A. D. MICHAL
A. B. MEWBORN

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# General projective differential geometry of paths

by

A. D. Michal and A. B. Mewborn

Pasadena, Cal.

## Introduction.

In a previous study, Michal [1]) has outlined a general projective geometry of paths for the case in which the coordinate Banach space was assumed to possess a postulated inner product and its Banach ring of linear functions was assumed to have a contraction operation. The present study omits these assumptions and developes the geometry of a space of paths upon the basis of two assumed elements of structure. These are a linear connection and a gauge form and their projective laws of transformation. These laws are a general form of corresponding transformations obtained by change of representation in the special case [2]) of a flat projective geometry. In terms of the above, we define a projective connection in our new Banach space $B_1$ of projective coordinates in a way which again is clearly related to a property of this geometric object in the flat case [2]).

We develop the properties of this projective connection under a restricted coprojective transformation in the second section. This problem may be regarded in a way as a converse problem to one we have considered elsewhere [2]). In the third section we define and study the curvature form of the projective coordinate space $B_1$.

## 1. Invariant properties of paths under projective transformations.

The space of paths whose geometry we consider in this paper will be defined in a Hausdorff topological space $H$ with allowable

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[1]) Michal (I), (II). Roman numerals refer to the bibliography at the end of the paper.

[2]) Michal and Mewborn (III), specially theorems 2.4 and 2.5.

$K^{(3)}$ coordinates [3]) in a Banach space $B$. We shall assume that there exist in $B$ two classes of projectively related geometric objects.

DEFINITION 1.1. *Linear connection.* The components of this geometric object are functions of class $C^{(1)}$ in $x$, and symmetric and bilinear [4]) in the contravariant vectors (c.v.) $\xi_1$ and $\xi_2$, which transform according to [5])

$$(1.1)\qquad \bar{\Gamma}(\bar{x}, \bar{\xi}_1, \bar{\xi}_2) = \bar{x}(x;\ \Gamma(x, \xi_1, \xi_2)) + \bar{x}(x;\ x(\bar{x};\ \bar{\xi}_1;\ \bar{\xi}_2)),$$

or the equivalent

$$(1.2)\qquad \bar{\Gamma}(\bar{x}, \bar{\xi}_1, \bar{\xi}_2) = \bar{x}(x;\ \Gamma(x, \xi_1, \xi_2)) - \bar{x}(x;\ \xi_1;\ \xi_2),$$

under allowable transformation of coordinates $\bar{x} = \bar{x}(x)$.

DEFINITION 1.2. *Projective change of connection.* The transformation

$$(1.3)\qquad \hat{\Gamma}(x, \xi_1, \xi_2) = \Gamma(x, \xi_1, \xi_2) + \varphi(x, \xi_1)\xi_2 + \varphi(x, \xi_2)\xi_1$$

where $\varphi(x, \xi)$ is any arbitrary scalar field valued form of class $C^{(1)}$ in $x$, linear in the contravariant vector $\xi$, defines a new linear connection from a given one and a function $\varphi(x, \xi)$. This transformation is called the projective change of connection, and establishes the first class of geometric objects mentioned above. The subclass for which $\varphi(x, \xi)$ is also integrable is important and will be considered in section 2.

DEFINITION 1.3. *Gauge form.* This is a geometric object whose components are scalar field valued functions of class $C^{(1)}$ in $x$, symmetric and bilinear in the arbitrary c.v. $\xi_1$ and $\xi_2$ so that

$$(1.4)\qquad \bar{\Gamma}^0(\bar{x}, \bar{\xi}_1, \bar{\xi}_2) = \Gamma^0(x, \xi_1, \xi_2)$$

under allowable change of coordinates $\bar{x} = \bar{x}(x)$.

DEFINITION 1.4. *Projective change of gauge form.* The transformation

$$(1.5)\qquad \hat{\Gamma}^0(x, \xi_1, \xi_2) = \Gamma^0(x, \xi_1, \xi_2) + \frac{1}{M}\{\tfrac{1}{2}[\psi(x, \xi_1;\ \xi_2) + \\ + \psi(x, \xi_2;\ \xi_1)] - \psi(x, \Gamma(x, \xi_1, \xi_2)) - \psi(x, \xi_1)\psi(x, \xi_2)\}$$

where $M$ is a fixed positive number, $\psi(x, \xi)$ is subject to the same

[3]) Michal and Hyers (IV), p. 5.

