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Kernel functions and conformal mapping [1])

by

S. Bergman and M. Schiffer

To our teacher, Erhard Schmidt, on the occasion of his 75th birthday

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Introduction.

The concept of a kernel function has found increasing application in the theory of functions which satisfy certain linear differential equations in a fixed domain. It has permitted a unified treatment of different important theories in analysis.

A particular role is played by the reproducing kernel of the class of functions considered which can easily be constructed by means of a complete orthonormal system in this class and is, on the other hand, closely related to such important domain functions as Green's and Neumann's functions. This kernel was originally introduced in the study of pseudo-conformal mapping by means of pairs of analytic functions of two complex variables (Bergman[1]). Its usefulness for the classical theory of analytic functions of one complex variable was soon realized (Bergman[2]) and, finally, its connection established with Green's function and the canonical map functions (Schiffer[3]). Its role was also studied from a general point of view by stressing its reproducing property in a linear function space with hermitian metric. Most of these results were extended to the theory of partial differential equations of elliptic type and a new approach to the boundary value problems was obtained (Bergman—Schiffer [1][2][3]). The dependence of the kernel functions upon the basic metric and its significance were investigated (Garabedian [1], Schiffer [5]).

At this occasion it became clear that certain other kernels should also be taken into consideration which are closely related to the reproducing kernel but have important properties of their

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own. It is the purpose of this paper to study such an additional kernel in great detail for the theory of analytic functions of one complex variable. We show how this new kernel leads to important inequalities for the reproducing kernel which is still considered as the fundamental one. From these inequalities numerous estimates for the coefficients of univalent functions in a given domain are derived and it is shown that Grunsky's necessary and sufficient conditions for univalence (Grunsky [1]) are an immediate consequence of the theory of these two kernels.

Since our new kernel does not have the reproducing property we are naturally led to study those functions which are reproduced by it except for a constant factor. This introduces a homogeneous integral equation the eigen values and eigen functions of which are to be determined. It appears that this integral equation is closely related to the classical one used in the treatment of boundary value problems by integral equations; thus, a connection is established between these different approaches to the theory of conformal mapping.

Using formal identities between the two kernels we establish a quickly convergent series for the reproducing kernel which seems to us of great importance for the numerical side of the theory of mapping of multiply-connected domains upon canonical domains.

Finally, we establish variation formulas which show the dependance of some of the quantities discussed upon the domain of definition if the latter varies. All our formulas show a great symmetry and simplicity which seems to justify the introduction of the new concepts.

1. Generalities and notations.

We consider in the complex z-plane a finite domain B which is bounded by n closed analytic curves $C_\nu(\nu = 1, 2, \ldots n)$; we denote the boundary $\sum_{\nu=1}^{n} C_\nu$ of B by C. If a complex-valued function $F(x, y)$ is differentiable in both arguments for every point $x + iy = z \in B$, we can define two complex differential operators on F:

$$(1.1) \quad \frac{\partial F}{\partial z} = \tfrac{1}{2}\left(\frac{\partial F}{\partial x} - i\frac{\partial F}{\partial y}\right), \quad \frac{\partial F}{\partial z^\dagger} = \tfrac{1}{2}\left(\frac{\partial F}{\partial x} + i\frac{\partial F}{\partial y}\right), \quad z^\dagger = x - iy.$$

Analytic functions $f(z)$ are characterized by the Cauchy-Riemann condition

$$\frac{\partial f}{\partial z^\dagger} = 0 \tag{1.2}$$

while anti-analytic functions $f(z^\dagger)$ satisfy correspondingly

$$\frac{\partial f}{\partial z} = 0. \tag{1.2a}$$

Let $f(z)$ and $g(z)$ be analytic in the closed region $B + C$; by means of (1.2) and (1.2a) we may establish the following simple rules on integration by parts:

$$\iint_B [f'(z)]^\dagger g(z) d\tau = \frac{1}{2i} \int_C [f(z)]^\dagger g(z) dz, \quad d\tau = dxdy, \tag{1.3}$$

$$\iint_B f'(z)[g(z)]^\dagger d\tau = -\frac{1}{2i} \int_C f(z)[g(z)dz]^\dagger. \tag{1.4}$$

Here and in the following the contour integration on C will be understood to be in the positive sense with respect to B.

We shall denote the complement of $B + C$ in the z-plane by $\overleftarrow{B}$, and $\overleftarrow{B}_\nu$ will be that component of B which is bounded by C_ν.

We assume that C is given in a parametric form $z(s)$ where s is the length parameter on C; thus $z' = \dfrac{dz}{ds}$ represents in each point of C the tangential unit vector. C has at each point $z(s)$ a normal and we denote by $\dfrac{\partial}{\partial n_z}$ the differential operator in the direction of the interior normal with respect to B.

In the following we shall often write $a(z)^\dagger$ instead of $[a(z)]^\dagger$.

2. Green's function.

Green's function $g(z, \zeta)$ of B is defined in the usual way by its three fundamental properties:

(a) $g(z, \zeta)$ is harmonic in z, for $\zeta \in B$ fixed, except for $z = \zeta$.

(b) $g(z, \zeta) + \log |z - \zeta|$ is harmonic in the neighbourhood of $z = \zeta$.

(c) $g(z, \zeta) \equiv 0$ for $z \in C$ and $\zeta \in B$.

The symmetry of $g(z, \zeta)$ in z and ζ follows easily from these properties; our assumptions on the analyticity of C ensure the harmonicity of Green's function even on the boundary C of B as the two argument points z and ζ stay apart.

From the identity

$$(2.1) \qquad g(z(s), \zeta) \equiv 0, \qquad z(s) \epsilon C, \qquad \zeta \epsilon B,$$

we derive by differentiation with respect to s:

$$(2.2) \qquad z' \frac{\partial g}{\partial z} + (z')^\dagger \frac{\partial g}{\partial z^\dagger} = 2\Re \left\{ z' \frac{\partial g}{\partial z} \right\} \equiv 0$$

i.e.

$$(2.2a) \qquad z'(s) \frac{\partial}{\partial z} g(z(s), \zeta) = i \,.\, \mathscr{F}(\xi, \eta; s), \quad \zeta = \xi + i\eta,$$

where $\mathscr{F}$ is a real-valued function of its arguments.

We now define the two functions

$$(2.3) \quad K(z, \zeta^\dagger) = -\frac{2}{\pi} \frac{\partial^2 g(z, \zeta)}{\partial z \partial \zeta^\dagger}, \quad L(z, \zeta) = -\frac{2}{\pi} \frac{\partial^2 g(z, \zeta)}{\partial z \partial \zeta}.$$

They are both analytic in their arguments which easily follows from the harmonicity of Green's function. From (2.2a) and (2.3) we conclude

$$(2.4) \quad z'(s)\, L(z(s), \zeta) = -[z'(s) K(z(s), \zeta^\dagger)]^\dagger, \quad z(s) \epsilon C, \zeta \epsilon B.$$

The function $K(z, \zeta^\dagger)$ is for fixed $\zeta \epsilon B$ regular in the closed region $B + C$; the logarithmic pole of Green's function has been destroyed by the particular process of differentiation leading to K. The function $L(z, \zeta)$, however, has a double-pole for $z = \zeta$ and may be written in the form

$$(2.5) \qquad L(z, \zeta) = \frac{1}{\pi(z - \zeta)^2} - l(z, \zeta)$$

where $l(z, \zeta)$ is, for $\zeta \epsilon B$, regular for $z \epsilon B + C$.

We further notice the symmetry relations which follows from the definitions:

$$(2.6) \quad [K(z, \zeta^\dagger)]^\dagger = K(\zeta, z^\dagger), \; L(z, \zeta) = L(\zeta, z), \; l(z, \zeta) = l(\zeta, z).$$

For instance in the case of the unit-circle $|z| < 1$ we have

$$(2.7) \quad \begin{aligned} g(z, \zeta) &= \log \left| \frac{1 - \zeta^\dagger z}{z - \zeta} \right|, \quad K(z, \zeta^\dagger) = \frac{1}{\pi(1 - \zeta^\dagger z)^2} \\ L(z, \zeta) &= \frac{1}{\pi(z - \zeta)^2}, \quad l(z, \zeta) = 0. \end{aligned}$$

The functions K and L play a central role in the theory of logarithmic potential and conformal mapping and it is the principal aim of this paper to investigate their properties and to show their applications. The following result (Schiffer [3]) illustrates their importance:

Let $\mathfrak{L}^2$ be the class of all functions $f(z)$ which are analytic in B and for which the Lebesgue integral

$$\iint_B |f(z)|^2 \, d\tau < \infty. \tag{2.8}$$

For each $f(z) \in \mathfrak{L}^2$ we have the identities

$$\iint_B K(z, \zeta^\dagger) \, f(\zeta) d\tau_\zeta = f(z) \tag{2.9}$$

and

$$\iint_B L(z, \zeta)^\dagger f(\zeta) d\tau_\zeta = 0. \tag{2.10}$$

Both integrals are to be understood in the Lebesgue sense, and the improper integral in (2.10) is the limit of an integral over the domain $B_{\varepsilon, z}$ which is obtained from B by elimination of a circle around z with radius ε.

3. The kernel functions.

We shall call $K(z, \zeta^\dagger)$ and $l(z, \zeta)$ the kernel functions of the first and second kind with respect to the class $\mathfrak{L}^2$, since they appear as kernels of certain integral operators applied to the class; $K(z, \zeta^\dagger)$ might also be called the reproducing kernel of the class because of (2.9). The significance of $l(z, \zeta)$ follows from the identity

$$\iint_B l(z, \zeta)^\dagger \, f(\zeta) d\tau_\zeta = \frac{1}{\pi} \iint_B [(z - \zeta)^{-2}]^\dagger \, f(\zeta) d\tau_\zeta \tag{3.1}$$

which is a consequence of (2.5) and (2.10). We see that $l(z, \zeta)$ is a kernel of the class $\mathfrak{L}^2$ which has on each function $f(z) \in \mathfrak{L}^2$ the same effect as the important but singular kernel $[\pi(z - \zeta)^2]^{-1}$. Numerous applications of this fact will be given in the following.

Let $w = \varphi(z)$ map the domain B univalently upon a domain B_1 with analytic boundary C_1. If $\omega = \varphi(\zeta)$ and $g_1(w; \omega)$ is Green's function with respect to B_1, we have the well-known identity

$$g_1(w, \omega) = g_1(\varphi(z), \;\; \varphi(\zeta)) = g(z, \zeta). \tag{3.2}$$

Differentiating with respect to z and ζ and denoting by K_1, L_1 and l_1 the kernels with respect to B_1 which correspond to K, L and l, we find in view of (2.3):

$$K_1(w, \omega^\dagger) \, \varphi'(z)[\varphi'(\zeta)]^\dagger = K(z, \zeta^\dagger) \tag{3.3}$$

and

$$(3.4) \qquad L_1(w, \omega)\, \varphi'(z)\, \varphi'(\zeta) = L(z, \zeta).$$

Hence, in view of definition (2.5)

$$(3.5) \quad l_1(w, \omega)\varphi'(z)\varphi'(\zeta) = l(z, \zeta) + \frac{1}{\pi}\left\{\frac{\varphi'(z)\varphi'(\zeta)}{(\varphi(z)-\varphi(\zeta))^2} - \frac{1}{(z-\zeta)^2}\right\}.$$

This formula is better understood if we introduce the expression

$$(3.6) \qquad \Phi(z, \zeta) = \frac{1}{\pi} \log \frac{\varphi(z) - \varphi(\zeta)}{z - \zeta}$$

which is analytic in the closed region $B + C$ because of the univalence of $\varphi(z)$. Then we may write instead of (3.5):

$$(3.5a) \qquad l_1(w, \omega)\varphi'(z)\varphi'(\zeta) = l(z, \zeta) + \frac{\partial^2 \Phi}{\partial z\, \partial \zeta}.$$

We notice the formal identity

$$(3.7) \quad \left.\frac{\partial^2 \Phi}{\partial z \partial \zeta}\right|_{z=\zeta} = -\frac{1}{6\pi}\left[\frac{d^2}{dz^2}\log\frac{d\varphi}{dz} - \frac{1}{2}\left(\frac{d}{dz}\log\frac{d\varphi}{dz}\right)^2\right] = -\frac{1}{6\pi}\{\varphi, z\}$$

where $\{\varphi, z\}$ denotes the well-known differential parameter of Schwarz. This shows the interest of the function $\Phi(z, \zeta)$ of two complex variables in connection with the conformal mapping produced by $\varphi(z)$.

We now make the following application of the transformation formulas (3.3) and (3.5a). Ley B_1 be a simply-connected domain in the w-plane and let $w = \varphi(z)$ be the map of the unit-circle $|z| < 1$ upon B_1. Since $\varphi(z)$ is still analytic on $|z| = 1$ we see from (2.7) that

$$(3.8) \qquad \begin{aligned} K_1(w, \omega^\dagger) &= (\pi\varphi'(\zeta)^\dagger\varphi'(z)(1-\zeta^\dagger z)^2)^{-1} \\ l_1(w, \omega) &= (\varphi'(z)\varphi'(\zeta))^{-1}\frac{\partial^2\Phi}{\partial z \partial \zeta} \end{aligned}$$

are still analytic on the boundary C_1 of B_1 if w and ω are separated. If $w = \omega$, however, K_1 becomes strongly infinite while l_1 remains regular even then. Thus, in the case of a simply-connected domain $l(z, \zeta)$ is regular in both arguments in the closed region $B + C$.

We now want to extend this result to the case of an arbitrary finite connectivity. We choose one boundary curve C_ν, say C_1, and consider the complement of the domain $\overleftarrow{B}_1$. This domain contains our initial domain B as subdomain; let $g_1(z, \zeta)$ be its Green's function. $g_1(z, \zeta)$ vanishes on C_1 and is still harmonic on the curve itself. If $g(z, \zeta)$ is again Green's function of B, the term $g_1(z, \zeta) - g(z, \zeta)$ is regular harmonic in B and may therefore be expressed by its boundary values and $g(z, \zeta)$. In fact, we have

$$
\begin{aligned}
g_1(z, \zeta) - g(z, \zeta) &= \frac{1}{2\pi} \int_C g_1(z, t) \frac{\partial g(t, \zeta)}{\partial n_t} ds_t = \\
&= \frac{1}{2\pi} \sum_{\nu \neq 1} \int_{C_\nu} g_1(z, t) \frac{\partial g(t, \zeta)}{\partial n_t} ds_t
\end{aligned} \tag{3.9}
$$

where $\frac{\partial}{\partial n}$ denotes differentiation in the direction of the interior normal. We notice that the integration in (3.9) runs over all boundary components of B except for C_1; the point $t \in C$ therefore never lies on C_1.

From (3.9) and (2.3), (2.5) we easily deduce

$$
l_1(z, \zeta) - l(z, \zeta) = \frac{1}{\pi^2} \sum_{\nu \neq 1} \int_{C_\nu} \frac{\partial g_1(z, t)}{\partial z} \frac{\partial^2 g(t, \zeta)}{\partial n_t \partial \zeta} ds_t. \tag{3.10}
$$

Now, $l_1(z, \zeta)$ is regular even on C_1 since it is the l-kernel of a simply-connected domain. In (3.10) the right-hand integral is regular on C_1 since t does not run over this particular curve. Hence we proved the following theorem:

The function $l(z, \zeta)$ is regular analytic in the closed region $B + C$.

