. Therefore $|x| < n \cdot b$, whence $x \in I(b)$. Conversely, if the I-ideals of G form a chain, then G must be simply ordered. In fact should G not

4) Lattice Theory p. 222.

5) G. Birkhoff, Lattice-ordered groups, Ann. of Math. 43 (1942), p. 312.

be simply ordered, then G would contain two proper L-ideals S and S' with intersection O. Following the construction of S' we see that $I(a^+) \subseteq S'$, while $I(a^+)$ is the smallest L-ideal containing a^+. In the same way $I(a^-) \subseteq S$. The intersection of S and S' consists only of the identity, therefore $I(a^-)$ and $I(a^+)$ have only the identity as a common element. Hence $I(a^-)$ and $I(a^+)$ are incomparable (e.g. neither $I(a^-) \subseteq I(a^+)$, nor $I(a^+) \subseteq I(a^-)$).

THEOREM 4.3: If G is a commutative lattice-ordered group the I-ideals of G form a distributive lattice S_G.

PROOF: We prove that for two I-ideals, $I(a)$ and $I(b)$, there exist a join and a meet, which are also I-ideals. For $a = 0$ of $b = 0$, a join and meet evidently exist. We now prove: $I(a) \cup I(b) = I(|a| \cup |b|)$; since $I(a) = I(|a|)$ and $|a| \leqq |a| \cup |b|$, we have

$$I(|a|) \subseteq I(|a| \cup |b|) \text{ and } I(|b|) \subseteq I(|a| \cup |b|).$$

Conversely if $I(|a|) \subseteq I(c)$ and $I(|b|) \subseteq I(c)$, then $|a| \leqq n_1 \cdot |c|$, $|b| \leqq n_2 \cdot |c|$, therefore a and b both satisfy $|a| \leqq n \cdot |c|$, $|b| \leqq n \cdot |c|$ with $n = \max(n_1, n_2)$.

Hence $|a| \cup |b| \leqq n|c|$ and $I(|a| \cup |b| \subseteq I(c)$.

In the same way $I(|a| \cap |b|) \subseteq I(|a|)$ and $I(|a| \cap |b|) \subseteq I(|b|)$.

If $I(c) \subseteq I(|a|)$ and $I(c) \subseteq I|b|)$ then $|c| \leqq n|a|$ and $|c| \leqq n|b|$ for suitably closen n. Hence by Theorem 4.1 $|c| \leqq n \cdot |a| \cap n \cdot |b| = n \cdot (|a| \cap |b|)$, and therefore $I(c) \subseteq I(|a| \cap |b|)$. Therefore: the I-ideals of G form a lattice S_G. It is now easy to prove that this lattice is distributive. To do this we need the property, that G itself is a distributive lattice:

$$I(a) \cap (I(b) \cup I(c)) = I(a) \cap (I(|b| \cup |c|) = I(|a| \cap (|b| \cup |c|))$$
$$= I((|a| \cap |b|) \cup (|a| \cap |c|)) = I(|a| \cap |b|) \cup I(|a| \cap |c|)$$
$$= \{I(a) \cap I(b)\} \cup \{I(a) \cap I(c)\}.$$

§ 5. Let G be a commutative simply ordered group. We prove

THEOREM 5.1: The elements a and b are of the same rank (§ 1) if and only if $I(a) = I(b)$.

PROOF: If $a = b = 0$, then $I(a) = I(b)$; therefore we suppose $a \neq 0$; then $b \neq 0$.

Without restricting the generality we suppose $a > 0$, $b > 0$. If $x \in I(a)$, then $|x| \leqq n \cdot |a| = n \cdot a$. Now $a \backsim b$ (§ 1), so we can find a natural m with $n \cdot a < m \cdot b$; hence $|x| \leqq n \cdot a < m \cdot b = m \cdot |b|$. Therefore $I(a) \subseteq I(b)$ and in the same way $I(b) \subseteq I(a)$.

Hence it follows from $a \backsim b$ that $I(a) = I(b)$. If conversely $I(a) = I(b)$, and we suppose $a > 0$, $b > 0$, then $a \in I(b)$. Therefore $a < n \cdot b$ and, in the same way $b < m \cdot a$ for proper natural m and n; hence $a \backsim b$.

Theorem 5.2: For the elements a and b of G, $a \prec b$ if and only if $I(a)$ is a proper subset of $I(b)$.