[4]) The terms linear and bilinear will be used here to mean additive and continuous, and hence homogeneous of degree one, in the variable or variables mentioned.

[5]) Michal (V).

restrictions as $\varphi(x, \xi)$ above, and $\Gamma(x, \xi_1 \xi_2)$ is a linear connection, will be called the projective change of gauge form. This establishes the second class of geometric objects in out space. We note that the laws of transformation (1.1), (1.2) and (1.3) may be combined into a law of transformation under simultaneous change of coordinate and projective change of connection in the following form

$$
(1.6) \quad \hat{\Gamma}(\bar{x}, \bar{\xi}_1, \bar{\xi}_2) = \bar{x}(x; \Gamma(x, \xi_1, \xi_2)) + \bar{x}(x; x(\bar{x}; \bar{\xi}_1; \bar{\xi}_2)) + \\
+ \varphi(x, \xi_1)\bar{x}(x; \xi_2) + \varphi(x, \xi_2)\bar{x}(x; \xi_1).
$$

Likewise the transformations (1.4) and (1.5) may be combined to give the law of transformation under simultaneous change of coordinates and projective change of gauge form

$$
(1.7) \quad \hat{\Gamma}^0(\bar{x}, \bar{\xi}_1, \bar{\xi}_2) = \Gamma^0(x, \xi_1, \xi_2) + \frac{1}{M}\{\tfrac{1}{2}[\psi(x, \xi_1; \xi_2) + \\
+ \psi(x, \xi_2; \xi_1)] - \psi(x, \Gamma(x, \xi_1, \xi_2)) - \psi(x, \xi_1)\psi(x, \xi_2)\}.
$$

DEFINITION 1.5. *Coprojective change of connection and gauge form.* If we obtain from a given $\Gamma(x, \xi_1, \xi_2)$ and $\Gamma^0(x, \xi_1, \xi_2)$ involving the same $\xi_1$, $\xi_2$ a new pair by means of (1.3) and (1.5) in which $\varphi(x, \xi) \equiv \psi(x, \xi)$; then this pair $\hat{\Gamma}(x, \xi_1, \xi_2)$, $\hat{\Gamma}^0(x, \xi_1, \xi_2)$ will be said to be coprojective to the given pair, or obtained by a coprojective change of connection and gauge form.

DEFINITION 1.6. *System of paths.* The solutions $x = x(t)$ of the differential equation [6])

$$
(1.8) \qquad \frac{d^2x}{dt^2} + \Gamma\left(x, \frac{dx}{dt}, \frac{dx}{dt}\right) = \alpha(t)\frac{dx}{dt}
$$

where $\Gamma\left(x, \frac{dx}{dt}, \frac{dx}{dt}\right)$ is a component of a linear connection and $\alpha(t)$ is a numerical valued scalar function of the undefined parameter $t$, are the coordinate representations of a system of curves called paths.

Clearly this definition of paths is equivalent to the definition of the paths as the system of autoparallel curves.

DEFINITION 1.7. *Affine parameter s.* We may define a new parameter $s$ along each member of the system of paths (1.8) by the differential equation:

$$
(1.9) \qquad \frac{d^2s}{dt^2} = \alpha(t)\frac{ds}{dt}.
$$

[6]) Michal (I).

This parameter, which is only determined up to a first degree transformation $s' = as + b$ will be called an affine parameter.

It is clear from (1.8) and (1.9) that the differential equation of the paths in terms of an affine parameter is

$$\frac{d^2x}{ds^2} + \Gamma\left(x, \frac{dx}{ds}, \frac{dx}{ds}\right) = 0. \tag{1.10}$$

Further, under projective change of connection (1.3) equation (1.8) becomes.

$$\frac{d^2x}{dt^2} + \hat{\Gamma}\left(x, \frac{dx}{dt}, \frac{dx}{dt}\right) = \hat{\alpha}(t)\frac{dx}{dt} \tag{1.11}$$

where $\hat{\alpha}(t) = \alpha(t) + 2\varphi\left(x(t), \frac{dx(t)}{dt}\right)$. That is (1.8) remains invariant in form.