This property of the l-kernel will be of of great use for the general theory; it is one of the main reasons for the importance of this kernel. The K-kernel with its reproducing proporty and its simple definition (2.3) attracted the interest much earlier than the l-kernel; it has not, however, the property of regularity in the closed region $B + C$ and its infinity on the boundary was of some difficulty in its theory. By establishing a simple relationship between the two kernel functions we will be able to overcome this difficulty and to remove the infinity of the kernel function by addition of an elementary function.

4. Identities and inequalities for the kernel functions.

The functions of the class $\mathfrak{L}^2$ form a linear space Λ and we may introduce into this space a hermitian metric based on the scalar product between two elements f and g:

$$
(f, g^\dagger) = \iint_B f(z) (g(z))^\dagger d\tau_z. \tag{4.1}
$$

Then it is important to determine the various scalar products between kernel functions.

From the reproducing property (2.9) of the K-kernel and the symmetry laws (2.6), we deduce immediately

$$(4.2)\quad \iint_B K(z,\zeta^\dagger)[K(z,w^\dagger)]^\dagger d\tau_z = \iint_B K(w,z^\dagger)K(z,\zeta^\dagger)d\tau_z = K(w,\zeta^\dagger)$$

and

$$(4.3)\quad \iint_B l(z,w)[K(z,\zeta^\dagger)]^\dagger d\tau_z = \iint_B K(\zeta,z^\dagger)\, l(z,w)d\tau_z = l(w,\zeta).$$

It is a little more difficult to determine the scalar products between l-kernels. Using the identity (3.1), we find

$$(4.4)\quad \iint_B l(z,\zeta)^\dagger l(z,w)d\tau_z = \frac{1}{\pi}\iint_B l(z,w)[(z-\zeta)^{-2}]^\dagger\, d\tau_z.$$

By integration by parts of the type (1.3), we transform this into

$$(4.5)\quad \iint_B l(z,\zeta)^\dagger l(z,w)d\tau_z = -\frac{1}{2\pi i}\int_C l(z,w)\frac{dz}{(z-\zeta)^\dagger}.$$

For $z \epsilon C$, we have by (2.5) and (2.4)

$$(4.6)\quad l(z,w)dz = \frac{1}{\pi}\frac{dz}{(z-w)^2} - L(z,w)dz = \frac{1}{\pi}\frac{dz}{(z-w)^2} + [K(z,w^\dagger)dz]^\dagger.$$

Hence, (4.5) obtains the form

$$(4.7)\quad \iint_B l(z,\zeta)^\dagger l(z,w)d\tau_z = \left[\frac{1}{2\pi i}\int_C \frac{K(z,w^\dagger)}{z-\zeta}dz\right]^\dagger - \frac{1}{2\pi^2 i}\int_C \frac{dz}{(z-w)^2(z-\zeta)^\dagger}.$$

The first right-hand integral may be computed by the residue theorem; the second integral may be transformed into an integral over the complement $\tilde{B}$ of B by means of (1.3). Finally, we arrive at the identity:

$$(4.8)\quad \iint_B l(z,\zeta)^\dagger l(z,w)d\tau_z = K(w,\zeta^\dagger) - \Gamma(w,\zeta^\dagger)$$

with

$$(4.9)\quad \Gamma(w,\zeta^\dagger) = \frac{1}{\pi^2}\iint_{\tilde{B}} \frac{d\tau_z}{(z-w)^2[(z-\zeta)^2]^\dagger}.$$

Hence the scalar product between two l-kernels leads to a K-kernel and a Γ-term. The characteristic property of the latter is that it can be computed by elementary integration over the exterior of the considered domain B. It does not depend on the solution of a boundary value problem in harmonic functions as do the K- and l-kernels. We shall call expressions of this type geometric quantities and consider a problem in harmonic functions solved if it can be reduced to the computation of such terms. The geometric quantities are elementary ones as compared with the function-theoretic terms involving Green's function.

We now make the following natural application of our identities: We choose $r+s$ points $\zeta_1, \zeta_2, \ldots \zeta_r, \eta_1, \ldots \eta_s$ in B and $r+s$ arbitrary constants $\alpha_1 \ldots \alpha_r, \beta_1 \ldots \beta_s$. We start from the obvious inequality

$$(4.10)\qquad \iint_B \Big| \sum_{\nu=1}^{r} \alpha_\nu^\dagger K(z, \zeta_\nu^\dagger) + \lambda \sum_{\mu=1}^{s} \beta_\mu l(z, \eta_\mu) \Big|^2 d\tau \geqq 0, \ \lambda \text{ real},$$

and compute the left-hand integral by means of the identities (4.2), (4.3) and (4.8). We obtain

$$(4.11)\qquad \sum_{\nu,\mu=1}^{r} \alpha_\nu \alpha_\mu^\dagger K(\zeta_\nu, \zeta_\mu^\dagger) + 2\lambda \Re \left\{ \sum_{\nu=1}^{r} \sum_{\mu=1}^{s} \alpha_\nu \beta_\mu l(\zeta_\nu, \eta_\mu) \right\} + \\ + \lambda^2 \sum_{\nu,\mu=1}^{s} \beta_\nu \beta_\mu^\dagger [K(\eta_\nu, \eta_\mu^\dagger) - \Gamma(\eta_\nu, \eta_\mu^\dagger)] \geqq 0.$$

For $\lambda = 0$ we obtain the well-known inequality

$$(4.11a)\qquad \sum_{\nu,\mu=1}^{r} \alpha_\nu \alpha_\mu^\dagger K(\zeta_\nu, \zeta_\mu^\dagger) \geqq 0$$

which is often expressed by the statement that $K(z, \zeta^\dagger)$ is a definite kernel. This property is characteristic for any kernel which has the reproducing property with respect to a certain Hilbert space, as has been stressed in the abstract theory of such kernels (Aronszajn [1]); the proof in the general case is also based on the fact that the norm of every element in a Hilbert space is non-negative.

We conclude further from (4.11) the inequality

$$(4.12)\qquad \sum_{\nu,\mu=1}^{s} \beta_\nu \beta_\mu^\dagger K(\eta_\nu, \eta_\mu^\dagger) \geqq \sum_{\nu,\mu=1}^{s} \beta_\nu \beta_\mu^\dagger \Gamma(\eta_\nu, \eta_\mu^\dagger).$$

This is a real improvement of (4.11a) since the kernel $\Gamma(z, \zeta^\dagger)$ is a positive-definite kernel, too. In fact, we may write

$$(4.13)\qquad \sum_{\nu,\mu=1}^{s} \beta_\nu \beta_\mu^\dagger \Gamma(\eta_\nu, \eta_\mu^\dagger) = \frac{1}{\pi^2} \iint_{\tilde{B}} \Big| \sum_{\nu=1}^{s} \frac{\beta_\nu}{(z-\eta_\nu)^2} \Big|^2 d\tau_z$$

which proves our assertion. By means of (4.12) we can estimate the hermitian forms connected with the kernel function in terms of geometric expressions.

Finally, we obtain from (4.11) the discriminant inequality

$$(4.14)\qquad \Re\left\{\sum_{\nu=1}^{r}\sum_{\mu=1}^{s}\alpha_\nu\beta_\mu l(\zeta_\nu,\eta_\mu)\right\}^2 \leqq$$
$$\leqq \sum_{\mu,\nu=1}^{r}\alpha_\nu\alpha_\mu^\dagger K(\zeta_\nu,\zeta_\mu^\dagger)\cdot\sum_{\nu,\mu=1}^{s}\beta_\nu\beta_\mu^\dagger[K(\eta_\nu,\eta_\mu^\dagger)-\Gamma(\eta_\nu,\eta_\mu^\dagger)].$$

If we replace in this inequality each β_μ by $\beta_\mu e^{i\sigma}$, the right-hand side remains unchanged while the left-hand side varies. The best possible inequality thus obtained is

$$(4.14a)\qquad \left|\sum_{\nu=1}^{r}\sum_{\mu=1}^{s}\alpha_\nu\beta_\mu l(\zeta_\nu,\eta_\mu)\right|^2 \leqq$$
$$\leqq \sum_{\mu,\nu=1}^{r}\alpha_\nu\alpha_\mu^\dagger K(\zeta_\nu,\zeta_\mu^\dagger)\cdot\sum_{\nu,\mu=1}^{s}\beta_\nu\beta_\mu^\dagger[K(\eta_\nu,\eta_\mu^\dagger)-\Gamma(\eta_\nu,\eta_\mu^\dagger)].$$

Because of the definite character of $\Gamma(z,\zeta^\dagger)$ this inequality implies

$$(4.14b)\qquad \left|\sum_{\nu=1}^{r}\sum_{\mu=1}^{s}\alpha_\nu\beta_\mu l(\zeta_\nu,\eta_\mu)\right|^2 \leqq$$
$$\leqq \sum_{\mu,\nu=1}^{r}\alpha_\nu\alpha_\mu^\dagger K(\zeta_\nu,\zeta_\mu^\dagger)\cdot\sum_{\nu,\mu=1}^{s}\beta_\nu\beta_\mu K(\eta_\nu,\eta_\mu^\dagger).$$

If we finally choose $r=s$, $\alpha_\nu=\beta_\nu$, $\zeta_\nu=\eta_\nu$, we arrive at

$$(4.14c)\qquad \left|\sum_{\mu,\nu=1}^{r}\alpha_\nu\alpha_\mu l(\zeta_\nu,\zeta_\mu)\right| \leqq \sum_{\mu,\nu=1}^{r}\alpha_\nu\alpha_\mu^\dagger K(\zeta_\nu,\zeta_\mu^\dagger).$$

Another very important consequence of (4.8) is the identity

$$(4.15)\qquad K(z,z^\dagger)-\Gamma(z,z^\dagger)=\iint_B |l(z,\zeta)|^2 d\tau_\zeta \geqq 0.$$

Since $l(z,\zeta)$ is regular and analytic in the closed region $B+C$, we conclude from (4.15) that $K(z,z^\dagger)-\Gamma(z,z^\dagger)$ is bounded in $B+C$. This shows that the geometric quantity $\Gamma(z,z^\dagger)$ has at the boundary C the same asymptotic behavior as $K(z,z^\dagger)$ and that their difference behaves quite regular. At the same time, this elementary term provides at each interior point z a lower bound for $K(z,z^\dagger)$.

We have further the important theorem:

The hermitian kernel $K(z,\zeta^\dagger)-\Gamma(z,\zeta^\dagger)$ is regular in the closed region $B+C$.

The irregular behavior of $K(z,\zeta^\dagger)$ on C led to the phenomenon that the homogeneous integral equation

$$(4.16)\qquad \varphi(z)=\lambda\iint_B K(z,\zeta^\dagger)\,\varphi(\zeta)d\tau_\zeta$$

had the value $\lambda = 1$ as eigenvalue of infinite order so that each analytic function $\varphi(z)$ was a corresponding eigenfunction. The classical theory for regular hermitian kernels is, however, applicable to the regularized kernel $K(z, \zeta^\dagger) - \Gamma(z, \zeta^\dagger)$ and we shall later study its eigenvalues and eigenfunctions. The close relation between the two hermitean kernels $K(z, \zeta^\dagger)$ and $\Gamma(z, \zeta^\dagger)$ is illustrated by the easily established fact that $K(z, \zeta^\dagger)[\Gamma(z, \zeta^\dagger)]^{-1}$ is invariant with respect to linear transformations of B.

We mention further the special instance of (4.14a)

(4.17) $$|l(z, \zeta)|^2 \leqq K(z, z^\dagger) \,.\, [K(\zeta, \zeta^\dagger) - \Gamma(\zeta, \zeta^\dagger)]$$

which implies

(4.17a) $$|l(z, z)| \leqq K(z, z^\dagger).$$

5. The l-transforms

The l-kernel transforms every analytic function $f(z)$ of the class $\mathfrak{L}^2$ into a new analytic function $\mathrm{T}\, f(z)$ by means of the operation

(5.1) $$\mathrm{T} f(z) = \iint_B l(z, \zeta)\, f(\zeta)^\dagger \, d\tau_\zeta.$$

We call $\mathrm{T}\, f$ the l-transform of f and want to study the class of all these transforms. Using Schwarz' inequality and (4.15), we find

(5.2) $$|\mathrm{T} f(z)|^2 \leqq [K(z, z^\dagger) - \Gamma(z, z^\dagger)] \iint_B |f(\zeta)|^2 \, d\tau_\zeta$$

while the same reasoning applied to (2.9) yields

(5.2a) $$|f(z)|^2 \leqq K(z, z^\dagger) \iint_B |f(\zeta)|^2 d\tau_\zeta.$$

We see from (5.2) that the class of all l-transforms of $\mathfrak{L}^2$ forms a proper subclass of $\mathfrak{L}^2$ which contains only bounded functions. One easily sees that all l-transforms are analytic in the closed region $B + C$.

Because of the fundamental property (3.1) of the l-kernel we may express the l-transform of $f(z)$ by means of the improper integral

(5.3) $$\mathrm{T} f(z) = \frac{1}{\pi} \iint_B f(\zeta)^\dagger (\zeta - z)^{-2} \, d\tau_\zeta.$$

This representation has the advantage of possessing an elementary kernel and of admitting simple transformations. Applying for example the integration rule (1.4) we obtain

(5.4) $$\mathrm{T} f(z) = \frac{1}{2\pi i} \int_C (\zeta - z)^{-1} f[(\zeta) d\zeta]^\dagger$$

in the case that $f(z)$ is continuous in the closed region $B + C$. Further we way use the representation (5.3) in order to define the transform $\mathrm{T}\, f(z)$ in the whole complex z-plane. In each domain $\overset{\frown}{B}_\nu$ the function $\mathrm{T}f(z)$ then represents an analytic function.

The different analytic functions $\mathrm{T}\, f(z)$ defined in the domains B and $\overset{\frown}{B}_\nu$ do not form a continuous function in the whole z-plane. In order to study their behavior on C let us assume that $f(z)$ is continuous in $B + C$, so that the representation (5.4) holds. According to a classical theorem by Plemelj (Plemelj [1]) the function $\mathrm{T}\, f(z)$ has a saltus of the value

$$\Delta(\mathrm{T}\, f(z)) = -\, [f(z)\,.\,z'^2]^\dagger \tag{5.5}$$

if we cross at the point $z \in C$ from B into the complementary region $\overset{\frown}{B}$.

Let us illustrate these formulas by the following example. We have

$$\mathrm{T}K(z, w^\dagger) = \iint_B K(\zeta, w^\dagger)^\dagger\, l(\zeta, z) d\tau_\zeta = l(z, w) \text{ for } z \in B \tag{5.6}$$

and

$$\mathrm{T}K(z, w^\dagger) = \frac{1}{\pi}\iint_B K(\zeta, w^\dagger)^\dagger (\zeta - z)^{-2}\, d\tau_\zeta = \frac{1}{\pi(z - w)^2} \text{ for } z \in \overset{\frown}{B}. \tag{5.6a}$$

The latter result follows from the fact that $(\zeta - z)^{-2}$ for $z \in \overset{\frown}{B}$ belongs to the class $\mathfrak{L}^2$ with respect to ζ and that, therefore, the reproducing property of the kernel function may be applied here. The saltus condition (5.5) takes the form

$$-\, [K(z, w^\dagger) z'^2]^\dagger = \frac{1}{\pi(z - w)^2} - l(z, w) = L(z, w) \tag{5.7}$$

which is just the important boundary relation (2.4).

