COMPOSITIO MATHEMATICA

EINAR HILLE

Bilinear formulas in the theory of the transformation of Laplace

Compositio Mathematica, tome 6 (1939), p. 93-102
<http://www.numdam.org/item?id=CM_1939__6__93_0>

Bilinear formulas in the theory of the transformation of Laplace[1])

by

Einar Hille

New Haven (Conn.)

---

1. *Introduction.* In the theory of linear integral equations the term bilinear formula refers to the expansion of a symmetric kernel in terms of its characteristic functions. But it can be used more loosely about any expansion of the form

$$K(s, t) \sim \Sigma_{n=1}^{\infty} \varphi_n(s) \psi_n(t) \tag{1.1}$$

which may be associated with the kernel. This is the sense given to the term bilinear formula in the present note. We shall be concerned with the kernel of the transformation of Laplace

$$f(z) = \mathfrak{L}[F] \equiv \int_0^{\infty} e^{-zu} F(u)\, du\,. \tag{1.2}$$

Any expansion of the form

$$e^{-zu} \sim \Sigma_{n=1}^{\infty} \varphi_n(u) \psi_n(z)\,, \tag{1.3}$$

in which $\{\varphi_n(u)\}$ is an orthonormal system for the interval $(0, \infty)$, gives rise to a pair of associated expansions

$$F(u) \sim \Sigma_{n=1}^{\infty} a_n \varphi_n(u)\,, \tag{1.4}$$

$$f(z) \sim \Sigma_{n=1}^{\infty} a_n \psi_n(z)\,. \tag{1.5}$$

Under suitable restrictions the second formula gives an absolutely convergent expansion of a Laplace transform in the right half-plane in terms of a system of Laplace transforms which constitute an orthogonal system on the imaginary axis, the a_n being the corresponding Fourier coefficients. Formula (1.4) is then to be regarded as an inversion formula for such Laplace integrals. Conversely, if $F(u)$ is given by its Fourier

---

[1]) An abstract of the present paper was presented to the American Mathematical Society, October 28, 1936.

series (1.4), then the corresponding Laplace transform $f(z)$ is given by the convergent series (1.5).

The present note is devoted to a study of the relations between the systems $\{\varphi_n(u)\}$ and $\{\psi_n(z)\}$, the associated expansions (1.4) and (1.5), and the transformation of Laplace. For the sake of simplicity we restrict ourselves to the case of quadratically integrable functions.

2. *The associated systems.* Let $\{\varphi_n(u)\}$ be an orthonormal system of real functions complete in $L_2(0, \infty)$. Put

$$\psi_n(z) = \mathfrak{L}[\varphi_n] = \int_0^\infty e^{-zu}\varphi_n(u)\,du, \tag{2.1}$$

where $z = x + iy$, $x > 0$. These functions are holomorphic for $x > 0$. Since

$$\psi_n(x+iy) = \int_0^\infty e^{-iyu}e^{-xu}\varphi_n(u)\,du,$$

we see that for a fixed $x > 0$, $\psi_n(x+iy)$ equals $\sqrt{2\pi}$ times the Fourier transform of $e^{-xu}\Phi_n(u)$, where $\Phi_n(u) = \varphi_n(u)$ for $u \geqq 0$ and $= 0$ for $u < 0$. Let us define

$$\psi_n(iy) = \text{l.i.m.}\int_{-a}^{a} e^{-iyu}\Phi_n(u)\,du, \tag{2.2}$$

where l.i.m. denotes the limit in the mean of order two as $a \to \infty$. Then by the Plancherel theorem

$$\begin{aligned}\int_{-\infty}^{\infty} |\psi_n(x+iy)|^2\,dy &= 2\pi\int_0^\infty e^{-2xu}|\varphi_n(u)|^2\,du \\ &\leqq 2\pi\int_0^\infty |\varphi_n(u)|^2\,du = 2\pi \\ &= \int_{-\infty}^{\infty} |\psi_n(iy)|^2\,dy,\end{aligned} \tag{2.3}$$

and

$$\int_{-\infty}^{\infty} |\psi_n(x+iy) - \psi_n(iy)|^2\,dy = 2\pi\int_0^\infty [1-e^{-xu}]^2|\varphi_n(u)|^2\,du \to 0 \text{ as } x \to 0.$$

It follows that $\psi_n(iy)$ is the boundary function of $\psi_n(z)$ on the imaginary axis in the sense of convergence in the mean.