Definition 1.8. *Affine parameter* $\hat{s}$. The differential equation

$$\frac{\frac{d^2\hat{s}}{ds^2}}{\frac{d\hat{s}}{ds}} = 2\varphi\left(x, \frac{dx}{ds}\right) \quad \text{or} \quad \frac{d\hat{s}}{ds} = e^{2\int\varphi\left(x, \frac{dx}{ds}\right)ds}, \tag{1.12}$$

where $\int\varphi\left(x, \frac{dx}{ds}\right)ds$ is to be interpreted as an ordinary integral of the scalar field valued function of (1.3) taken along the paths, defines a new affine parameter $\hat{s}$ associated with $\hat{\Gamma}(x, \xi_1, \xi_2)$.

It is readily verified that (1.10) remains invariant in form under projective change of connection (1.3) and the corresponding change of parameter (1.12), taking the form

$$\frac{d^2x}{d\hat{s}} + \hat{\Gamma}\left(x, \frac{dx}{d\hat{s}}, \frac{dx}{d\hat{s}}\right) = 0. \tag{1.13}$$

Definition 1.9. *Projective normal parameter* $\pi$. This parameter is defined by the differential equation

$$\{\pi, s\} = -2M\,\Gamma^0\left(x, \frac{dx}{d\hat{s}}, \frac{dx}{d\hat{s}}\right) \tag{1.14}$$

where $M$ is the positive constant of (1.7), $\Gamma^0\left(x, \frac{dx}{ds}, \frac{dx}{ds}\right)$ is a component of a gauge form, and $\{\pi, s\}$ is the Schwarzian derivative [7])

$$\{\pi, s\} = \frac{\frac{d^3\pi}{ds^3}}{\frac{d\pi}{ds}} - \frac{3}{2}\left(\frac{\frac{d^2\pi}{ds^2}}{\frac{d\pi}{ds}}\right)^2. \tag{1.15}$$

[7]) Berwald (VI).

Three properties of the Schwarzian derivative (1.15) will be needed later in the paper and are exhibited here for reference.

$$(1.16)\qquad \{w, v\} = \{u, v\} \text{ if and only if } w = \frac{\alpha u + \beta}{\gamma u + \delta}, \quad \begin{vmatrix} \alpha & \beta \\ \gamma & \delta \end{vmatrix} \neq 0.$$

$$(1.17)\qquad \{w, v\} = \left(\frac{du}{dv}\right)^2 [\{w, u\} - \{v, u\}].$$

$$(1.18)\qquad \{w, v\} = -\left(\frac{dw}{dv}\right)^2 \{v, w\}.$$

From the property (1.16) it follows at once that $\pi$ is defined only up to a non-singular linear fractional transformation.

Definition 1.10. *Projectively invariant.* Any quantity which undergoes at most a non-singular linear fractional transformation under an operation, is said to be projectively invariant under that operation.

Theorem 1.1. *The projective normal parameter $\pi$ is projectively invariant under allowable transformation of coordinates $\bar{x} = \bar{x}(x)$.*

Proof: Let $\bar{s} = as + b$ be an affine parameter $s$ in $\bar{x}$ coordinates, then by definitions 1.8 and 1.9

$$\{\bar{\pi}, \bar{s}\} = -2M\,\bar{\Gamma}^0\left(\bar{x}, \frac{d\bar{x}}{d\bar{s}}, \frac{d\bar{x}}{d\bar{s}}\right) = -2M\,\Gamma^0\left(x, \frac{dx}{ds}, \frac{dx}{ds}\right)\left(\frac{ds}{d\bar{s}}\right)^2,$$

$$\{\bar{\pi}, \bar{s}\} = \{\pi, s\}\left(\frac{ds}{d\bar{s}}\right)^2.$$

Since $\{\bar{s}, s\} = 0$, we see from (1.17) that $\{\bar{\pi}, s\} = \{\pi, s\}$, which, by (1.16) completes the proof. Q. E. D.