Proof: Suppose $a \prec b$ $(a > 0,\ b > 0)$, then for $x \in I(a)$ we have $|x| \leqq n \cdot |a| = n \cdot a$ and $n \cdot a < b$ (for all natural n). Therefore $|x| < |b|$, hence $x \in I(b)$. But not every element of $I(b)$ is contained in $I(a)$; for, if $b \in I(a)$, then $|b| \leqq n \cdot |a|$ or $b \leqq n \cdot a$, contrary to the supposition that $n \cdot a < b$ for all natural n. Hence $I(a)$ is a proper subset of $I(b)$.

Conversely, if $I(a)$ is a proper subset of $I(b)$ there is an element y of $I(b)$ and not in $I(a)$, such that no natural multiple $n \cdot a$ of a exists with $y \leqq n \cdot a$. Therefore $n \cdot a < y$ for all natural n, and since $y \in I(b)$, we have $y < m_0 \cdot b$ for some natural m_0. It follows now $n \cdot a < m_0 \cdot b$ for all natural n, therefore $n \cdot a < b$ for all natural n or $a \prec b$.

Therefore in a commutative simply ordered group G we have $a \backsim b$ if and only if $I(a) = I(b)$ and $a \prec b$ if and only if $I(a) \subset I(b)$. If the element a is contained in the class A, then A corresponds to the I-ideal $I(a)$ of some arbitrary $a \in A$; and in addition, there are no other elements g in G, except the elements a of A, such that $I(g) = I(a)$. Furthermore $A \prec B$ implies $I(a) \subset I(b)$, if $a \in A$, $b \in B$.

Every I-ideal is generated by an element a, and therefore every I-ideal $I(a)$ corresponds to a class A, containing the element a. If $I(a) = I(b)$, then we have proved: $a \backsim b$. If $I(a) \subset I(b)$, then $a \prec b$; hence for the corresponding classes A and B we have $A \prec B$. Therefore we have the following result:

Theorem 5.3: If G is a commutative simply ordered group, there is a one to one correspondence preserving the orderrelations between the class-set Λ of G and the set of the I-ideals of G.

While the intersection of the classes of G is always empty, the I-ideals form a chain. For example, if Λ is the chain $O \prec A \prec B \prec C \prec D$ and $a \in A$, $b \in B$, $c \in C$, $d \in D$, we have $I(0) \subset I(a) \subset I(b) \subset I(c) \subset I(d)$.

§ 6. To generalize the preceding results for commutative lattice-ordered groups, we compare the I-ideals of G. Suppose that a and b are two elements of G which are not necessarily comparable with 0. We now define a and b to be of the same I-rank if and only if $I(a) = I(b)$; and we define a to be of a lower I-rank than b, if $I(a)$ is a proper subset of $I(b)$. We only use the notation $a \backsim b$ for the equality of rank as defined in § 2. That definition was only given for elements comparable with 0. Likewise we use the notation $a \prec b$ only for the cases we specified in § 2. However, it will appear that there is a close connection between the two types of relations of rank. First of all we give an example: G is the group of pairs $(m; n)$ (see ex. 2, § 3). $I(0; 0) = O, I(0; 1) = A$, consisting of all elements $(0; n)$ with n an integer; $I(1; 0) = B$, consisting of all elements $(n; 0)$ with n an integer; $I(1; 1) = C$, consisting of all elements of G. The Hasse-diagram of the I-ideals is shown in fig. 3.

If G consists of all cardinally ordered triples (m, n, p), with m, n and p integers and $(m_1, n_1, p_1) + (m_2, n_2, p_2) = (m_1 + m_2, n_1 + n_2, p_1 + p_2)$

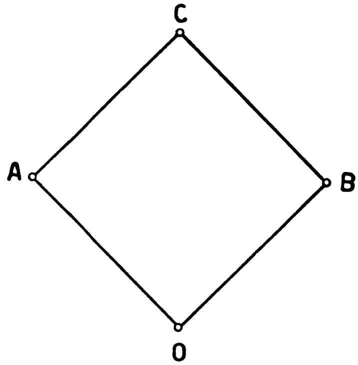

Fig. 3.

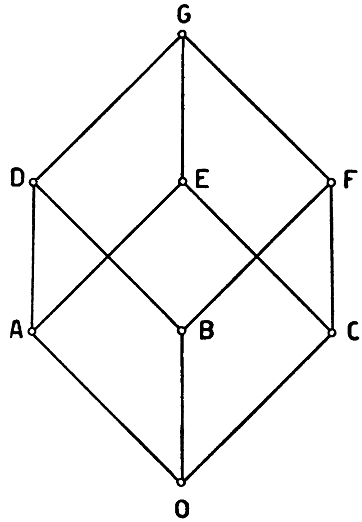

Fig. 4.

and we indicate $I(0, 0, 0) = 0$, $I(0, 0, 1) = A$, $I(0, 1, 0) = B$, $I(1, 0, 0) = C$, $I(0, 1, 1) = D$, $I(1, 0, 1) = E$, $I(1, 1, 0) = F$, and $I(1, 1, 1) = G$, then the Hasse-diagram of the I-ideals is given by fig. 4.