Formula (2.3) shows that $\psi_n(z)$ belongs to the class $H_2(0)$ in the terminology of Hille and Tamarkin. [2])

Let us define

$$(2.4) \qquad \Phi_{-n}(u) = \Phi_n(-u),\ \psi_{-n}(iy) = \psi_n(-iy), \quad n = 1, 2, 3, \ldots$$

It is then easily seen that (2.2) holds for all values of n. Let Φ_+ denote the system $\{\Phi_n(u)\}$, $n \geqq 1$, and Φ_- the system $\{\Phi_{-n}(u)\}$, $n \geqq 1$; put $\Phi = \Phi_+ + \Phi_-$, and let the letters Ψ_+, Ψ_-, and Ψ have similar significance for the functions $\psi_n(iy)$. The system Φ_+ is closed in $L_2(0, \infty)$, hence Φ_- is closed in $L_2(-\infty, 0)$, and Φ is closed in $L_2(-\infty, \infty)$. Since a Fourier transformation preserves orthogonality and closure, we conclude that Ψ is an orthogonal system closed in $L_2(-\infty, \infty)$.

We have consequently shown that Ψ_+ is an orthogonal system, i.e., that the functions $\psi_n(z)$ are orthogonal to each other on the imaginary axis. The system is of course not complete in $L_2(-\infty, \infty)$, but we shall see in § 3 that it is closed with respect to boundary functions of the class $H_2(0)$.

We have now the bilinear formula

$$(2.5) \qquad e^{-zu} \sim \textstyle\sum_{n=1}^{\infty} \varphi_n(u)\, \psi_n(z)\,.$$

Here $u \geqq 0$ and $\Re(z) > 0$. For a fixed z, the series is the Fourier series of e^{-zu} in the system Φ_+ since $e^{-zu} \in L_2(0, \infty)$. Hence by the closure relation

$$(2.6) \qquad \textstyle\sum_{n=1}^{\infty} |\psi_n(x+iy)|^2 = \frac{1}{2x}, \quad x > 0\,.$$

But (2.5) is a biorthogonal series since the functions $\psi_n(z)$ form an orthogonal system on the imaginary axis. It is not a Fourier series in the system Ψ_+, however, since e^{-zu} does not belong to any Lebesgue class on the imaginary axis or on any vertical line. Nevertheless, it is easy to see that it is the derived series of such a Fourier series, the differentiation being with respect to u. Formal integration gives

$$(2.7) \qquad \frac{1}{z}[1-e^{-zu}] = \textstyle\sum_{n=1}^{\infty} \psi_n(z) \displaystyle\int_0^u \varphi_n(t)\, dt\,.$$

This series is absolutely convergent for $x > 0$ by virtue of (2.6) and the relation

$$u = \textstyle\sum_{n=1}^{\infty} \left\{ \displaystyle\int_0^u \varphi_n(t)\, dt \right\}^2,$$

[2]) On the theory of Laplace integrals [Proc. Nat. Acad. Sci. 19 (1933), 908–912]. Cf. the introduction to § 3 below.

which is a consequence of the closure relation for the system Φ_+. On the imaginary axis the right hand side is the Fourier series in Ψ_+ of the function on the left.

3. *Functions of* $H_2(0)$. Let $f(z) \in H_2(0)$, i.e., $f(z)$ is holomorphic for $\Re(z) > 0$ and there exists a constant M such that

$$\int_{-\infty}^{\infty} |f(x+iy)|^2 \, dy \leqq M, \quad x > 0.$$

Such a function admits of boundary values on the imaginary axis, $f(iy) \in L_2(-\infty, \infty)$, and

$$\int_{-\infty}^{\infty} |f(x+iy) - f(iy)|^2 \, dy \to 0 \text{ as } x \to 0.$$

Moreover

$$\int_{-\infty}^{\infty} |f(x+iy)|^2 \, dy \leqq \int_{-\infty}^{\infty} |f(iy)|^2 dy, \quad x > 0. \tag{3.1}$$

These properties together with the Plancherel theory of Fourier transforms and the simplest properties of orthogonal series suffice for the subsequent discussion. [3])

Now let the Fourier series of $f(iy)$ in the Ψ-system be

$$f(iy) \sim \Sigma_{-\infty}^{\infty} a_n \, \psi_n(iy),$$

$$a_n = \frac{1}{2\pi} \int_{-\infty}^{\infty} f(it) \, \overline{\psi_n(it)} \, dt, \tag{3.2}$$

and form the inverse Fourier transform of

$$r_N(iy) = f(iy) - f_N(iy) = f(iy) - \Sigma_{-N}^{N} a_n \, \psi_n(iy),$$

i.e.,

$$R_N(u) = \text{l.i.m.} \frac{1}{\sqrt{2\pi}} \int_{-a}^{a} r_N(y) \, e^{iuy} \, dy.$$

By (2.3)

$$\Phi_n(u) = \text{l.i.m.} \frac{1}{2\pi} \int_{-a}^{a} \psi_n(iy) e^{iuy} \, dy,$$

so that defining

$$F(u) = \text{l.i.m.} \frac{1}{2\pi} \int_{-a}^{a} f(iy) \, e^{iuy} \, dy, \tag{3.3}$$