At this point we notice that (5.6) may also be written in the form

$$l(z, w) = \frac{1}{\pi}\iint_B K(\zeta, w^\dagger)^\dagger\, (\zeta - z)^{-2}\, d\tau_\zeta \text{ for } z \in B \tag{5.6b}$$

which shows that $l(z, w)$ may be computed elementarily, once the K-kernel has been determined.

Similarly, we have

$$\mathrm{T}l(z, w) = \iint_B l(\zeta, w)^\dagger l(\zeta, z) d\tau_\zeta = K(z, w^\dagger) - \Gamma(z, w^\dagger) \text{ for } z \in B \tag{5.8}$$

and

$$\mathrm{T}l(z, w) = \frac{1}{\pi}\iint_B l(\zeta, w)^\dagger (\zeta - z)^{-2}\, d\tau_\zeta = \tag{5.8a}$$

$$= \frac{1}{\pi^2}\iint_B [(\zeta - w)^{-2})]^\dagger (\zeta - z)^{-2}\, d\tau_\zeta,\ z \in \overset{\frown}{B}.$$

One can show that in this example, too, the saltus condition (5.5) leads to the boundary relation (2.4) between the two kernels. If we bring (5.8) into the form

$$(5.8b)\quad K(z, w^\dagger) = \Gamma(z, w^\dagger) + \frac{1}{\pi}\iint_B l(\zeta, z)[(\zeta - w)^{-2}]^\dagger\, d\tau_\zeta, \quad z \in B,$$

we find that the K-kernel can be expressed by elementary computations in terms of the l-kernel. One sees that it is sufficient to find a construction for either kernel and that the other is then easily obtained.

Now let $f(z)$ and $g(z)$ be a pair of functions of the class $\mathfrak{L}^2$. Defining $\mathrm{T}\, f$ and $\mathrm{T}\, g$ by (5.1), we can easily compute by means of (4.8) the scalar product:

$$(5.9)\quad \iint_B \mathrm{T}f \cdot (\mathrm{T}g)^\dagger d\tau = \iint_B \iint_B [K(\zeta, \eta^\dagger) - \Gamma(\zeta, \eta^\dagger)] f(\zeta)^\dagger g(\eta) d\tau_\zeta d\tau_\eta.$$

Using the reproducing property (2.9) of the K-kernel and the definitions (4.9), (5.3), we may bring (5.9) into the elegant form

$$(5.10)\qquad \iint_{B+\widetilde{B}} \mathrm{T}f \cdot (\mathrm{T}g)^\dagger d\tau = \iint_B f^\dagger g\, d\tau.$$

This result suggests the following concepts. Just as the metric in the space Λ was based on the scalar product (4.1), we may base a metric in the linear space Λ_T of transforms on the metric

$$(5.11)\qquad [\mathrm{T}f,\ (\mathrm{T}g)^\dagger] = \iint_{B+\widetilde{B}} \mathrm{T}f \cdot (\mathrm{T}g)^\dagger\, d\tau.$$

In fact, the transforms being defined in the whole complex plane, it is natural to integrate in their scalar products over this whole region. In this notation, we now may express (5.10) in the form

$$(5.10a)\qquad [\mathrm{T}f,\ (\mathrm{T}g)^\dagger] = (f^\dagger, g).$$

The close relation between the linear spaces Λ and Λ_T becomes more evident by the following inversion formulas: Given a function $\mathrm{T}f \in \Lambda_\mathrm{T}$, we want to find its generating function $f \in \Lambda$. For this purpose, we determine

$$(5.12)\quad \mathrm{T}[\mathrm{T}f(z)] = \iint_B l(\zeta, z)[\iint_B l(w, \zeta)\, f(w)^\dagger d\tau_w]^\dagger d\tau_\zeta, \quad z \in B.$$

Using (4.8) and (5.3), we have

$$(5.13)\quad \frac{1}{\pi}\iint_B [\mathrm{T}f(\zeta)]^\dagger \frac{d\tau_\zeta}{(\zeta - z)^2} = \iint_B [K(z, \zeta^\dagger) - \Gamma(z, \zeta^\dagger)] f(\zeta)\, d\tau_\zeta.$$

By virtue of (2.9) and (4.9), this may be written in the form

$$(5.14) \qquad f(z) = \frac{1}{\pi} \iint\limits_{B+\overset{\frown}{B}} [\mathrm{T} f(\zeta)]^\dagger \frac{d\tau_\zeta}{(\zeta - z)^2}, \qquad z \in B.$$

This formula shows the great symmetry existing between the spaces Λ and Λ_{T}; the corresponding elements transform into each other by an integral operation with the same kernel, extended in each case over the proper domain of definition only.

The meaning of (5.14) becomes very clear if we transform the domain integral into a contour integral along C by means of the integration rule (1.4). We arrive at

$$(5.14\mathrm{a}) \quad \frac{1}{\pi} \iint\limits_{B+\overset{\frown}{B}} [\mathrm{T}f(\zeta)]^\dagger \frac{d\tau_\zeta}{(\zeta - z)^2} = \frac{1}{2\pi i} \int\limits_C - [\Delta\,(\mathrm{T}f(\zeta))]^\dagger \frac{d\zeta^\dagger}{\zeta - z}$$

where $\Delta(\mathrm{T}f)$ is the discontinuity of $\mathrm{T}f$ on C as given by (5.5). Thus, (5.14) is nothing but the identity

$$(5.15) \qquad f(z) = \frac{1}{2\pi i} \int\limits_C \frac{f(\zeta)}{\zeta - z} d\zeta$$

in the case that $f(z)$ is continuous in the closed region $B \epsilon C$. This transformation shows also clearly that the value of the right-hand integral in (5.14) has the value zero for $z \epsilon \overset{\frown}{B}$.

Each function $\mathrm{T}f \epsilon \Lambda_{\mathrm{T}}$ is also a function of Λ and has a norm $(\mathrm{T}f, (\mathrm{T}f)^\dagger)$. We may compare its norm in Λ_{T} with that in Λ and find by (5.11)

$$(5.16) \quad [\mathrm{T}f, (\mathrm{T}f)^\dagger] = (\mathrm{T}f, (\mathrm{T}f)^\dagger) + \iint\limits_{\overset{\frown}{B}} |\mathrm{T}f|^2 d\tau \geqq (\mathrm{T}f, (\mathrm{T}f)^\dagger).$$

There arises the question under what circumstances equality might hold in (5.16). It is obvious that in this case necessarily

$$(5.17) \qquad \mathrm{T}f(z) \equiv 0 \quad \text{for} \quad z \epsilon \overset{\frown}{B}.$$

From (5.14) we then conclude that

$$(5.18) \qquad f(z) = \mathrm{T}\,(\mathrm{T}f(z)), \quad \text{for} \quad z \epsilon B,$$

i.e. $f(z)$ belongs also to the space Λ_{T} and is, therefore, analytic in $B + C$. Hence we may apply the saltus condition (5.5) to $\mathrm{T}f(z)$ and since $\mathrm{T}f(z)$ vanishes in $\overset{\frown}{B}$, we simply obtain for the limit of $\mathrm{T}f$ an interior approach to $z \epsilon C$:

$$(5.19) \qquad \mathrm{T}f(z) = -\,[f(z)z'^2]^\dagger, \quad z \epsilon C.$$

We write this result in the more symmetric forms

(5.19a) $\quad (\mathrm{T}f(z) + f(z))z' = -[(\mathrm{T}f(z) + f(z)z']^\dagger_*, \quad z \in C$

(5.19b) $\quad i(\mathrm{T}f(z) - f(z))z' = -[i(\mathrm{T}f(z) - f(z))z']^\dagger, \quad z \in C.$

We introduce two real harmonic functions $\Omega_1(x, y)$ and $\Omega_2(x, y)$ such that

$$(5.20) \qquad \mathrm{T}f(z) + f(z) = \frac{\partial \Omega_1}{\partial z}, \quad i(\mathrm{T}f(z) - f(z)) = \frac{\partial \Omega_2}{\partial z}.$$

The formulas (5.19a) and (5.19b) then simply state that on C

$$(5.21) \qquad \frac{d\Omega_1}{ds} = 0, \quad \frac{d\Omega_2}{ds} = 0.$$

Hence Ω_1 and Ω_2 are two real harmonic functions in B which are constant on each boundary curve C_ν. Therefore, they may be linearly composed of the harmonic measure functions

$$(5.22), \qquad \omega_\nu(x, y) = \frac{1}{2\pi} \int_{C_\nu} \frac{\partial g(z, \zeta)}{\partial n_\zeta} ds_\zeta, \quad z = x + iy,$$

which have the value 1 on C_ν and 0 on the rest of C. We introduce the analytic functions

$$(5.23) \qquad w_\nu'(z) = 2\frac{\partial}{\partial z} \omega_\nu(x, y), \qquad w_\nu'(z) \equiv \frac{dw_\nu(z)}{dz}$$

which clearly satisfy the boundary conditions

$$(5.23') \qquad w_\nu'(z)z' = -[w_\nu'(z)z']^\dagger.$$

Then it follows from our considerations above that

$$(5.24) \qquad f(z) = \sum_{\nu=1}^{n} a_\nu w_\nu'(z)$$

with complex coefficients a_ν.

If inversely $f(z)$ has the form (5.24), it is analytic in $B + C$ and we may apply the formula (5.4). Because of (5.23a) this leads to

$$(5.25) \quad \mathrm{T}f(z) = \frac{1}{2\pi i} \int_C \sum_{\nu=1}^{n} a_\nu^\dagger [w_\nu'(\zeta)\zeta']^\dagger (\zeta - z)^{-1} ds = -\frac{1}{2\pi i} \int_C a_\nu^\dagger w_\nu'(\zeta) \frac{d\zeta}{\zeta - z}$$

i.e.

$$(5.25a) \qquad \mathrm{T}f(z) = -\sum_{\nu=1}^{n} a_\nu^\dagger w_\nu'(z) \qquad \text{for } z \in B$$

$$(5.25b) \qquad \mathrm{T}f(z) = 0 \qquad \text{for } z \in \widetilde{B}.$$

We see that in this case (5.17) and (5.18) indeed are fulfilled and that, therefore the equality sign in (5.16) holds. Since the harmonic measures have non-vanishing derivatives only in multiply-connected domains B, we see that equality in (5,16) is impossible in simply-connected domains, except for identically vanishing $f(z)$.

We notice finally that each function

$$f(z) = i \sum_{\nu=1}^{n} \bar{a}_\nu w_\nu'(z), \qquad a_\nu \text{ real} \tag{5.26}$$

satisfies the condition

$$f(z) = \mathrm{T}f(z) \tag{5.27}$$

and that every solution of (5.27) must necessarily have the form (5.26).

6. The eigen functions of the l-kernel

It is natural to ask for those functions in Λ which coincide with their l-transforms except for a numerical factor, i.e. which satisfy the integral equation

$$\varphi_\nu(z) = \lambda_\nu \iint_B \varphi_\nu(\zeta)^\dagger\, l(\zeta, z) d\tau_\zeta. \tag{6.1}$$

Every multiple of $\varphi_\nu(z)$ will have the same property, since we may put (6.1) into the form

$$a\varphi_\nu(z) = \lambda_\nu a(a^{-1})^\dagger \iint_B [a\varphi_\nu(\zeta)]^\dagger l(\zeta, z) d\tau_\zeta. \tag{6.1a}$$

We use this fact in order to put normalizing restrictions on the functions $\varphi_\nu(z)$ which we will consider. It is sufficient to deal with functions $\varphi_\nu(z)$ for which

$$\iint_B |\varphi_\nu(z)|^2 d\tau_z = 1 \tag{6.2}$$

and for which the corresponding λ_ν satisfies the condition

$$\lambda_\nu \geqq 0. \tag{6.3}$$

We shall call such a function $\varphi_\nu(z)$ an eigen function and the corresponding value λ_ν an eigen value of the kernel l.

At the end of the preceding section we saw that the value 1 is an eigen value of the l-kernel in each domain of connectivity $n \geqq 1$ and that it belongs to the $n-1$ linearly independent eigen functions $i\, w_1'(z), \ldots i\, w_{n-1}'(z)$. In such a case we say that the eigen value $\lambda_\nu = 1$ is of degeneracy $n - 2$.

The study of the integral equation (6.1) may easily be reduced to the classical theory of integral equations with hermitian definite kernels. In fact, iterating the integral equation (6.1) we obtain

$$(6.4) \qquad \varphi_\nu(z) = \lambda_\nu \iint_B [\lambda_\nu \iint_B \varphi_\nu(w)^\dagger l(w, \zeta) d\tau_w]^\dagger l(\zeta, z) d\tau_\zeta,$$

which leads because of (4.8) to the new integral equation

$$(6.5) \qquad \varphi_\nu(z) = \lambda_\nu^2 \iint_B [K(z, \zeta^\dagger) - \Gamma(z, \zeta^\dagger)] \varphi_\nu(\zeta) d\tau_\zeta.$$

Hence every eigen function of (6.1) is also a solution of the simpler integral equation (6.5) and to each eigen value λ_ν of (6.1) corresponds an eigen value λ_ν^2 of (6.5). Now we shall show that the converse of this statement is also true and derive from this fact the existence of solutions of (6.1).

The kernel $K(z, \zeta^\dagger) - \Gamma(z, \zeta^\dagger)$ is hermitian, regular in $B + C$ and positive definite, since we have for an arbitrary continuous function $\mu(z)$ in B in view of (4.8):

$$(6.6) \qquad \iint_B \iint_B [K(z, \zeta^\dagger) - \Gamma(z, \zeta^\dagger)]\, \mu(z)^\dagger \mu(\zeta) d\tau_z d\tau_\zeta = \iint_B \left| \iint_B l(z, w) \mu(z)^\dagger\, d\tau_z \right|^2 d\tau_w \geqq 0.$$

Thus, the existence theorems for such kernels become applicable. We conclude:

a) There exists a sequence of positive eigen values λ_ν^2 for the kernel $K(z, \zeta^\dagger) - \Gamma(z, \zeta^\dagger)$.

b) The corresponding eigen functions $\psi_\nu(z)$ are analytic in the closed region $B + C$.

c) We have the orthogonality relation for two eigen functions ψ_ν and ψ_μ which belong to different eigen values λ_ν^2, λ_μ^2:

$$(6.7) \qquad \iint_B \psi_\nu \psi_\mu^\dagger\, d\tau = 0 \quad \text{if} \quad \lambda_\nu^2 \neq \lambda_\mu^2.$$

d) The eigen functions $\psi_\nu^{(\varrho)}(z)$ ($\varrho = 1, 2, \ldots m$) which belong to an eigen value λ_ν^2 of degeneracy $m - 1$ may be supposed orthonormalized, i.e.

$$(6.7a) \qquad \iint_B \psi_\nu^{(\varrho)}\, [\psi_\nu(\sigma)]^\dagger\, d\tau = \delta_{\varrho\sigma}.$$

However, this condition fixes the $\psi_\nu^{(\varrho)}$, only up to a unitary transformation.