Theorem 1.2. *The parameter $\pi$ is projectively invariant under coprojective change of connection and gauge form.*

Proof: Under projective change of gauge form (1.5), (1.14) becomes

$$(1.19)\quad \{\pi, s\} = -2M\,\hat{\Gamma}^0\left(x, \frac{dx}{ds}, \frac{dx}{ds}\right) + \left[2\varphi\left(x, \frac{dx}{ds}, \frac{dx}{ds}\right) - \right.$$
$$\left. -2\varphi\left(x, \Gamma\left(x, \frac{dx}{ds}, \frac{dx}{ds}\right)\right) - 2\left\{\varphi\left(x, \frac{dx}{ds}\right)\right\}^2\right].$$

Since $x$ must satisfy the equation (1.10) of the paths, a direct computation from (1.12) shows that the expression in square brackets of (1.19) is simply $\{\hat{s}, s\}$. From this and (1.17)

$$\{\pi, \hat{s}\} = -2M\hat{\Gamma}^0\left(x, \frac{dx}{d\hat{s}}, \frac{dx}{d\hat{s}}\right)$$

where $\hat{s}$ by definition 1.8 is the affine parameter $s$ associated with $\hat{\Gamma}\left(x, \frac{dx}{d\hat{s}}, \frac{dx}{d\hat{s}}\right)$.

Definition 1.11. *Gauge variable* $x^0$. A numerical variable $x^0$, called gauge variable, is defined along all paths by the equation

$$x^0 = -\frac{1}{2M} \log \frac{ds}{d\pi} \tag{1.20}$$

where $s$ and $\pi$ are corresponding parameters [definitions 1.7 and 1.9].

Definition 1.12. *Gauge variable* $\hat{x}^0$. A second gauge variable $\hat{x}^0$ corresponding to $\hat{s}$ and hence to $\hat{\Gamma}(x, \xi_1, \xi_2)$ is defined along each path by

$$d\hat{x}^0 = dx^0 - \frac{1}{M}\varphi(x, dx), \qquad \text{or} \tag{1.21}$$

$$\hat{x}^0 = x^0 - \frac{1}{M}\int_q^x \varphi\left(x, \frac{dx}{dt}\right)dt + C(q)$$

where the integral is taken along the path with arbitrary parameter $t$ from any initial point $q$.

In terms of the gauge variable $x^0$ and the parameter $\pi$ it is now possible to establish a representation of the system of paths, which does not involve the parameter $s$, as follows

$$\{s, \pi\} = 2M\, \Gamma^0\left(x, \frac{dx}{d\pi}, \frac{dx}{d\pi}\right) \tag{1.22}$$

whence

$$\frac{d^2x^0}{d\pi^2} + M\left(\frac{dx^0}{d\pi}\right)^2 + \Gamma^0\left(x, \frac{dx}{d\pi}, \frac{dx}{d\pi}\right) = 0 \tag{1.23}$$

and, by eliminating $s$ between (1.10) and (1.20),

$$\frac{d^2x}{d\pi^2} + 2M\frac{dx^0}{d\pi}\frac{dx}{d\pi} + \Gamma\left(x, \frac{dx}{d\pi}, \frac{dx}{d\pi}\right) = 0. \tag{1.24}$$

Definition 1.13. *The Banach space* $B_1$. The space of all couples of the form $X = (x, x^0)$ where $x$ is in $B$ and $x^0$ is a real number, form a Banach space under suitable definition of operations and norm. This space will be denoted by $B_1$ and its elements by capital letters.

Definition 1.14. *Projective connection* $\Pi(X, Y, Z)$. The components of this geometric object are functions with arguments and values in $B_1$ given by

$$(1.25)\ \Pi(X, Y, Z) = (\Gamma(x, y, z) + Mz^0y + My^0z, \Gamma^0(x, y, z) + My^0z^0)$$

where $M$ is the same positive constant as in (1.5) and (1.14).

The following properties of the projective connection are practically immediate consequences of its definition:

$$(1.26)\quad \begin{cases} \text{a)} & \Pi(X, Y, Z) \text{ is independent of } x^0 \\ \text{b)} & \Pi(X, (y, 0), (z, 0)) = (\Gamma(x, y, z), \Gamma^0(x, y, z)) \\ \text{c)} & \Pi(X, Y, (0, z^0)) = Mz^0Y \\ \text{d)} & \Pi(X, Y, Z) = \Pi(X, Y, (z, 0)) + Mz^0Y \\ & \qquad\qquad\;\; = \Pi(X, (y, 0), Z) + My^0Z \\ \text{e)} & \Pi(X, Y, Z) = \Pi(X, Z, Y) \\ \text{f)} & a\Pi(X, Y, Z) = \Pi(X, aY, Z) = \Pi(X, Y, aZ) \end{cases}$$

where $a$ is any real number.