§ 7. Now we try to find the relation between the class-set Λ (of § 2) and the I-ideals of a commutative lattice-ordered group G.

Theorem 7.1: For $a, b \in G$ and $a > 0$, $b > 0$, we have $a \backsim b$ if and only if $I(a) = I(b)$.

Proof: Suppose $a \backsim b$ and $a > 0$, $b > 0$. If $x \in I(a)$, and therefore $|x| \leqq n \cdot a < m \cdot b$ for some natural m and n, then $I(a) \subseteq I(b)$, and in the same way $I(b) \subseteq I(a)$. Hence $I(a) = I(b)$. Conversely, we must show, if $I(a) = I(b)$, and $a > 0$, $b > 0$, then $a \backsim b$. Indeed, since $|a| = a < n \cdot b$ and $b < m \cdot a$ for some natural m and n, we have $a \backsim b$.

Theorem 7.2: From $a \prec b$ we conclude $I(a) \subset I(b)$, but not conversely.

PROOF: If $a \prec b$, then $n \cdot a < m_0 \cdot b$ for all natural $n (a > 0, b > 0)$. Thus we have for any $x \in I(a)$, $|x| \leqq n \cdot a < m_0 \cdot b$, therefore $x \in I(b)$. But we have not $I(b) \subseteq I(a)$ for if $b \in I(a)$, then we should have $b \leqq n \cdot a$ and $m_0 \cdot b \leqq m_0 n \cdot a$ contrary to our supposition. Therefore $I(a) \subset I(b)$. That the opposite of the theorem is not true, appears from the ex. 2, § 3; in fact, we have $I(0, 1) \subset I(1, 1)$, but not $(0, 1) \prec (1,1)$.

With every element a of a class of G there corresponds an I-ideal $I(a)$, and $I(a') = I(a)$ for all $a' \in A$. Therefore, a class A of G corresponds with an I-ideal $I(a)$, generated by a representing element a of A. Furthermore $A \prec B$ implies $I(a) \subset I(b)$ (proper subset), if $a \in A$, $b \in B$. Conversely an I-ideal, generated by an element a of G, corresponds to a class A of G, viz. the class A of which a is a member (we may suppose, that $a \geqq 0$, since $I(a) = I(|a|)$). The class A, corresponding to an I-ideal of G, does not depend on the choice of the generating element a of I (this follows from Theorem 8.1). Therefore we have:

THEOREM 7.3; If G is a commutative lattice-ordered group, then the set of the classes (formed by the elements of $G^{\pm}$) corresponds one to one with the set of the I-ideals of G. The correspondence preserves the order-relation in one direction, i.e. $A \prec B$ implies $I(a) \subset I(b)$, if $a \in A$, $b \in B$.

The last result enables us to decide whether or not there are

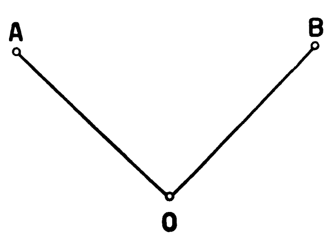

Fig. 5.

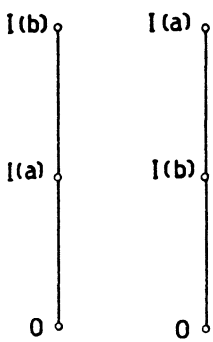

Fig. 6.

commutative lattice-ordered groups with a prescribed class-diagram. We prove that there is no commutative lattice-ordered group G with a class-diagram as shown in fig. 5. In fact, for such a group G the lattice of the I-ideals is a lattice consisting of three elements, e.g. this lattice is one the chains $O — I(a) — I(b)$ or $O — I(b) — I(a)$ (fig. 6). Other lattices of three elements do not exist. If, however, the I-ideals from a chain. G must be a simply ordered group (theorem 4.2), and the class-set Λ must be a simply ordered set too. Therefore the diagram of fig. 5 cannot be the class-diagram of G. Finally we put two questions:

1. Is the commutative lattice-ordered group uniquely defined but for isomorphism by the lattice of the I-ideals?

2. What conditions must be satisfied by this lattice if a distributive lattice with smallest element is the lattice of the I-ideals of a commutative lattice-ordered group?

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