To a set $\psi_\nu^{(\varrho)}(z)$ of m eigen functions belonging to the eigen value λ_ν^2 we introduce a new set of functions by the definition

$$(6.8) \qquad \Psi_\nu^{(\varrho)}(z) = \lambda_\nu \iint_B \psi_\nu^{(\varrho)}(\zeta)^\dagger l(\zeta, z)\, d\tau_\zeta, \qquad \lambda_\nu > 0.$$

In view of the integral equation (6.5) satisfied by $\psi_\nu^{(\varrho)}$, we have also

$$(6.8a) \qquad \psi_\nu^{(\varrho)}(z) = \lambda_\nu \iint_B \Psi_\nu^{(\varrho)}(\zeta)\, l(\zeta, z) d\tau_\zeta.$$

One sees easily from the definition (6.8) that the $\Psi_\nu^{(\varrho)}(z)$ form an orthonormalized set of eigen functions for the same eigen value λ_ν^2 with respect to the integral equation (6.5). Therefore, there exists a unitary matrix $U = (u_{\varrho\sigma})$ such that

$$(6.9) \qquad \Psi_\nu^{(\varrho)}(z) = \sum_{\sigma=1}^{m} u_{\varrho\sigma} \psi_\nu^{(\sigma)}(z).$$

Introducing this representation for $\Psi_\nu^{(\sigma)}(z)$ into (6.8a) we obtain

$$(6.9a) \qquad \psi_\nu^{(\varrho)}(z) = \sum_{\sigma=1}^{m} u_{\varrho\sigma}^\dagger \Psi_\nu^{(\sigma)}(z),$$

which gives the matrix formula

$$(6.10) \qquad U \,.\, U^\dagger = E, \qquad E = \text{unit matrix},$$

for the unitary matrix U. Because of the unitary property of U this is equivalent to the symmetry of U.

Now, it is well-known that every symmetric unitary matrix U may be expressed by means of a unitary matrix V in the form

$$(6.11) \qquad U = VV', \qquad V' = \text{transposed matrix of } V,$$

where V is only determined up to a real-orthogonal matrix factor.

If we introduce another orthonormal system $\varphi_\nu^{(\varrho)}$ of eigen functions for λ_ν^2 which is obtained from the system $\psi_\nu^{(\varrho)}$ by means of a unitary matrix W, we see easily that their corresponding functions $\Phi_\nu^{(\varrho)}$, obtained by a transformation (6.8), evolve from the $\Psi_\nu^{(\varrho)}$ by means of the unitary matrix $W^\dagger$. One concludes then immediately from (6.9) that the eigen functions $\Phi_\nu^{(\varrho)}(z)$ and $\varphi_\nu^{(\varrho)}(z)$ are interrelated by a linear transformation with the unitary matrix $W^\dagger U W^{-1}$. If we now choose the arbitrary unitary matrix W by the condition

$$(6.12) \qquad W = V'$$

one sees that we have the identity

$$(6.13) \qquad \Phi_\nu^{(\varrho)}(z) = \lambda_\nu \iint_B \varphi_\nu^{(\varrho)}(\zeta)^\dagger\, l(\zeta, z) d\tau_\zeta = \varphi_\nu^{(\varrho)}(z).$$

Hence we have proved that *each eigen value λ_ν^2 of the integral equation* (6.5) *leads to an eigen value $\lambda_\nu > 0$ of the integral equation* (6.1). *An appropriate complete set of orthonormalized eigen functions of* (6.5) *can be chosen such that it is simultaneously a complete set of orthonormal eigen functions with respect to* (6.1).

The complete equivalence of the two integral equations (6.1) and (6.5) is, therefore, proved.

The set of eigen functions $\{\varphi_\nu(z)\}$ may be a complete orthonormal set with respect to the function space Λ, i.e. every $\mathfrak{L}^2$-integrable function $f(z)$ in B may be expressed by the Fourrier development

$$(6.14) \qquad f(z) = \sum_{\nu=1}^{\infty} a_\nu \varphi_\nu(z), \qquad a_\nu = \iint_B f(z)\varphi_\nu(z)^\dagger d\tau_z$$

which converges uniformly in every closed subdomain of B. In case of incompleteness there exist functions $f(z) \in \Lambda$ which do not vanish identically and are orthogonal to all eigen functions $\varphi_\nu(z)$. Hence these functions are, also orthogonal to the kernel $l(z, \zeta)$ and may be considered as eigen functions of (6.1) and (6.5) to the eigen value $\lambda = \infty$. We may then complete our system $\{\varphi_\nu(z)\}$ by addition of further eigen functions to this eigen value. We will not exclude in this paper the possibility of the eigen value $\lambda = \infty$ and may, therefore, always assume a complete orthonormal system of eigen functions $\{\varphi_\nu(z)\}$. We arrange the eigen functions in such order that the corresponding eigen values λ_ν form a non-decreasing sequence.

We now may express every function $f \in \Lambda$ in terms of these eigen functions. In view of the integral equation (6.1) it is now exceedingly simple to express the l-transform $\mathrm{T}f$ of f. In fact, we have in view of definition (5.1), (6.14) and (6.1):

$$(6.15) \qquad \mathrm{T}f(z) = \sum_{\nu=1}^{\infty} \frac{1}{\lambda_\nu} a_\nu^\dagger \varphi(z).$$

From (5.10a) and (5.16), we have the inequality

$$(6.16) \qquad (\mathrm{T}f, (\mathrm{T}f)^\dagger) \leqq (f, f^\dagger)$$

which may be expressed in terms of the Fourier coefficients a_ν as follows:

$$(6.16\text{a}) \qquad \sum_{\nu=1}^{\infty} \frac{1}{\lambda_\nu^2} | a_\nu |^2 \leqq \sum_{\nu=1}^{\infty} | a_\nu |^2.$$

As we noticed already in the beginning of this section we know $n-1$ linearly independent eigen functions $i w_\nu'(z)$ to the eigen value $\lambda = 1$. Let us orthonormalize these $n - 1$ functions in the way described before and we obtain the first $n - 1$ eigen functions $\varphi_\nu(z)$. For every function which is linearly independent of these initial eigen functions the inequality (6.16a) must be a proper one. Hence we conclude:

All eigen values λ_ν of $l(z, \zeta)$ are $\geqq 1$ and only the derivatives of the harmonic measures belong to the eigen value 1.

Let us now consider an eigen function $\varphi_\nu(z)$ with $\nu \geqq n$. This function is orthogonal to all functions $w_i'(z)$, $i = 1, 2, \dots n$, i.e.

$$\iint_B \varphi_\nu(z) w_1'(z)^\dagger d\tau_z = 0. \tag{6.17}$$

Using the definition (5.23) of $w_i'(z)$ we obtain by integration by parts

$$\int_C \varphi_\nu(z)\omega_i(x, y)dz = 0. \tag{6.17a}$$

Since $\omega_i(x, y) = 0$ on C except for the boundary component C_i where $\omega_i(x, y) = 1$, we may write instead of (6.17a)

$$\int_{C_i} \varphi_\nu(z)dz = 0. \tag{6.18}$$

In other words: The eigen functions $\varphi_\nu(z)$ belonging to the eigen values $\lambda_\nu > 1$ possess single-valued integrals $\Phi_\nu(z)$ in B.

The subspace Λ_s of Λ, consisting of all functions with single-valued integrals, may also be defined as consisting of all functions $f(z) \in \Lambda$ which are orthogonal to all $w_i'(z)$; this is shown by the same reasoning which leads from (6.17) to (6.18). Hence it is evident that the functions $\varphi_\nu(z)$ which belong to the eigen values $\lambda_\nu > 1$ form a complete orthonormal system for this subspace Λ_s.

Finally we apply the orthonormal system $\{\varphi_\nu(z)\}$ in order to express our kernel functions in Fourier form. From the reproducing property (2.9) we obtain for the K-kernel the typical form, valid in every complete orthonormal system:

$$K(z, \zeta^\dagger) = \sum_{\nu=1}^{\infty} \varphi_\nu(z)\varphi_\nu(\zeta)^\dagger. \tag{6.19}$$

Using the integral equation (6.1) in order to compute the Fourier coefficients of the l-kernel, we obtain the development:

$$l(z, \zeta) = \sum_{\nu=1}^{\infty} \frac{1}{\lambda_\nu} \varphi_\nu(z)\varphi_\nu(\zeta). \tag{6.20}$$

From (6.20) and (4.8) we conclude next

$$K(z, \zeta^\dagger) - \Gamma(z, \zeta^\dagger) = \sum_{\nu=1}^{\infty} \frac{1}{\lambda_\nu^2} \varphi_\nu(z) \varphi_\nu(\zeta)^\dagger \tag{6.21}$$

a result which also follows immediately from the general theory of positive definite kernels. From (6.19) and (6.21) we derive further

$$\Gamma(z, \zeta^\dagger) = \sum_{\nu=1}^{\infty} \left(1 - \frac{1}{\lambda_\nu^2}\right) \varphi_\nu(z) \varphi_\nu(\zeta)^\dagger. \tag{6.22}$$

This shows that the eigen functions $\varphi_\nu(z)$ may also be considered as belonging to the purely geometric kernel $\Gamma(z, \zeta^\dagger)$ with the eigen value $(1 - \lambda_\nu^{-2})$.

The positive definite character of $\Gamma(z, \zeta^\dagger)$ and the development (6.22) provide a new proof for the fact that all λ_ν are greater or equal to one.

7. Discussion of the eigen functions

The significance of the eigen functions of the integral equation (6.1) and their connection with a classical problem of potential theory become clear by the following considerations. In view of (5.4) we may write the integral equation (6.1) in the form

$$\varphi_\nu(z) = \frac{\lambda_\nu}{2\pi i} \int_C (\zeta - z)^{-1} [\varphi_\nu(\zeta) d\zeta]^\dagger. \tag{7.1}$$

From Cauchy's theorem we have, on the other hand, immediately

$$\lambda_\nu \varphi_\nu(z) = \frac{\lambda_\nu}{2\pi i} \int_C (\zeta - z)^{-1} \varphi_\nu(\zeta) d\zeta. \tag{7.2}$$

Adding these two equations and introducing the harmonic real function $h_\nu(x, y)$ for which

$$\varphi_\nu(z) = \frac{\partial}{\partial z} h_\nu(x, y) \tag{7.3}$$

we obtain

$$\varphi_\nu(z) = \frac{\lambda_\nu}{2\pi i(1 + \lambda_\nu)} \int_C \frac{dh_\nu(\xi, \eta)}{\zeta - z} = \frac{\lambda_\nu}{2\pi i(1 + \lambda_\nu)} \int_C \frac{h_\nu(\xi, \eta)}{(\zeta - z)^2} d\zeta. \tag{7.4}$$

Integrating this equation, we arrive at the integral equation for $h_\nu(x, y)$

$$h_\nu(x, y) = \frac{\lambda_\nu}{\pi(1 + \lambda_\nu)} \int_C h_\nu(\xi, \eta) \Re\left\{\frac{\zeta'}{i(\zeta - z)}\right\} ds_\zeta + \text{const.} \tag{7.5}$$

Now it is well known that

$$\frac{\partial}{\partial n_\zeta} \log \frac{1}{|\zeta - z|} = \Re\left\{\frac{\zeta'}{i(\zeta - z)}\right\}. \tag{7.6}$$

Thus, (7.5) obtains the form

$$h_\nu(x, y) = \frac{\lambda_\nu}{\pi(1+\lambda_\nu)} \int_C h_\nu(\xi, \eta) \frac{\partial}{\partial n_\zeta}\left(\log \frac{1}{|\zeta - z|}\right) ds_\zeta + \text{const.} \tag{7.7}$$

The integral equation (7.7) for $h_\nu(x, y)$ contains an arbitrary constant of integration which is evident since the definition (7.3) of h_ν determines this function only up to an additive constant.

Let us now suppose that we know a solution $h_\nu(x, y)$ of (7.7). The function $h_\nu(x, y) + a$ will also be a solution of the same integral equation because of the well-known fact that for $z \in B$ we have the identity

$$\frac{1}{2\pi} \int_C \frac{\partial}{\partial n_\zeta}\left(\log \frac{1}{|\zeta - z|}\right) ds_\zeta = 1. \tag{7.8}$$

Thus, we conclude from (7.7):

$$h_\nu(x, y) + a = \frac{\lambda_\nu}{\pi(1+\lambda_\nu)} \int_C [h_\nu(\xi, \eta) + a] \frac{\partial}{\partial n_\zeta}\left(\log \frac{1}{|\zeta - z|}\right) ds_\zeta + {} \\ + a\frac{1 - \lambda_\nu}{1 + \lambda_\nu} + \text{const.} \tag{7.7a}$$

We see that for $\lambda_\nu > 1$ we are able to introduce such a constant a into our function $h_\nu(x, y)$ that it satisfies the simpler integral equation

$$h_\nu(x, y) = \frac{\lambda_\nu}{\pi(1+\lambda_\nu)} \int_C h_\nu(\xi, \eta) \frac{\partial}{\partial n_\zeta}\left(\log \frac{1}{|\zeta - z|}\right) ds_\zeta, \quad z \in B. \tag{7.9}$$

We may derive from (7.9) an integral equation for the function $h_\nu(s) = h_\nu(x(s), y(s))$ considered as a function of the arc length on C. Using the discontinuity behavior of the dipole potential on the charged line C, we find

$$h_\nu(s_z) = \frac{\lambda_\nu}{\pi} \int_C h_\nu(s_\zeta) \frac{\partial}{\partial n_\zeta}\left(\log \frac{1}{|\zeta - z|}\right) ds_\zeta. \tag{7.10}$$

This result gives a clear understanding of the significance of the eigen functions $\varphi_\nu(z)$. The inhomogeneous integral equation

$$f(s_z) = \varphi(s_z) - \frac{\lambda}{\pi} \int_C \varphi(s_\zeta) \frac{\partial}{\partial n_\zeta}\left(\log \frac{1}{|\zeta - z|}\right) ds_\zeta, \tag{7.11}$$

plays a central role in the boundary value problem of harmonic functions if treated with the Fredholm theory. Now we see that our eigen functions $\varphi_\nu(z)$ are closely connected with the eigen functions of the corresponding homogeneous integral equation (7.10). Their importance for general theory thus becomes obvious.

To illustrate the theory we shall now determine the eigen functions $\varphi_\nu(z)$ and their corresponding eigen values λ_ν for a few simple domains.

Let the domain B be mapped by the linear transformation

$$w = \varphi(z) = \frac{\alpha z + \beta}{\gamma z + \delta} \tag{7.12}$$

into a new domain B_1. The function $\Phi(z, \zeta)$, defined in (3.6), is in this case

$$\Phi(z, \zeta) = \frac{1}{\pi}\{\log(\alpha\delta - \beta\gamma) - \log(\gamma z + \delta) - \log(\gamma\zeta + \delta)\} \tag{7.12a}$$

whence

$$\frac{\partial^2 \Phi}{\partial z \partial \zeta} = 0. \tag{7.13}$$

Hence the transformation formula (3.5a) for the l-kernel now takes the simple form

$$l_1(w, \omega)\varphi'(z)\varphi'(\zeta) = l(z, \zeta), \tag{7.14}$$

and in view of (6.20) we have the series development

$$l_1(w, \omega) = \sum_{\nu=1}^{\infty} \frac{1}{\lambda_\nu}[\varphi_\nu(z)\varphi'(z)^{-1}] \cdot [\varphi_\nu(\zeta)\varphi'(\zeta)^{-1}]. \tag{7.15}$$

Now it is easily verified that the functions

$$\psi_\nu(w) = \varphi_\nu(z(w))\varphi'(z(w))^{-1} \tag{7.16}$$

form a complete orthonormal set of analytic functions in B_1. Hence

$$l_1(w, \omega) = \sum_{\nu=1}^{\infty} \frac{1}{\lambda_\nu} \psi_\nu(w)\, \psi_\nu(\omega) \tag{7.15a}$$

and this clearly shows that the $\psi_\nu(w)$ are the eigen functions in B_1 with the same eigen values λ_ν. We proved, therefore:

If a domain B is mapped into a domain B_1 by a linear transformation (7.12) *the eigen functions of both domains are related by* (7.16) *and the eigen values are the same.*

For the case of the unit circle we have $l(z, \zeta) = 0$; hence all eigen values are infinite and because of the previous theorem this is true for every circle. We may now also consider domains B

which contain the point at infinity since we may always transform such domains into finite domains by linear transformation.