THEOREM 1.3. *The projective connection $\Pi(X, Y, Z)$ is symmetric and bilinear in the arguments $Y$ and $Z$.*

*Proof*: Symmetry follows from e of (1.26); hence our proof will consist in showing additivity and continuity in $Y$.

From definitions 1.13 and 1.14 and the linearity properties in definitions 1.1 and 1.3 we have by a simple calculation that

$$\Pi(X, Y_1 + Y_2, Z) = \Pi(X, Y_1, Z) + \Pi(X, Y_2, Z).$$

By the definition of norm in $B_1$ and the bilinearity in definitions 1.1 and 1.3 we have

$$\begin{aligned} ||\,\Pi(X,Y,Z)\,||^2 &= ||\,\Gamma(x,y,z)+My^0z+Mz^0y\,||^2+|\,\Gamma^0(x,y,z)+My^0z^0\,|^2 \\ &\leqq \{M_\Gamma\,||\,y\,||\cdot||\,z\,|| + M\,|\,y^0\,|\cdot||\,z\,|| + M\,|\,z^0\,|\cdot||\,y\,||\}^2 + \\ &\qquad + \{M_{\Gamma^0}\,||\,y\,||\cdot||\,z\,|| + M\,|\,y^0\,|\cdot|\,z^0\,|\}^2. \end{aligned}$$

Hence, for a fixed $X_0$ and $Z_0$, we can make the last member as small as we choose by taking $\bar{Y}$ with small enough norm. Thus for any $\varepsilon > 0$ there exists $\delta > 0$ such that $||\,Y\,|| = \sqrt{||\,y\,||^2 + |\,y^0\,|^2} < \delta$ implies

$$||\,\Pi\,(X_0, Y, Z_0)\,|| = ||\,\Pi(X_0, Y, Z^0) - \Pi(X_0, 0, Z_0)\,|| \leqq \varepsilon$$

since $\Pi(X, 0, Z) = 0$. This shows continuity at $Y = 0$, and completes the proof. Q. E. D.

THEOREM 1.4. *The differential equation*

$$1.27)\qquad \frac{d^2X}{d\pi^2} + \Pi\left(X, \frac{dX}{d\pi}, \frac{dX}{d\pi}\right) = 0$$

*is equivalent to the pair* (1.23) (1.24) *and hence represents the system of paths* (1.10) *in terms of the projective normal parameter* $\pi$.

*Proof*: In general, if $X$ is a function of any scalar parameter $t$,

$$\frac{d^2X}{d\pi^2} = \left(\frac{d^2x}{d\pi^2}, \frac{d^2x^0}{d\pi^2}\right), \text{ and hence}$$

$$\Pi\left(X, \frac{dX}{d\pi}, \frac{dX}{d\pi}\right) = \left(\Gamma\left(x, \frac{dx}{d\pi}, \frac{dx}{d\pi}\right) + 2M\frac{dx^0}{d\pi}\frac{dx}{d\pi}, \Gamma^0\left(x, \frac{dx}{d\pi}, \frac{dx}{d\pi}\right) + M\left(\frac{dx^0}{d\pi}\right)^2\right).$$

If we substitute these values in (1.27) and equate separately the Banach elements and real elements to zero, we get (1.23) and (1.24). The steps are all reversible, so the proof is complete.

Q. E. D.

## 2. The Integrable Case.

Certain interesting and important aspects of our theory involve the subclass of linear connections obtainable by a projective change of connection (1.3) in which $\varphi(x, \xi)$ is the exact Fréchet differential of a scalar numerical valued function.

We shall express the latter function in terms of the logarithm of a positive real valued function $\varrho(x)$ of class $C^{(2)}$,

$$\varphi(x, \xi) = -M d^x_\xi \log \varrho(x). \tag{2.1}$$

Definition 2.1. *Change of representation.* This is a change of coordinate representation in the $B_1$ space from $X = (x, x^0)$ to $\overline{X} = (\bar{x}, \bar{x}^0)$ such that

$$\left\{\begin{array}{ll} \bar{x} = \bar{x}(x) & \text{is an allowable } K^{(3)} \text{ transformation} \\ & \text{of coordinates in } B, \\ \bar{x}^0 = x^0 + \log \varrho(x) & \text{is the integrable case of (1.21).} \end{array}\right. \tag{2.2}$$

A change of representation *necessarily* entails a simultaneous coprojective change of connection and gauge form since (1.21) implies that $\bar{x}^0$ is associated with the coprojective connection and gauge form.