Consider the domain B_1 obtained by mapping the exterior of the unit circle by means of

$$(7.17)\qquad w = \varphi(z) = z + \frac{a}{z}, \qquad a < 1.$$

B_1 is the exterior of an ellipse with the principal axes $(1 + a)$ and $(1 - a)$. Using (3.8), we have immediately a formula for the l-kernel of this simply-connected domain:

$$(7.18)\quad l_1(w, \omega) = (\varphi'(z)\varphi'(\zeta))^{-1} \cdot \frac{-a}{\pi(z\zeta - a)^2} = \sum_{\nu=1}^{\infty} a^{\nu}\psi_{\nu}(w)\psi_{\nu}(\omega)$$

with

$$(7.18a)\qquad \psi_{\nu}(w) = i\sqrt{\frac{\nu}{\pi}} \cdot z^{-(\nu+1)}\varphi'(z)^{-1}, \quad w = \varphi(z)$$

Since the functions $i\sqrt{\frac{\nu}{\pi}} z^{-(\nu+1)}$ $(\nu = 1, 2, \ldots)$ form a complete orthonormal system in $|z| > 1$ the $\psi_{\nu}(w)$ do the same in B_1. The representation of the l-kernel shows that the $\psi_{\nu}(w)$ are the eigen functions of the exterior of the ellipse and we have in this case:

$$(7.19)\qquad \lambda_{\nu} = a^{-\nu}.$$

The domain B_1 has an interesting extremum property with respect to the eigen values λ_{ν}. Consider an arbitrary simply-connected domain B which is bounded by a closed analytic curve C and contains the point at infinity. One shows by a linear transformation that even for such a domain B the development (6.22) for $\Gamma(z, \zeta^{\dagger})$ is valid; it is also obvious that $\lambda_1 > 1$ because of the simple connectivity of B. Hence we derive from (6.22) the inequality

$$(7.20)\quad \Gamma(z, z^{\dagger}) \geqq \left(1 - \frac{1}{\lambda_1^2}\right) \sum_{\nu=1}^{\infty} |\, \varphi_{\nu}(z)\,|^2 = \left(1 - \frac{1}{\lambda_1^2}\right) K(z, z^{\dagger})$$

This inequality assumes a simple meaning if we let $z \to \infty$. Let

$$(7.21)\qquad z = d(\zeta + c_0 + c_1\zeta^{-1} + \ldots)$$

be the function which maps the domain $|\zeta| > 1$ upon B. The constant d is called the mapping radius of B and plays a considerable role in the conformal geometry of B. One easily verifies the limit relations, which follow from (3.8), (7.21) and (4.9):

$$(7.22)\qquad \lim_{z\to\infty} |\, z\,|^4 K(z, z^{\dagger}) = \frac{1}{\pi} d^2$$

and

$$\lim_{z\to\infty} |z|^4 \Gamma(z, z^\dagger) = \frac{1}{\pi^2} A \tag{7.23}$$

where A is the area of the finite complement $\breve{B}$ of B.

From (7.20), (7.22) and (7.23) we obtain the inequality

$$A \geqq \left(1 - \frac{1}{\lambda_1^2}\right) \pi d^2. \tag{7.24}$$

It is well known that between the mapping radius d and the area A of $\breve{B}$ the following inequality holds:

$$\pi d^2 \geqq A. \tag{7.25}$$

Hence we can transform (7.24) into

$$\lambda_1^2 \leqq \frac{\pi d^2}{\pi d^2 - A}. \tag{7.26}$$

We see that the lowest eigen value λ_1 provides an upper bound for the excess of πd^2 over A. In the case of a circle we have $\lambda_1 = \infty$ and $\pi d^2 = A$.

Now it is interesting that the inequality (7.26) is an equality for all ellipses; these may therefore be considered as the extremum domains with respect to (7.26). In fact we have in the case of the ellipse $B_1 : d = 1$, $A = \pi(1 - a^2)$ and $\lambda_1^2 = a^{-2}$ which shows that equality holds in (7.26).

Another interesting result may be obtained for the eigen functions $\varphi_\nu(z)$ of a simply-connected domain B. In this case the whole plane is divided into two complementary domains B and $\breve{B}$ and let $\varphi_\nu(z)$, λ_ν and $\breve{\varphi}_\nu(z)$, $\breve{\lambda}_\nu$ denote the corresponding eigen functions and eigen values. The eigen functions $\varphi_\nu(z)$ of B have an l-transform $\mathrm{T}\varphi_\nu$ which is defined in B and in $\breve{B}$. In view of the integral equation we have in B

$$\mathrm{T}\varphi_\nu(z) = \frac{1}{\lambda_\nu} \varphi_\nu(z) \qquad \text{for } z \in B. \tag{7.27}$$

Hence we may write the identity (5.10) in the form

$$\iint_{\breve{B}} \mathrm{T}\varphi_\nu \cdot (\mathrm{T}\varphi_\mu)^\dagger d\tau + \frac{1}{\lambda_\nu \lambda_\mu} \iint_B \varphi_\nu \varphi_\mu^\dagger \, d\tau = \iint_B \varphi_\nu^\dagger \varphi_\mu \, d\tau. \tag{7.28}$$

Because of the orthogonality relations between the eigen functions we obtain

$$\iint_{\breve{B}} \mathrm{T}\varphi_\nu (\mathrm{T}\varphi_\mu)^\dagger \, d\tau = \left(1 - \frac{1}{\lambda_\nu^2}\right) \delta_{\nu\mu}. \tag{7.29}$$

Hence the transforms of the eigen functions $\varphi_\nu(z)$ of B create an orthonormal system of analytic functions in $\breve{B}$:

$$\psi_\nu(z) = i\left(1 - \frac{1}{\lambda_\nu^2}\right)^{-\frac{1}{2}} \mathrm{T}\varphi_\nu(z) \qquad \text{for } z \in \breve{B}. \tag{7.30}$$

Because of (7.27) and the saltus condition (5.5) for $\mathrm{T}\varphi_\nu$, we find for the boundary value of $\psi_\nu(z)$ at a point $z \in C$:

$$\psi_\nu(z) = i\left(1 - \frac{1}{\lambda_\nu^2}\right)^{-\frac{1}{2}} \left[\frac{1}{\lambda_\nu} \varphi_\nu(z) - (\varphi_\nu(z) z'^2)^\dagger\right], \quad z \in C. \tag{7.31}$$

We have, therefore, integrating along C in the positive sense with respect to $\breve{B}$:

$$\frac{1}{2\pi i} \int_C \frac{[\psi_\nu(\zeta)\, d\zeta]^\dagger}{\zeta - z} = -i\left(1 - \frac{1}{\lambda_\nu^2}\right)^{-\frac{1}{2}} \left\{\frac{1}{\lambda_\nu} \cdot \frac{1}{2\pi i} \int_C \frac{[\varphi_\nu(\zeta) d\zeta]^\dagger}{\zeta - z} - \frac{1}{2\pi i} \int_C \frac{\varphi_\nu(\zeta)\, d\zeta}{\zeta - z}\right\}, \quad z \in \breve{B}. \tag{7.32}$$

The last right-hand integral vanishes because of Cauchy's theorem, and using the definition (7.30) of $\psi_\nu(z)$ we find by means of (5.4):

$$\psi_\nu(z) = \frac{\lambda_\nu}{2\pi i} \int_C \frac{[\psi_\nu(\zeta)\, d\zeta]^\dagger}{\zeta - z} = \lambda_\nu \mathrm{T}\psi_\nu(z). \tag{7.33}$$

This proves that the functions $\psi_\nu(z)$ are the eigen functions $\breve{\varphi}_\nu$ of $\breve{B}$ and that the sequences λ_ν and $\breve{\lambda}_\nu$ are identical. Hence

Two complementary simply-connected domains have the same set of eigen values. This is a very useful result since the determination of the eigen values of one domain may be much easier than those of the other. We see for example that the interior of an ellipse with principal axes $(1 + a)$ and $(1 - a)$ has the eigen values $a^{-\nu}$; while the mapping function of a circle on the exterior of this ellipse is an elementary function, the map of the circle upon the interior of the ellipse is given by quite complicated elliptic functions. The importance of our result for the conformal geometry of domains becomes quite obvious from this example.

We understand the last result better if we notice that the λ_ν are the eigen values of the integral equation (7.10) which is defined on C alone and does not indicate which adjacent domain of C is to be considered. For the same reason the treatment by integral equations of the boundary value problem for harmonic functions leads to a simultaneous solution for the so-called interior and exterior problems.

If the eigen functions $\varphi_\nu(z)$ of a simply-connected domain B are known, we may easily construct the kernel functions and by their aid map B upon the unit-circle. At the same time, we can compute by elementary integrations the functions $\breve{\varphi}_\nu(z)$ of the complementary domain $\breve{B}$ and determine its kernel functions, too. Hence knowledge of the $\varphi_\nu(z)$ permits at the same time the conformal mapping of the two complementary domains upon the unit circle. We have the formulas for the kernel functions $\breve{K}(z, \zeta^\dagger)$ and $\breve{l}(z, \zeta)$:

$$\breve{K}(z, \zeta^\dagger) = \sum_{\nu=1}^{\infty} (1 - \lambda_\nu^{-2})^{-1} \cdot \mathrm{T}\varphi_\nu(z)[\mathrm{T}\varphi_\nu(\zeta)]^\dagger, \tag{7.34}$$

$$\breve{l}(z, \zeta) = -\sum_{\nu=1}^{\infty} (\lambda_\nu - \lambda_\nu^{-1})^{-1}\, \mathrm{T}\varphi_\nu(z) \cdot \mathrm{T}\varphi_\nu(\zeta). \tag{7.35}$$

We notice also the elegant formula:

$$\breve{\Gamma}(z, \zeta^\dagger) = \sum_{\nu=1}^{\infty} \mathrm{T}\varphi_\nu(z)[\mathrm{T}\varphi_\nu(\zeta)]^\dagger \tag{7.36}$$

which follows easily from (6.22) and (7.30).

8. The space Λ_s and its kernel functions.

Let

$$F(z; u, v) = \log \frac{z-u}{z-v} + \text{regular terms} \tag{8.1}$$

and

$$G(z; u, v) = \log \frac{z-u}{z-v} + \text{regular terms} \tag{8.2}$$

be the logarithms of two univalent functions in B which map this domain on the whole complex plane slit along concentric circular arcs around the origin or along rectilinear slits directed towards the origin, respectively. The points $u, v \in B$ shall correspond to the origin and the point at infinity after the mapping. The functions F and G are determined by this description up to an additive constant. On each contour C_ν of B, we have

$$F(z; u, v) = a_\nu + iK_\nu(s), \quad z \in C_\nu, \tag{8.3}$$

$$G(z; u, v) = l_\nu(s) + ib_\nu, \quad z \in C_\nu \tag{8.4}$$

where a_ν and b_ν are constants and $k_\nu(s)$, $l_\nu(s)$ real-valued functions of the arc length s.

Let us define two functions

$$(8.5)\qquad P(z; u, v) = \frac{1}{2\pi}\{F(z; u, v) - G(z; u, v)\}$$

and

$$(8.6)\quad Q(z; u, v) = \frac{1}{2\pi}\{F(z; u, v) + G(z; u, v)\} = \frac{1}{\pi}\log\frac{z-u}{z-v} - q(z; u, v).$$

They are both analytic in B, except for two simple logarithmic poles of Q at u and v. On each C_ν one has because of (8.3) and (8.4)

$$(8.7)\qquad P(z; u, v) = -Q(z; u, v)^\dagger + c_\nu, \qquad z \in C_\nu.$$

The functions $P'(z; u, v)$ and $q'(z; u, v)$ (where the dash denotes the differentiation with respect to the first argument) are both of the class Λ_s and we want to develop them in Fourier series with respect to the complete orthonormal system $\varphi_\nu(z)$ with $\lambda_\nu > 1$, i.e. $\nu \geqq n$. We have by virtue of (1.4)

$$(8.8)\qquad \iint_B P'(z; u, v)\varphi_\nu(z)^\dagger d\tau_z = -\frac{1}{2i}\int_C P(z; u, v)[\varphi_\nu(z)dz]^\dagger.$$

Using (6.18) and (8.7) this may also be written in the form

$$(8.9)\qquad \frac{1}{2i}\int_C [Q(z; u, v)\varphi_\nu(z)dz]^\dagger = \left[\frac{1}{2i}\int_C Q'(z; u, v)\Phi_\nu(z)dz\right]^\dagger$$

where $\Phi_\nu(z)$ denotes again the single-valued integral of $\varphi_\nu(z)$. From (8.6) and the residue theorem we finally find

$$(8.10)\qquad \iint_B P'(z; u, v)\varphi_\nu(z)^\dagger d\tau_z = [\Phi_\nu(u) - \Phi_\nu(v)]^\dagger$$

whence the Fourier series

$$(8.11)\qquad P'(z; u, v) = \sum_{\nu=n}^{\infty} \varphi_\nu(z)[\Phi_\nu(u) - \Phi_\nu(v)]^\dagger$$

and integrating this identity between z and ζ, we finally obtain:

$$(8.12)\quad P(z; u, v) - P(\zeta; u, v) = \sum_{\nu=n}^{\infty} [\Phi_\nu(z) - \Phi(\zeta)] \cdot [\Phi_\nu(u) - \Phi_\nu(v)]^\dagger.$$

It should be noticed that in this derivation no use was made of the integral equation satisfied by the $\varphi_\nu(z)$ so that the representation (8.12) will hold for each complete orthonormal system $\varphi_\nu(z)$ with single-valued integrals $\Phi_\nu(z)$.

Next, we compute

$$(8.13)\qquad \iint_B q'(z; u, v)\varphi_\nu(z)^\dagger d\tau_z = -\frac{1}{2i}\int_C q(z; u, v)[\varphi_\nu(z)dz]^\dagger.$$

Using the definition (8.6) of q and the boundary relation (8.7), we obtain

$$(8.14)\quad \iint_B q'(z;u,v)\varphi_\nu(z)^\dagger d\tau_z = -\frac{1}{2\pi i}\int_C \log\frac{z-u}{z-v}[\varphi_\nu(z)dz_\nu]^\dagger + \left[\frac{1}{2i}\int_C P(z;u,v)\varphi_\nu(z)dz\right]^\dagger.$$

The last integral vanishes by Cauchy's theorem, while the integral equation (7.1) yields by integration between u and v:

$$(8.15)\qquad \Phi_\nu(u)-\Phi_\nu(v) = \frac{-\lambda_\nu}{2\pi i}\int_C \log\frac{z-u}{z-v}[\varphi_\nu(z)dz]^\dagger.$$

Thus, we finally arrive at the Fourier series

$$(8.16)\qquad q'(z;u,v) = \sum_{\nu=n}^{\infty}\frac{1}{\lambda_\nu}\varphi_\nu(z)(\Phi_\nu(u)-\Phi_\nu(v)).$$

Integrating again between z and ζ, we obtain at last

$$(8.17)\quad q(z;u,v)-q(\zeta;u,v)=\sum_{\nu=n}^{\infty}\frac{1}{\lambda_\nu}(\Phi_\nu(z)-\Phi_\nu(\zeta))(\Phi_\nu(u)-\Phi_\nu(v)).$$

Most of the important domain functions, as for example Green's and Neumann's functions of B and many others may easily be expressed in terms of P and Q (Garabedian—Schiffer [1]). The formulas (8.12) and (8.17) show the simple construction of these functions in terms of the $\varphi_\nu(z)$.