Lemma 2.1. *A necessary and sufficient condition that $\Xi = (\xi, \xi^0)$ be a contravariant vector with components in $B_1$ is that $\xi$ be a contravariant vector with components in $B$, and that $\xi^0$ transform according to*

$$\bar{\xi}^0 = \xi^0 + d^x_\xi \log \varrho(x). \tag{2.3}$$

*Proof*: Assume that $\Xi$ is a c.v. and hence transforms

(2.4) $\qquad \overline{\Xi} = \overline{X}(X; \Xi)$ under change of representation.

Clearly

$$\overline{\Xi} = (\bar{x}((x, x^0); (\xi, \xi^0)), \bar{x}^0((x, x^0); (\xi, \xi^0))).$$

Since by the hypotheses and the definitions of $B_1$

$$\bar{\xi} = \bar{x}(x; \xi) = \bar{x}((x, x^0); (\xi, \xi^0)) \text{ and}$$

$$\bar{\xi}^0 = \xi^0 + \frac{\varrho(x; \xi)}{\varrho(x)} = \bar{x}^0\,((x, x^0); (\xi, \xi^0)),$$

the condition is necessary. The steps of this argument are reversible, hence the condition is also sufficient. Q. E. D.

THEOREM 2.1. *The projective connection $\Pi(X, \Xi_1, \Xi_2)$ in the arbitrary c.v. $\Xi_1$ and $\Xi_2$ transforms as a component of linear connection under a change of representation. That is*

$$\begin{aligned} \overline{\Pi}(\overline{X}, \overline{\Xi}_1, \overline{\Xi}_2) &= \overline{X}(X; \Pi(X, \Xi_1, \Xi_2)) \\ &\quad + \overline{X}(X; X(\overline{X}; \overline{\Xi}_1; \overline{\Xi}_2)) \end{aligned} \tag{2.5}$$

*under* $\overline{X} = \overline{X}(X)$.

*Proof*: By Kerner's theorem on the symmetry of the second successive Fréchet differential with independent increments $y$, $z$

$$\varphi(x, y; z) = \varphi(x, z; y) = -M\frac{\varrho(x; y; z)}{\varrho(x)} + M\frac{\varrho(x; y)}{\varrho(x)}\frac{\varrho(x; z)}{\varrho(x)}. \tag{2.6}$$

Hence in the present case (1.6) and (1.7) become

$$\begin{aligned} &\text{a)} \quad \widehat{\overline{\Gamma}}(\bar{x}, \bar{\xi}_1, \bar{\xi}_2) = \bar{x}(x; \Gamma(x, \xi_1, \xi_2)) + \bar{x}(x; x(\bar{x}; \bar{\xi}_1; \bar{\xi}_2)) - \\ &\qquad\qquad - M\frac{\varrho(x; \xi_1)}{\varrho(x)}\bar{\xi}_2 - M\frac{\varrho(x; \xi_2)}{\varrho(x)}\bar{\xi}_1 \\ &\text{b)} \quad \widehat{\overline{\Gamma}}{}^0(\bar{x}, \bar{\xi}_1, \bar{\xi}_2) = \Gamma^0(x, \xi_1, \xi_2) - d^x_{\xi_2}\, d^x_{\xi_1} \log \varrho(x) + \\ &\qquad\qquad + \frac{\varrho(x; \Gamma(x, \xi_1, \xi_2))}{\varrho(x)} - M\frac{\varrho(x; \xi_1)}{\varrho(x)}\frac{\varrho(x; \xi_2)}{\varrho(x)}. \end{aligned} \tag{2.7}$$

Since $\varrho(x)$ is a scalar and hence $\log \bar{\varrho}(\bar{x}) = \log \varrho(x)$, the inverse of (2.4) and (2.3) is

$$\begin{aligned} \Xi &= X(\overline{X}; \overline{\Xi}) = (x(\bar{x}; \bar{\xi}), \bar{\xi}^0 - d^{\bar{x}}_{\bar{\xi}} \log \bar{\varrho}(\bar{x})) \\ \text{and} \quad X(\overline{X}; \overline{\Xi}_1; \overline{\Xi}_2) &= (x(\bar{x}; \bar{\xi}_1; \bar{\xi}_2), -d^{\bar{x}}_{\bar{\xi}_2} d^{\bar{x}}_{\bar{\xi}_1} \log \bar{\varrho}(\bar{x})). \end{aligned} \tag{2.8}$$