Let further

$$(8.18)\qquad f_0(z,u) = \frac{1}{z-u} + \text{regular terms},$$

$$g_0(z,u) = \frac{1}{z-u} + \text{regular terms}$$

be univalent in B, mapping the domain upon the whole plane slit along straight segments parallel to the real and the imaginary axis, respectively. The point $z=u$ obviously corresponds to infinity.

Using the well-known relations between f_0, g_0 on the one hand and P, Q on the other, it is possible to show that

$$(8.19)\quad \frac{1}{2\pi}[f_0'(z,u)-g_0'(z,u)]=\sum_{\nu=n}^{\infty}\varphi_\nu(z)\varphi_\nu(u)^\dagger,\qquad f_0'(z,u)=\frac{df_0(z,u)}{dz},$$

and

$$(8.20)\quad \frac{1}{2\pi}[f_0'(z,u)+g_0'(z,u)]+\frac{1}{\pi(z-u)^2}=\sum_{\nu=n}^{\infty}\frac{1}{\lambda_\nu}\varphi_\nu(z)\varphi_\nu(u).$$

These results could also be obtained directly by applying to f_0 and g_0 the same reasoning as we did before to F and G.

We see. that the kernel functions of the Λ_s-space

$$K_s(z, \zeta^\dagger) = \sum_{\nu=n}^{\infty} \varphi_\nu(z)\varphi_\nu(\zeta)^\dagger \tag{8.21}$$

and

$$l_s(z, \zeta) = \sum_{\nu=n}^{\infty} \frac{1}{\lambda_\nu} \varphi_\nu(z)\, \varphi_\nu(\zeta) \tag{8.22}$$

have an important geometric significance.

It is of interest to notice that K_s and l_s lead to the same algebra under scalar multiplication as did K and l. In fact, we clearly have:

$$\iint_B K_s(z, \zeta^\dagger)K_s(\zeta, w^\dagger)d\tau_\zeta = K_s(z, w^\dagger), \tag{8.23}$$

$$\iint_B K_s(z, \zeta^\dagger)\, l_s(\zeta, w)d\tau_\zeta = l_s(z, w), \tag{8.24}$$

and in view of (8.22) and (6.22):

$$\iint_B l_s(z, \zeta)\, l_s(\zeta, w)^\dagger d\tau_\zeta = K_s(z, w^\dagger) - \Gamma(z, w^\dagger). \tag{8.25}$$

In fact, in the series development (6.22) for $\Gamma(z, w^\dagger)$ the first $n-1$ eigen functions do not appear. Thus, all inequalities which we deduced in section 4 for the kernel functions K and l of the space Λ remain valid if we replace those kernels by K_s and l_s. It is also easily verified that K_s and l_s behave under conformal transformations just as K and l and that analogous formulas to (3.3) and (3.5) hold for them.

Since the developments (6.19) and (6.20) may be expressed in the form

$$K(z, \zeta^\dagger) = -\frac{2}{\pi}\frac{\partial^2 g(z, \zeta)}{\partial z\, \partial \zeta^\dagger} = \sum_{i,k=1}^{n-1} p_{ik} w_i'(z) w_k'(\zeta)^\dagger + K_s(z, \zeta^\dagger) \tag{8.26}$$

and

$$l(z, \zeta) = \frac{1}{\pi(z-\zeta)^2} + \frac{2}{\pi}\frac{\partial^2 g(z, \zeta)}{\partial z\, \partial \zeta} = -\sum_{i,k=1}^{n-1} p_{ik} w_i'(z) w_k'(\zeta) + l_s(z, \zeta) \tag{8.27}$$

with real coefficients p_{ik}, we have

$$K_s(z, \zeta^\dagger) = -\frac{2}{\pi}\frac{\partial^2 G(z, \zeta)}{\partial z\, \partial \zeta^\dagger}, \quad l_s(z, \zeta) = \frac{1}{\pi(z, \zeta)^2} + \frac{2}{\pi}\frac{\partial^2 G(z, \zeta)}{\partial z\, \partial \zeta} \tag{8.28}$$

with

(8.29) $$G(z, \zeta) = g(z, \zeta) + 2\pi \sum_{i,k=1}^{n-1} p_{ik}\omega_i(z)\omega_k(\zeta).$$

It is easily seen that $G(z, \zeta)$ is the real part of the logarithm of a univalent function which maps the domain B upon the exterior of a circle which is slit along concentric circular arcs. The point ζ corresponds in this map to infinity.

The fact that in the series development for $K_s(z, \zeta^\dagger)$ only eigen functions occur with $\lambda_\nu > 1$ leads to the following important application. We define the ϱ-th iterated Γ-kernel by the formula

(8.30) $$\Gamma^{(\varrho)}(z, \zeta^\dagger) = \sum_{\nu=n}^{\infty} \left(1 - \frac{1}{\lambda_\nu^2}\right)^{\varrho} \varphi_\nu(z)\varphi_\nu(\zeta)^\dagger.$$

Obviously, we have

(8.31) $$\Gamma^{(\varrho+1)}(z, \zeta^\dagger) = \iint_B \Gamma^{(\varrho)}(z, w^\dagger)\, \Gamma(w, \zeta^\dagger) d\tau_w, \quad \Gamma^{(1)}(z, \zeta^\dagger) = \Gamma(z, \zeta^\dagger).$$

Hence all kernels $\Gamma^{(\varrho)}(z, \zeta^\dagger)$ are geometric integrals and may be computed elementarily.

Next we consider the kernels

(8.32) $$\Delta_\mu(z, \zeta^\dagger) = \sum_{\varrho=0}^{\mu} (-1)^{\varrho} \binom{\mu}{\varrho} \Gamma^{(\varrho+1)}(z, \zeta^\dagger) =$$
$$= \sum_{\nu=n}^{\infty} \left(1 - \frac{1}{\lambda_\nu^2}\right) \frac{1}{\lambda_\nu^{2\mu}} \varphi_\nu(z)\, \varphi_\nu(\zeta)^\dagger.$$

The $\Delta_\mu(z, \zeta^\dagger)$ are also elementary expressions being linear combinations of the $\Gamma^{(\varrho)}$-kernels. It is obvious that they are positive-definite kernels.

Finally, we construct the sum

(8.33) $$\sum_{\mu=0}^{\infty} \Delta_\mu(z, \zeta^\dagger) = \sum_{\nu=n}^{\infty} \varphi_\nu(z)\varphi_\nu(\zeta)^\dagger = K_s(z, \zeta^\dagger).$$

We see that we can express the kernel $K_s(z, \zeta^\dagger)$ as an infinite sum of elementary integrals. It is of particular interest that

(8.34) $$K_s(z, z^\dagger) = \sum_{\mu=0}^{\infty} \Delta_\mu(z, z^\dagger)$$

appears as a sum of positive terms. This leads to an infinity of inequalities for $K_s(z, z^\dagger)$. Since $\Delta_0(z, z^\dagger) = \Gamma(z, z^\dagger)$, we see that the inequalities

(8.35) $$K_s(z, z^\dagger) \geqq \Gamma(z, z^\dagger)$$

is only the first in a series of improving inequalities for the kernel function.

From (8.32) and (6.22) we obtain the inequality

$$(8.36)\qquad \Delta_\mu(z, z^\dagger) \leqq \frac{1}{\lambda_n^{2\mu}} \Gamma(z, z^\dagger);$$

by Schwarz' inequality we have on the other hand:

$$(8.37)\qquad |\Delta_\mu(z, \zeta^\dagger)|^2 \leqq \Delta_\mu(z, z^\dagger)\, \Delta_\mu(\zeta, \zeta^\dagger) \leqq \frac{1}{\lambda_n^{4\mu}} \Gamma(z, z^\dagger)\, \Gamma(\zeta, \zeta^\dagger).$$

Hence we see that the series development (8.33) for the K_s-kernel converges geometrically. It seems that it leads to a very useful numerical method in conformal mapping.

Each domain B can be mapped into a canonical domain which plays a distinguished role with respect to the kernel functions K_s and l_s. It is well known that the function

$$(8.38)\qquad h(z, u) = \tfrac{1}{2}[f_0(z, u) + g_0(z, u)] = \frac{1}{z - u} + \text{regular terms}$$

maps the domain B univalently upon a domain B_1 which solves the following extremum problem: Among all domains which are obtained from B by a conformal map with a pole of residue 1 at $z = u$, B_1 possesses a complement $\widetilde{B}_1$ with maximal area A_μ (Schiffer [1]). B_1 is a canonical domain for all domains B which can be mapped into each other by means of a univalent function $\varphi(z)$ with the normalization

$$(8.39)\qquad \varphi(u) = u, \qquad \varphi'(u) = 1.$$

In fact, let $w = \varphi(z)$ map our original domain B into a new domain B_2. Let $f_{02}(w, u)$, $g_{02}(w, u)$ and $h_2(w, u)$ denote analogous univalent functions with respect to B_2 as were $f_0(z, u)$, $g_0(z, u)$ and $h(z, u)$ with respect to B. We clearly may put

$$(8.40)\qquad f_{02}(\varphi(z), u) = f_0(z, u), \quad g_{02}(\varphi(z), u) = g_0(z, u)$$

and hence

$$(8.41)\qquad h_2(\varphi(z), u) = h(z, u).$$

Hence the same procedure (8.38) leads to the same canonical domain B_1 for all domains B_2 which are equivalent to B by means of a function (8.39).

The function

$$(8.42)\qquad w = H(z, u) = u + h(z, u)^{-1}$$

has clearly the normalization (8.39) and maps B upon a canonical domain D which is more suitable for our purposes. The transition from D to B_1 is given by the linear transformation

$$h_D(w, u) = (w - u)^{-1}. \tag{8.43}$$

On the other hand, we have because of (8.38), (8.20) and (8.22) for each domain B the identity

$$\dot{h}'(z, u) = \pi l_s(z, u) - \frac{1}{(z-u)^2}. \tag{8.44}$$

Thus, in the particular case of the domain D we conclude from (8.43) and (8.44)

$$l_s(w, u) \equiv 0, \qquad w \in D. \tag{8.45}$$

From (8.25) we easily obtain

$$K_s(w, u^\dagger) = \Gamma(w, u^\dagger). \tag{8.46}$$

This shows that in the case of the canonical domain D the series development (8.33) for $K_s(w, u^\dagger)$ may be stopped after the first term.

From the series development (8.22) and (8.45) we conclude that in the case of the domain D all eigen function $\varphi_\nu(w)$ which do not belong to the eigen value ∞ vanish at the distinguished point u. Since $\Gamma(u, u^\dagger) \neq 0$ we conclude also that in the case of the canonical domain D there exists at least one eigen value ∞.

We now define the following concept: Let R be a domain in the z-plane which does not contain the point u. We call the expression

$$A_u(R) = \iint_R \frac{d\tau_t}{|t-u|^4} \tag{8.47}$$

the area of R with respect to u. Clearly, $A_u(R)$ represents the area of the image of R under the linear transformation $\dfrac{1}{z-u}$.

From (8.46) we conclude

$$K_s(u, u^\dagger) = \Gamma(u, u^\dagger) = \frac{1}{\pi^2} A_u(\widetilde{D}). \tag{8.48}$$

By definition of D, we clearly have $A_u(\widetilde{D}) = A_u$. Because of the behavior of $K_s(u, u^\dagger)$ under conformal transformation, its value is the same for all equivalent domains obtained from each other by means of a function (8.39). From (6.22), we obtain on the other hand the inequality

$$\Gamma(u, u^\dagger) \geqq \left(1 - \frac{1}{\lambda_n^2}\right) K(u, u^\dagger) = \frac{1}{\pi^2}\left(1 - \frac{1}{\lambda_n^2}\right) A_u. \tag{8.49}$$

Clearly, $\Gamma(u, u^\dagger) = \frac{1}{\pi^2} A_u(\breve{B})$; hence we proved the inequality

$$A_u(\breve{B}) \geqq \left(1 - \frac{1}{\lambda_u^2}\right) A_u. \tag{8.50}$$

Because of the extremum property of B_1, mentioned above, we always have $A_\mu(\breve{B}) \leqq A_\mu$ and hence (8.50) leads to

$$\lambda_n^2 \leqq \frac{A_u}{A_u - A_u(\breve{B})}. \tag{8.51}$$

A somewhat different approach is necessary if the point $u = \infty$ lies in B. In this case the class of univalent functions with the normalization

$$\varphi(\infty) = \infty, \qquad \varphi'(\infty) = 1 \tag{8.39a}$$

must be considered. Let B_1 be that domain obtained from B by means of a function (8.39a) which has a complement $\breve{B}_1$ with the largest possible area A. This maximum area is related to the span S of B by means of the identity $A = \frac{\pi}{2} S$. The span plays a role in various problems of conformal mapping of multiply-connected domains (Schiffer [1]). One shows easily that (8.51) tends for $u \to \infty$ to the inequality

$$\lambda_n^2 \leqq \frac{A}{A - A(\breve{B})} = \frac{\pi s}{\pi s - 2A(\breve{B})} \tag{8.52}$$

where $A(\breve{B})$ is the area of the complement $\breve{B}$ of B. This is the generalization of (7.26) to the case of multiply-connected domains.

9. Applications to the theory of univalent functions:

We proved in section 4 inequalities of the type (4.14a) between the kernel functions K and l; exactly the same inequalities can be derived from (8.23—8.25) for the kernel functions K_s and l_s of the class A_s. Since the functions K and l show a very different behavior under conformal transformation these inequalities represent also important inequalities for the univalent mapping functions in the domain.

Let $w = \varphi(z)$ be univalent and analytic in the closed region $B + C$. It maps B upon a domain B_1 of the same type as B and we have, therefore, between the kernels K_1 and l_1 the following inequalities in analogy to (4.14):

$$\left| \sum_{\nu, \mu=1}^{r} \alpha_\nu \alpha_\mu l_1(\omega_\nu, \omega_\mu) \right| \leqq \sum_{\nu, \mu=1}^{r} \alpha_\nu \alpha_\mu^\dagger K_1(\omega_\nu, \omega_\mu^\dagger) \tag{9.1}$$

for every choice of the complex numbers α_ν and points $\omega_\nu \in B_1$. Now let ζ_ν be the point in B corresponding to ω_ν, i.e. $\omega_\nu = \varphi(\zeta_\nu)$. Using the transformation formulas (3.3) and (3.5a) for the kernel functions, we obtain

$$(9.2) \qquad \left| \sum_{\nu, \mu=1}^{r} \alpha_\nu \alpha_\mu [l(\zeta_\nu, \zeta_\mu) + U(\zeta_\nu, \zeta_\mu)] \right| \leqq \sum_{\nu, \mu=1}^{r} \alpha_\nu \alpha_\mu^\dagger K(\zeta_\nu, \zeta_\mu^\dagger)$$

with

$$(9.3) \qquad U(z, \zeta) = \frac{\partial^2 \Phi(z, \zeta)}{\partial z \, \partial \zeta}.$$

If we assume our domain B and its kernel functions well-known and fixed, we have in (9.2) an important condition on all univalent functions in B.