Consider now the right member of (2.5) which can be writter in the form

$$\overline{X}(X; \Pi(X, \Xi_1, \Xi_2) + X(\overline{X}; \overline{\Xi}_1; \overline{\Xi}_2)) =$$
$$= (\bar{x}(x; \Gamma(x, \xi_1, \xi_2) + M\xi_2^0 \xi_1 + M\xi_1^0 \xi_2 + x(\bar{x}; \bar{\xi}_1; \bar{\xi}_2),$$

(2.9)

$$\Gamma^0(x, \xi_1, \xi_2) + M\xi_1^0 \xi_2^0 - d^{\bar{x}}_{\bar{\xi}_2} d^{\bar{x}}_{\bar{\xi}_1} \log \bar{\varrho}(\bar{x}) +$$
$$+ \frac{1}{\varrho(x)} \varrho(x; \Gamma(x, \xi_1, \xi_2) + M\xi_2^0\xi_1 + M\xi_1^0\xi_2 + x(\bar{x}; \bar{\xi}_1; \bar{\xi}_2))).$$

In (2.6), $y$ and $z$ were independent increments, but in evaluating the above expression we must bear in mind that $\bar{\xi}_1 = \bar{x}(x; \xi_1)$ is also a function of $x$, hence

$$(2.10) \qquad d^{\bar{x}}_{\bar{\xi}_2} d^{\bar{x}}_{\bar{\xi}_1} \log \bar{\varrho}(\bar{x}) = d^x_{\xi_2} d^x_{\xi_1} \log \varrho(x) - \frac{\bar{\varrho}(\bar{x}; \bar{x}(x; \xi_1; \xi_2))}{\bar{\varrho}(\bar{x})} =$$
$$= d^x_{\xi_2} d^x_{\xi_1} \log \varrho(x) + \frac{\varrho(x; x(\bar{x}; \bar{\xi}_1; \bar{\xi}_2))}{\varrho(x)}.$$

By definition 1.14, we have in the $\overline{X}$ representation

$$\overline{\Pi}(\overline{X}, \overline{\Xi}_1, \overline{\Xi}_2) = (\hat{\overline{\Gamma}}(\bar{x}, \bar{\xi}_1, \bar{\xi}_2) + M\bar{\xi}_2^0 \bar{\xi}_1 + M\bar{\xi}_1^0 \bar{\xi}_2,$$
$$\hat{\overline{\Gamma}}{}^0(\bar{x}, \bar{\xi}_1, \bar{\xi}_2) + M\bar{\xi}_1^0 \bar{\xi}_2^0).$$

If now we substitute in the right members of (2.11) according to (2.7), (2.8), and (2.10) and collect terms, we obtain the $B_1$ element which is the right member of (2.9), which completes the proof. Q. E. D.

## 3. The Curvature Form in $B_1$.

DEFINITION 3.1. *The curvature form* $B_{(1)}(X, \Xi_1, \Xi_2, \Xi_3)$. The components of this geometric object are defined in terms of the components of the projective connection by the relation

$$(3.1) \qquad \begin{aligned} B_{(1)}(X, \Xi_1, \Xi_2, \Xi_3) &= \Pi(X, \Xi_1, \Xi_2; \Xi_3) - \\ &- \Pi(X, \Xi_1, \Xi_3; \Xi_2) + \Pi(X, \Pi(X, \Xi_1, \Xi_2), \Xi_3) - \\ &- \Pi(X, \Pi(X, \Xi_1, \Xi_3), \Xi_2). \end{aligned}$$

THEOREM 3.1. *The curvature form* $B_{(1)}(X, \Xi_1, \Xi_2, \Xi_3)$ *is trilinear in the arbitrary* $\Xi_1$, $\Xi_2$, $\Xi_3$ *and is skew-symmetric in the last two.*

THEOREM 3.2. *The curvature form* $B_{(1)}(X, \Xi_1, \Xi_2, \Xi_3)$ *vanishes unless* $\xi_1$, $\xi_2$ *and* $\xi_3$ *are all different from zero in* $B$.