We illustrate our result by considering special cases of (9.2). Let B be the unit circle $|z| < 1$. In this case K and l are given by (2.7). Hence we have the inequality

$$(9.2a) \qquad \left| \sum_{\nu, \mu=1}^{r} \alpha_\nu \alpha_\mu U(\zeta_\nu, \zeta_\mu) \right| \leqq \frac{1}{\pi} \sum_{\nu, \mu=1}^{r} \alpha_\nu \alpha_\mu^\dagger \frac{1}{(1 - \zeta_\mu^\dagger \zeta_\nu)^2}.$$

Specializing further to $r = 1$, we obtain the interesting necessary condition for univalence in the unit circle, expressed in terms of Schwarz' differential parameter (3.7) For a similar sufficient condition see (Nehari [1]):

$$(9.2b) \qquad \{\varphi, z\} \leqq \frac{6}{(1 - |z|^2)^2}.$$

We may generalize this result to the case of multiple connectivity by use of (9.2) for $r = 1$:

$$(9.2c) \qquad \left| l(z, z) - \frac{1}{6\pi} \{\varphi, z\} \right| \leqq K(z, z^\dagger).$$

Let Σ be a closed rectifiable curve in B and $p(s)$ a complex-valued function of the length parameter on Σ; we obtain from (9.2) by a limit process

$$(9.4) \quad \left| \int_{z \in \Sigma} \int_{\zeta \in \Sigma} [l(z, \zeta) + U(z, \zeta)] p(s_z) p(s_\zeta) ds_z \, ds_\zeta \right| \leqq \int_{z \in \Sigma,} \int_{\zeta \in \Sigma} K(z, \zeta^\dagger) p(s_z) p(s_\zeta)^\dagger ds_z ds_\zeta$$

Let us now assume that the analytic functions l, U, K are developed into power series of their variables around the origin 0 which we suppose in B:

$$(9.5) \qquad U(z, \zeta) = \sum_{\mu, \nu=0}^{\infty} c_{\mu\nu} z^\mu \zeta^\nu, \qquad l(z, \zeta) = \sum_{\mu, \nu=0}^{\infty} l_{\mu\nu} z^\mu \zeta^\nu,$$

$$K(z, \zeta^\dagger) = \sum_{\mu, \nu=0}^{\infty} k_{\mu\nu} z^\mu (\zeta^\nu)^\dagger.$$

Because of the symmetries of these three functions, we find for their coefficients

(9.5a) $$c_{\mu\nu} = c_{\nu\mu}, \qquad l_{\mu\nu} = l_{\nu\mu}, \qquad k_{\mu\nu} = k^{\dagger}_{\nu\mu}.$$

We now choose for Σ a curve in the common domain of convergence of all three developments (9.5) which surrounds the origin. Let

(9.6) $$q(z) = \frac{1}{2\pi i}\sum_{\nu=0}^{N} \alpha_\nu z^{-(\nu+1)}, \quad p(s) = q(z(s))\, z'(s).$$

We clearly have

(9.7) $$\iint_{z,\,\zeta\in\Sigma} [l(z,\zeta) + U(z,\zeta)]p(s_z)p(s_\zeta)ds_z ds_\zeta = \sum_{\mu,\,\nu=0}^{N} (c_{\mu\nu} + l_{\mu\nu})\alpha_\mu \alpha_\nu$$

and

(9.8) $$\iint_{z,\,\zeta\in\Sigma} K(z,\zeta^\dagger)p(s_z)p(s_\zeta)^\dagger ds_z ds_\zeta = \sum_{\mu,\,\nu=0}^{N} k_{\mu\nu}, \alpha_\mu \alpha^\dagger_\nu.$$

Hence the inequality (9.4) leads to relations between certain quadratic and hermitian forms which are connected with the coefficient matrices of the kernel functions:

(9.9) $$\left| \sum_{\mu,\,\nu=0}^{N} (\overline{c_{\mu\nu}} + l_{\mu\nu})\alpha_\mu\alpha_\nu \right| \leqq \sum_{\mu,\,\nu=0}^{N} k_{\mu\nu}\, \alpha_\mu \alpha^\dagger_\nu.$$

Similar inequalities are obtained for the coefficient matrices of the kernels in the Λ_s-space. These inequalities were first discovered by Grunsky (Grunsky [1]) who gave them a somewhat different formulation. Compare also (Schiffer [4]).

Grunsky showed also that when the necessary conditions (9.9) are satisfied for every N and every choice of the α_ν, the univalence of the function $\varphi(z)$ considered is ensured; i.e. all conditions (9.9) are a sufficient condition for univalence. We want to give here a new and shorter proof for this fact which is based on an important result concerning the kernel function. We first announce the following theorem:

Let $V(z,\zeta)$ be symmetric and analytic in both arguments in a neighbourhood of the origin; let

(9.10) $$V(z,\zeta) = \sum_{m,\,n=0}^{\infty} d_{mn} z^m \zeta^n, \quad K(z,\zeta^\dagger) = \sum_{m,\,n=0}^{\infty} k_{mn} z^m (\zeta^n)^\dagger$$

be the series for V and the K-kernel around the origin. If for every complex vector $\alpha_0, \alpha_1, \ldots \alpha_N$

(9.11) $$\left| \sum_{m,\,n=0}^{N} d_{mn}\alpha_m\alpha_n \right| \leqq \sum_{m,\,n=0}^{N} k_{mn}\alpha_m\alpha^\dagger_n$$

then $V(z,\zeta)$ is analytic in the whole domain B.

In order to prove this theorem we introduce a complete set of orthonormal functions $\chi_\nu(z)$ which is very useful in dealing with power series developments. Each $\chi_\nu(z)$ has around the origin the series development

$$(9.12) \qquad \chi_\nu(z) = \sum_{\mu=\nu}^{\infty} \beta_{\nu\mu} z^\mu, \qquad \nu = 0, 1, 2, \ldots$$

The condition that the matrix $(\beta_{\nu\mu})$ be triangular determines the set $\chi_\nu(z)$ in a unique way (Bergman [2]) Since the K-kernel can be expressed in the forms

$$(9.13) \qquad K(z, \zeta^\dagger) = \sum_{\nu=0}^{\infty} \chi_\nu(z)\chi_\nu(\zeta)^\dagger = \sum_{\mu,\nu=0}^{\infty} k_{\mu\nu} z^\mu (\zeta^\nu)^\dagger$$

we conclude from (9.12) the identities

$$(9.14) \qquad k_{\mu\nu} = \sum_{\varrho=0}^{\infty} \beta_{\varrho\mu} \beta_{\varrho\nu}^\dagger.$$

Since $\beta_{\varrho\mu} = 0$ for $\varrho > \mu$, the matrix $(k_{\mu\nu})$ consists of finite combinations of β-terms.

The relation (9.12) between the z^μ and $\chi_\nu(z)$ can easily be inverted:

$$(9.15) \qquad z^\nu = \sum_{\mu=\nu}^{\infty} b_{\nu\mu} \chi_\mu(z), \qquad \nu = 0, 1, \ldots$$

The matrix $(b_{\nu\mu})$ is of the same triangular form as its inverse $(\beta_{\nu\mu})$. Introducing (9.15) into (9.13) and comparing the coefficients of $\chi_\mu(z)\chi_\nu(\zeta)^\dagger$, we find

$$(9.16) \qquad \delta_{\mu\nu} = \sum_{\varrho,\sigma}^{\infty} k_{\varrho\sigma} b_{\varrho\mu} b_{\sigma\nu}^\dagger;$$

again, all sums (9.16) are only of finite range.

We rearrange formally the series (9.10) for $V(z, \zeta)$ by means of (9.15) and obtain

$$(9.17) \qquad \sum_{m,n=0}^{\infty} d_{mn} z^m \zeta^n = \sum_{\mu,\nu=0}^{\infty} t_{\mu\nu} \chi_\mu(z) \chi_\nu(\zeta), \quad t_{\mu\nu} = \sum_{m,n=0}^{\infty} d_{mn} b_{m\mu} b_{n\nu}.$$

We do not know if and where the second sum (9.17) converges. But the series for $t_{\mu\nu}$ are finite expressions and well-defined.

Introduce an arbitrary complex vector $a_\nu (\nu = 0, \ldots, N)$ and let

$$(9.18) \qquad \alpha_\nu = \sum_{\mu=0}^{N} b_{\nu\mu} a_\mu.$$

Consider now the expression

$$(9.19) \qquad \sum_{\mu,\nu=0}^{N} t_{\mu\nu} a_\mu a_\nu = \sum_{m,n=0}^{N} d_{mn} \alpha_m \alpha_n.$$

Using the assumption (9.11) of our theorem, we find by virtue of (9.16) and (9.18)

$$(9.20) \qquad \left| \sum_{\mu,\nu=0}^{N} t_{\mu\nu} a_\mu a_\nu \right| \leqq \sum_{\nu=0}^{N} | a_\nu |^2$$

for arbitrary choice of the complex vector a_ν.

From (9.20) we derive easily that for any two vectors a_ν and a'_ν

$$(9.20a) \qquad \left| \sum_{\mu,\nu=1}^{N} t_{\mu\nu} a_\mu a'_\nu \right| \leqq \sum_{\nu=0}^{N} | a_\nu |^2 + \sum_{\nu=0}^{N} | a'_\nu |^2$$

holds.

Now let B' be a closed subdomain of B. In this domain the kernel function $K(z, z^\dagger)$ is uniformly bounded, say by the constant M. Hence, in view of (9.13), (9.20a) leads to the inequality

$$(9.21) \qquad \left| \sum_{\mu,\nu=0}^{N} t_{\mu\nu} \chi_\mu(z) \chi_\nu(\zeta) \right| \leqq 2M$$

for arbitrary choice of z and ζ in B'. Hence the functions

$$(9.22) \qquad V_N(z, \zeta) = \sum_{\mu,\nu=1}^{N} t_{\mu\nu} \chi_\mu(z) \chi_\nu(\zeta)$$

are uniformly bounded in B' and form a normal family there. Therefore we can select subsequences of our set which converge uniformly in each closed subdomain of B'. But in view of (9.17) and (9.22) the limit functions will always coincide with $V(z, \zeta)$ in the neighbourhood of the origin. Hence the whole sequence $V_N(z, \zeta)$ possesses the same limit and converges uniformly in each closed subdomain of B'. The limit function is the analytic continuation of the power series $V(z, \zeta)$ over the whole domain. Hence our theorem is proved.

The application of this result to the theory of univalent functions is immediate. Grunsky's conditions (9.9) guarantee the regularity of the function $U(z, \zeta)$ in B, which shows that $\Phi(z, \zeta)$ is regular is B and that except for $z = \zeta$ we never have $\varphi(z) = \varphi(\zeta)$. This is just the univalence property required. It is remarkable how closely the proof of necessity and sufficiency of (9.9) is connected with the kernel functions.

Finally we want to study the extremum problem, for which functions the Grunsky inequalities (9.9) may become equalities. Since these inequalities have been derived from the more general inequalities (4.14c) it will be sufficient to determine these domains B for which equality can hold in (4.14c) under an appropriate choice of points ζ_ν and constants α_ν. If we go back in the derivation of these inequalities we see that they can only become precise if the corresponding non-negative integral (4.10) vanishes, i.e. if there exists a real constant λ such that

$$(9.29) \qquad \sum_{\nu=1}^{r} \alpha_\nu^\dagger K(z, \zeta_\nu^\dagger) + \lambda \sum_{\nu=1}^{r} \alpha_\nu l(z, \zeta_\nu) \equiv 0 \quad \text{for } z \in B.$$

In proceeding from (4.14a) to (4.14c) we furthermore neglected the term

$$(9.30) \qquad \sum_{\nu, \mu=1}^{r} \alpha_\nu \alpha_\mu^\dagger \Gamma(\zeta_\nu, \zeta_\mu^\dagger) = \frac{1}{\pi^2} \iint_{\tilde{B}} \Big| \sum_{\nu=1}^{r} \frac{\alpha_\nu}{z - \zeta_\nu} \Big|^2 d\tau_z.$$

This integral can only vanish if the area of $\tilde{B}$ is zero, i.e. if B is a slit domain. It is true that we developed our theory only for domains B which are bounded by closed analytic curves; at this stage, however, the consideration of more general domains becomes inevitable. Using the continuity of K and l in dependence of their domain of definition B it can be shown that the identities (4.2), (4.3) and (4.8) hold in the most general case. For slit domains, the term Γ is to be taken as zero in (4.8).

We multiply (9.29) with $l(w, s)^\dagger$ and integrate the identity over all $z \in B$. Using (4.3) and (4.8), we obtain

$$(9.29\text{a}) \quad \sum_{\nu=1}^{r} \alpha_\nu^\dagger l(w, \zeta_\nu)^\dagger + \lambda \sum_{\nu=1}^{r} \alpha_\nu K(w, \zeta_\nu^\dagger)^\dagger \equiv 0 \quad \text{for } w \in B.$$

From (9.29) and (9.29a) we conclude the identity

$$(9.31) \qquad \sum_{\nu=1}^{r} [\alpha_\nu^\dagger K(z, \zeta_\nu^\dagger) + \alpha_\nu l(z, \zeta_\nu)] \equiv 0.$$

In view of (2.5) this may also be written as

$$(9.31\text{a}) \quad \sum_{\nu=1}^{r} [\alpha_\nu^\dagger K(z, \zeta_\nu^\dagger) - \alpha_\nu L(z, \zeta_\nu)] + \frac{1}{\pi} \sum_{\nu=1}^{r} \frac{\alpha_\nu}{(z - \zeta_\nu)^2} \equiv 0.$$

We want to study this expression at the boundary C of B; however, this boundary may be a very complicated one and we prefer, therefore, to map B upon an auxiliary domain B_1 with smooth boundary C_1. Let $z = f(w)$ give the map of B_1 into B. We multiply (9.31a) with $f'(w)$ and have by virtue of (3.3) and (3.4)

$$(9.32) \sum_{\nu=1}^{r} [A_\nu^\dagger K_1(w, \omega_\nu^\dagger) - A_\nu L_1(w, \omega_\nu)] + \frac{1}{\pi} \sum_{\nu=1}^{r} \frac{\alpha_\nu f'(w)}{(f(w) - \zeta_\nu)^2} \equiv 0, \quad w \in B_1,$$

where $\zeta_\nu = f(\omega_\nu)$ and $A_\nu = \alpha_\nu [f'(\omega_\nu)^{-1}]$. Let $w'(s)$ be the tangent vector at the point $w \in C_1$; multiplying (9.32) with $w'(s)$ and using the boundary relation (2.4) between K and L, we obtain

$$(9.33) \qquad \frac{d}{ds} \sum_{\nu=1}^{r} \frac{\alpha_\nu}{f(w(s)) - \zeta_\nu} = \text{real}, \qquad w(s) \in C_1.$$

Integrating with respect to s, we find at last

$$(9.34) \qquad \Im\left\{\sum_{\nu=1}^{r} \frac{\alpha_\nu}{z-\zeta_\nu}\right\} = \text{const. on } C.$$

This proves that C consists of analytic slits with the algebraic equation (9.34).

It is evident that the same treatment leads to the extremum domains in the particular case of Grunsky's coefficient inequalities (9.9).