*Proof*: From (1.25) we have by differentiation

$$\Pi(X, \Xi_1, \Xi_2, \Xi_3) = (\Gamma(x, \xi_1, \xi_2; \xi_3), \; \Gamma^0(x, \xi_1, \xi_2; \xi_3))$$

and by iteration in the second argument

$$\Pi(X, \Pi(X, \Xi_1, \Xi_2), \Xi_3) = (\Gamma(x, \Gamma(x, \xi_1, \xi_2), \xi_3) + \\ + M[\xi_1^0\Gamma(x, \xi_2, \xi_3) + \xi_2^0\Gamma(x, \xi_3, \xi_1) + \xi_3^0\Gamma(x, \xi_1, \xi_2) + \\ + \Gamma^0(x, \xi_1, \xi_2)\xi_3 + M\{\xi_1^0\xi_2^0\xi_3 + \xi_2^0\xi_3^0\xi_1 + \xi_3^0\xi_1^0\xi_2\}],$$

$$\Gamma^0(x, \Gamma(x, \xi_1, \xi_2), \xi_3) + M[\xi_1^0\Gamma^0(x, \xi_2, \xi_3) + \xi_2^0\Gamma^0(x, \xi_3, \xi_1) + \\ + \xi_3^0\Gamma^0(x, \xi_1, \xi_2) + M\xi_1^0\xi_2^0\xi_3^0]).$$

If we substitute these values in (3.1), collect terms, and introduce the notation

$$(3.2) \qquad B_{(1)}(X, \Xi_1, \Xi_2, \Xi_3) = (b_{(1)}, b^0_{(1)})$$

we obtain

$$b_{(1)} = B(x, \xi_1, \xi_2, \xi_3) + \Gamma^0(x, \xi_1, \xi_2)\xi_3 - \Gamma^0(x, \xi_1, \xi_3)\xi_2$$

$$b^0_{(1)} = \Gamma^0(x, \xi_1, \xi_2; \xi_3) - \Gamma^0(x, \xi_1, \xi_3; \xi_2) + \\ + \Gamma^0(x, \Gamma(x, \xi_1, \xi_2), \xi_3) - \Gamma^0(x, \Gamma(x, \xi_1, \xi_3), \xi_2)$$

where $B(x, \xi_1, \xi_2, \xi_3)$ is the curvature form based on $\Gamma(x, \xi_1, \xi_2)$ in the space $B$. Clearly $b_{(1)}$ and $b^0_{(1)}$ both vanish for any $\xi_i = 0$, $(i = 1, 2, 3)$, hence the theorem. Q. E. D.

Corollary 3.1. *The curvature form is c.v.f. valued and independent of* $x$, $\xi_1^0$, $\xi_2^0$ *and* $\xi_3^0$ *hence*

$$b_{(1)} = b_{(1)}(x, \xi_1, \xi_2, \xi_3) \text{ and } b^0_{(1)} = b^0_{(1)}(x, \xi_1, \xi_2, \xi_3).$$

Corollary 3.2 *The real element* $b^0_{(1)}$ *of* (3.2) *is expressible in terms of covariant differentials* [8]) *in the form*

$$b^0_{(1)}(x, \xi_1, \xi_2, \xi_3) = \Gamma^0(x, \xi_1, \xi_2 \mid \xi_3) - \Gamma^0(x, \xi_1, \xi_3 \mid \xi_2).$$

Corollary 3.3 *Both* $b_{(1)}$ *and* $b^0_{(1)}$ *satisfy a cyclic indentity of the form*

$$b(x, \xi_1, \xi_2, \xi_3) + b(x, \xi_2, \xi_3, \xi_1) + b(x, \xi_3, \xi_1, \xi_2) = 0.$$

Should we impose the further restrictions that $\Pi(X, Y, Z)$ have a differential with the $\delta$-property [9]) with respect to $Y$, $B_{(1)}(X, \Xi_1, \Xi_2, \Xi_3)$ vanish identically and $M = 1$, then our space would be a *locally flat projective space* and its geometry a *locally flat projective geometry* in the sense we have developed more fully elsewhere [9]).

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[8]) Michal (V), Theorem 3, p. 400.

[9]) Michal and Mewborn (III).

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