10. Variation formulas for the eigen functions $\varphi_\nu(z)$.

We now want to study the dependance of some of the important domain functions on the varying domain B. Let $\nu(s)$ be continuous on C and $\varepsilon\nu(s)$ denote the shift of each boundary point $z(s) \in C$ along the interior normal direction at this point. This defines a deformation of the boundary $C \equiv z(s)$ into a new curve system C^* with the parametric representation

$$(10.1) \qquad z^*(s) = z(s) + iz'(s) \,.\, \varepsilon\nu(s) = z(s) + iz'(s)\delta n(s).$$

C^* is the boundary of a new domain B^* which differs very little from B for small ε. We may choose the deformation function $\nu(s)$ in such a way that C^* is a system of closed analytic curves.

Let $g^*(z, \zeta)$ be Green's function of B^*; according to a classical formula by Hadamard (Hadamard [1], Lévy [1]), we have

$$(10.2) \quad g^*(z, \zeta) = g(z, \zeta) - \frac{1}{2\pi} \int_C \frac{\partial g(z, t)}{\partial n_t} \frac{\partial g(t, \zeta)}{\partial n_t} \delta n(t) ds_t + \varepsilon^2 \gamma_\varepsilon(z, \zeta)$$

where $\gamma_\varepsilon(z, \zeta)$ is bounded and harmonic in each closed subdomain of B. Using the definitions (2.3) and (2.5) for the kernels K and l, we obtain from (10.2) by differentiation the following formulas for their first order variations:

$$(10.3) \qquad \delta K(z, \zeta^\dagger) = \frac{1}{\pi^2} \int_C \frac{\partial^2 g(z, t)}{\partial z \, \partial n_t} \frac{\partial^2 g(t, \zeta)}{\partial n_t \, \partial \zeta^\dagger} \partial n_t \, ds_t$$

and

$$(10.3a) \qquad \delta l(z, \zeta) = -\frac{1}{\pi^2} \int_C \frac{\partial^2 g(z, t)}{\partial z \, \partial n_t} \frac{\partial^2 g(t, \zeta)}{\partial n_t \, \partial \zeta} \partial n_t \, ds_t.$$

Further it is easily seen that

$$(10.4) \qquad \frac{\partial g(z, t)}{\partial n_t} = \frac{2}{i} \frac{\partial g(z, t)}{\partial t} t' = -\frac{2}{i} \frac{\partial g(z, t)}{\partial t^\dagger} (t')^\dagger.$$

Using this identity in (10.3) and (10.3a) we find

$$(10.5)\qquad \delta K(z,\zeta^\dagger) = \int\limits_C K(z,t^\dagger)K(t,\zeta^\dagger)\delta n_t\,ds_t = \int\limits_C L(z,t)L(t,\zeta)^\dagger\delta n_t\,ds_t$$

$$(10.5a)\qquad \delta l(z,\zeta) = \int\limits_C L(z,t)\,L(t,\zeta)t'^2\delta n_t ds_t = \int\limits_C K(z,t^\dagger)\,K(\zeta,t^\dagger)(t'^2)^\dagger\delta n_t\,ds$$
$$= -\int\limits_C L(z,t)\,K(\zeta,t^\dagger)\delta n_t\,ds_t = -\int\limits_C K(z,t^\dagger)\,L(t,\zeta)\delta n_t\,ds_t.$$

Let now $\zeta_k(\nu = 1, 2, \ldots, r)$ be an arbitrary set of points of B and $\alpha_\nu(\nu = 1, 2, \ldots, r)$ a set of complex numbers. We have by virtue of (10.5) and (10.5a)

$$(10.6)\qquad \delta\left\{\sum_{\nu,\mu=1}^{r}\alpha_\nu\alpha_\mu^\dagger K(\zeta_\nu,\zeta_\mu^\dagger)\right\} = \int\limits_C \left|\sum_{\mu=1}^{r}\alpha_\mu^\dagger K(t,\zeta_\mu^\dagger)\right|^2\delta n_t\,ds_t$$

$$(10.6a)\qquad \delta\left\{\sum_{\nu,\mu=1}^{r}\alpha_\nu\alpha_\mu l(\zeta_\nu,\zeta_\mu)\right\} = \int\limits_C \left(\sum_{\mu=1}^{r}\alpha_\mu K(\zeta_\mu,t^\dagger)\right)^2(t'^2)^\dagger\,\delta n_t\,ds_t.$$

If the domain B decreases under variation, $\delta n_t \geqq 0$ on C and we see from (10.6) and (10.6a) that the expressions

$$(10.7)\qquad \sum_{\nu,\mu=1}^{r}\alpha_\nu\alpha_\mu^\dagger K(\zeta_\nu,\zeta_\mu^\dagger) \pm \left|\sum_{\nu,\mu=1}^{r}\alpha_\nu\alpha_\mu l(\zeta_\nu,\zeta_\mu)\right|$$

increase with decreasing domain. The terms of the inequality (4.14c) have, therefore, the following behavior; if the domain decreases, the bigger term increases quicker than the smaller term and the inequality becomes continually stronger.

We now want to determine the Fourier coefficients of the functions δK and δl with respect to a given orthonormal system in B. For this purpose, we have to prove the identity (2.9) for the case that the analytic function $f(z)$ is continuous in the closed region $B + C$ and for $z \epsilon C$. In this case, we may apply the boundary condition (2.4) and we obtain

$$(10.8)\qquad \iint\limits_B K(z,\zeta^\dagger)\,f(\zeta)d\tau_\zeta = -\,[z'^2\iint\limits_B L(z,\zeta)\,f(\zeta)^\dagger d\tau_\zeta]^\dagger,\quad z\,\epsilon\,C.$$

Because of (2.5), we have

$$(10.9)\qquad \iint\limits_B L(z,\zeta)\,f(\zeta)^\dagger d\tau_\zeta = \frac{1}{\pi}\iint\limits_B \frac{f(\zeta)^\dagger}{(z-\zeta)^2}d\tau_\zeta - \iint\limits_B l(z,\zeta)\,f(\zeta)^\dagger d\tau_\zeta.$$

Using the definitions (5.1) and (5.3), the regularity of $l(z,\zeta)$ in the closed region $B + C$ and elementary properties of improper integrals, we have

$$(10.10) \qquad \frac{1}{\pi}\iint\limits_{B} f(\zeta)^{\dagger}(z-\zeta)^{-2}d\tau_{\zeta} = \lim_{w\to z} \mathrm{T}f(w), \qquad w \in \tilde{B},$$

$$(10.10a) \qquad \iint\limits_{B} f(\zeta)^{\dagger}\, l(z,\zeta)d\tau_{\zeta} = \lim_{w\to z} \mathrm{T}f(w), \qquad w \in B$$

Because of the saltus condition (5.5) we find therefore

$$(10.11) \qquad \iint\limits_{B} L(z,\zeta)\, f(\zeta)^{\dagger}d\tau_{\zeta} = -\,[f(z)z'^{2}]^{\dagger}, \qquad z \in C$$

and hence from (10.8):

$$(10.12) \qquad \iint\limits_{B} K(z,\zeta^{\dagger})f(\zeta)d\tau_{\zeta} = f(z), \qquad z \in C$$

Hence the identity (2.9) has been extended to the closed region $B+C$.

The eigen functions $\varphi_{\nu}(z)$ are continuous in the closed region $B+C$ and form a complete orthonormal system in B. Using (10.5), (10.5a) and (10.12), we compute the following identities:

$$(10.13) \quad \iint\limits_{B}\iint\limits_{B} \delta K(z,\zeta^{\dagger})\varphi_{\nu}(z)^{\dagger}\varphi_{\mu}(\zeta)d\tau_{z}d\tau_{\zeta} = \int\limits_{C} \varphi_{\nu}(t)^{\dagger}\varphi_{\mu}(t)\,\delta n_{t}ds_{t}$$

$$(10.13a) \quad \iint\limits_{B}\iint\limits_{B} \delta l(z,\zeta)\varphi_{\nu}(z)^{\dagger}\varphi_{\mu}(\zeta)d^{\dagger}\tau_{z}d\tau_{\zeta} = \Big[\int\limits_{C} \varphi_{\nu}(t)\varphi_{\mu}(t)t'^{2}\delta n_{t}ds_{t}\Big]^{\dagger}.$$

From the definition of the eigen functions $\varphi_{\nu}(z)$ as solutions of the integral equation (6.5) it can easily be shown that each $\varphi_{\nu}(z)$ varies continuously with the domain B, if it does not belong to a degenerate eigenvalue λ_{ν}; its first order variation is of the class $\mathfrak{L}^{2}$ in B and we put:

$$(10.14) \qquad \delta\varphi_{\nu}(z) = \sum_{\mu=1}^{\infty} v_{\nu\mu}\,\varphi_{\mu}(z).$$

We denote further the first order variation of the non-degenerate eigen value λ_{ν} by $\delta\lambda_{\nu}$.

In view of (6.19) and (6.20), the notation (10.14) leads to the formulas:

$$(10.15) \quad \delta K(z,\zeta^{\dagger}) = \sum_{\nu,\mu=1}^{\infty} v_{\nu\mu}^{\dagger}\varphi_{\nu}(z)\varphi_{\mu}(\zeta)^{\dagger} + \sum_{\nu,\mu=1}^{\infty} v_{\nu\mu}\varphi_{\mu}(z)\varphi_{\nu}(\zeta)^{\dagger},$$

$$(10.16a) \quad \delta l(z,\zeta) = \sum_{\nu,\mu=1}^{\infty}\frac{1}{\lambda_{\nu}} v_{\nu\mu}\varphi_{\nu}(z)\varphi_{\mu}(\zeta) + $$

$$+ \sum_{\nu,\mu=1}^{\infty}\frac{1}{\lambda_{\nu}} v_{\nu\mu}\varphi_{\mu}(z)\varphi_{\nu}(\zeta) - \sum_{\nu=1}^{\infty}\frac{\delta\lambda_{\nu}}{\lambda_{\nu}^{2}}\varphi_{\nu}(z)\varphi_{\nu}(\zeta).$$

Introducing these expressions into (10.13) and (10.13a) respectively and using the orthonormality of the system $\{\varphi_\nu(z)\}$, we find the following equations for $v_{\nu\mu}$ and $\delta\lambda_\nu$:

$$(10.16)\qquad v_{\nu\mu}^\dagger + v_{\mu\nu} = \int_C \varphi_\nu(t)^\dagger \varphi_\mu(t)\,\delta n_t\,ds_t$$

$$(10.16\text{a})\qquad \frac{1}{\lambda_\nu} v_{\nu\mu} + \frac{1}{\lambda_\nu} v_{\mu\nu} - \frac{\delta\lambda_\nu}{\lambda_\nu^2}\delta_{\nu\mu} = \left[\int_C \varphi_\nu(t)\varphi_\mu(t)t'^2\delta n_t\,ds_t\right]^\dagger.$$

These equations determine completely the variation of an eigen function $\varphi_\nu(z)$ and its corresponding eigen value λ_ν provided that λ_ν is non-degenerate.

From (10.16) and (10.16a) we conclude

$$(10.17)\qquad -\frac{\delta\lambda_\nu}{\lambda_\nu^2} = \delta\left(\frac{1}{\lambda_\nu}\right) = \int_C \left(\Re\{\varphi_\nu(t)^2 t'^2\} - \frac{1}{\lambda_\nu}|\varphi_\nu(t)|^2\right) dn_t\,\delta s.$$

We may write

$$(10.18)\quad \Re\{\varphi_\nu(t)^2 t'^2\} = \frac{1}{2}\left(\lambda_\nu + \frac{1}{\lambda_\nu}\right)|\varphi_\nu(t)|^2 - \frac{\lambda_\nu}{2}\left|\frac{\varphi_\nu(t)}{\lambda_\nu} - (\varphi_\nu(t)t'^2)^\dagger\right|^2.$$

If we introduce in the complementary domain $\breve{B}$ the function $\psi_\nu(z)$ defined by (7.30) we may express the variation formula (10.17) by means of (10.18) and (7.31) in the form

$$(10.19)\qquad \delta\left(\frac{1}{\lambda_\nu}\right) = \frac{1}{2}\left(\lambda_\nu - \frac{1}{\lambda_\nu}\right)\int_C \left(|\varphi_\nu(t)|^2 - |\psi_\nu(t)|^2\right)\delta n_t\,ds_t.$$

Since we proved in section 7 that in the case of a simply-connected domain B the function $\psi_\nu(z)$ is an eigen function of the complementary domain $\breve{B}$ with the same eigen value λ_ν, the great symmetry of (10.19) is obvious.

Further interesting formulas appear when the type of variation (10.1) is specialized. The following kind of variation has been of great use in the general theory (Schiffer [2]); let z_0 be an arbitrary fixed point in B. Let the boundary C be subjected to the variation

$$(10.20)\qquad \delta z = \frac{a\varrho^2}{z - z_0}, \qquad 0 < \varrho,\ z_0 \in B,$$

which is for ϱ small enough of the type (10.1). One sees immediately that the normal shift of a point $t \in C$ is given by the formula

$$(10.21)\qquad \delta n = \Re\left\{\frac{1}{it'}\,\frac{a\varrho^2}{t - z_0}\right\}.$$

Under this particular variation the formula (10.17) may be transformed into

$$(10.22)\quad \delta\left(\frac{1}{\lambda_\nu}\right) = \Re\left\{\frac{a\varrho^2}{2i}\int_C \{(\varphi_\nu t')^2 + [(\varphi_\nu t')^2]^\dagger - \frac{2}{\lambda_\nu}|\varphi_\nu|^2\}\frac{dt}{t-z_0}\right\}.$$

A slight rearrangement of this formula and the residue theorem yield

$$(10.23)\quad \delta\left(\frac{1}{\lambda_\nu}\right) = \Re\{\pi a\varrho^2\left(1-\frac{1}{\lambda_\nu^2}\right)\varphi_\nu(z_0)^2\} + $$
$$+\Re\left\{\frac{a\varrho^2}{2i}\int_C\left(\frac{\varphi_\nu}{\lambda_\nu}-(\varphi_\nu t'^2)^\dagger\right)^2\frac{dt}{t-z_0}\right\}.$$

Now we remember that the function $T\varphi_\nu$ is regular in each complementary domain $\tilde{B}_\varrho$ and has the boundary values $\frac{\varphi_\nu(t)}{\lambda_\nu} - (\varphi_\nu\,.\,t'^2)^\dagger$. Hence the integrand of the integral (10.23) is regular inside each boundary curve C_ϱ and the integral vanishes. Therefore, finally:

$$(10.24)\qquad \delta\left(\frac{1}{\lambda_\nu}\right) = \Re\{a\varrho^2\pi\left(1-\frac{1}{\lambda_\nu^2}\right)\varphi_\nu(z_0)^2\}.$$

This is a variation formula of the „interior" type where all boundary integrals have been eliminated.

A similar result is obtained if the point z_0 in the variation (10.20) is chosen in a complementary domain $\tilde{B}_\sigma$. One finds easily by the same considerations

$$(10.25)\qquad \delta\left(\frac{1}{\lambda_\nu}\right) = \Re\{a\varrho^2\pi\left(1-\frac{1}{\lambda_\nu^2}\right)\psi_\nu(z_0)^2\}$$

where the function $\psi_\nu(z_0)$ is connected with the l-transform $T\varphi_\nu$ by (7.30).

It is not difficult to determine the variation of the eigen functions $\varphi_\nu(z)$ under a variation (10.20). The corresponding formulas for the kernels K and l have been given in (Schiffer [3]). Formulas of this type are of particular use if one extends the definition of the functions and functionals considered to domains of the most general type; in this case, the boundary C may be so involved that a description (10.1) of the domain variation becomes impossible.

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