---

[3]) For the properties of functions of the classes $H_p(0)$ consult E. Hille and J. D. Tamarkin, On the absolute integrability of Fourier transforms [Fundamenta Math. **25** (1935), 329–352], esp. § 2.

we have

$$R_N(u) = \sqrt{2\pi}[F(u) - \Sigma_{-N}^{N} a_n \Phi_n(u)].$$

The function $R_N(u) \in L_2(-\infty, \infty)$ and

$$\int_{-\infty}^{\infty} |R_N(u)|^2 du = \int_{-\infty}^{\infty} |r_N(iy)|^2 dy.$$

The right-hand side is known to tend to zero as $N \to \infty$, hence the same is true of the left. It follows that

$$F(u) \sim \Sigma_{-\infty}^{\infty} a_n \Phi_n(u)$$

is the Fourier series of $F(u)$ in Φ.

By a theorem of N. Wiener [4]) $F(u) \equiv 0$ for $u < 0$. Since

$$\int_{-\infty}^{0} |R_N(u)|^2 du = 2\pi \Sigma_{-N}^{-1} |a_n|^2 \to 0,$$

we conclude that $a_n = 0$ for $n < 0$. Consequently

$$f(iy) \sim \Sigma_1^{\infty} a_n \psi_n(iy), \tag{3.4}$$

$$F(u) \sim \Sigma_1^{\infty} a_n \varphi_n(u) \tag{3.5}$$

are the Fourier series of $f(iy)$ in Ψ_+ and of $F(u)$ in Φ_+ resp. By virtue of (3.5) we have also

$$a_n = \int_0^{\infty} F(u) \varphi_n(u) du. \tag{3.6}$$

Let us now form

$$r_N(z) = f(z) - f_N(z) = f(z) - \Sigma_1^N a_n \psi_n(z), \quad \Re(z) > 0.$$

This function belongs to $H_2(0)$ and by (3.1)

$$\int_{-\infty}^{\infty} |r_N(x+iy)|^2 dy \leqq \int_{-\infty}^{\infty} |r_N(iy)|^2 dy$$

$$= \int_{-\infty}^{\infty} |f(iy)|^2 dy - 2\pi \Sigma_1^N |a_n|^2,$$

and as $N \to \infty$ the last member tends to zero. It follows that $r_N(x+iy)$ converges in the mean of order two to the limit zero. On the other hand using (2.6) and the convergence of $\Sigma |a_n|^2$, we see that $f_N(z)$ converges absolutely and uniformly to a limit in the half-plane $x \geqq \delta > 0$. It follows that $r_N(x+iy)$ converges

[4]) The operational calculus [Math. Annalen **95** (1926), 557—584], 580.

to zero in the ordinary sense as well as in the mean. Hence

$$f(z) = \Sigma_1^\infty a_n \psi_n(z), \tag{3.7}$$

where the series converges absolutely for $\Re(z) > 0$ and uniformly for $\Re(z) \geqq \delta > 0$. On the imaginary axis it reduces to the Fourier series of the boundary function.

Putting

$$F_N(u) = \Sigma_1^N a_n \varphi_n(u),$$

we note that $F_N(u)$ converges to $F(u)$ in the mean as $N \to \infty$. But by (2.1)

$$f_N(z) = \int_0^\infty e^{-zu} F_N(u)\, du,$$

and letting $N \to \infty$ gives

$$f(z) = \int_0^\infty e^{-zu} F(u)\, du, \quad \Re(z) > 0.$$

Thus $f(z)$ is the Laplace transform of $F(u)$, a function which belongs to $L_2(-\infty, \infty)$.

On the other hand, if we start with a ψ-series

$$\Sigma_1^\infty a_n \psi_n(z) \text{ with } \Sigma_1^\infty |a_n|^2 < \infty, \tag{3.8}$$

then it converges absolutely and uniformly in every half-plane $\Re(z) \geqq \delta > 0$ to a function $f(z)$ holomorphic in the right half-plane. Denoting the Nth partial sum of the series by $f_N(z)$, we first observe that $f_N(z) \in H_2(0)$ for every N. Further for fixed $x > 0$

$$\int_{-\infty}^\infty |f_m(x+iy) - f_n(x+iy)|^2 dy \leqq \int_{-\infty}^\infty |f_m(iy) - f_n(iy)|^2 dy,$$

whence it follows that $f_N(x+iy)$ converges in the mean to $f(x+iy)$, uniformly in x, and that

$$\int_{-\infty}^\infty |f(x+iy)|^2 dy \leqq \int_{-\infty}^\infty |f(iy)|^2 dy.$$

Consequently $f(z) \in H_2(0)$ and is the Laplace transform of

$$F(u) \sim \Sigma_1^\infty a_n \varphi_n(u), \tag{3.9}$$

a function in $L_2(0, \infty)$.

Finally, if we start with an arbitrary function $F(u)$ in $L_2(0, \infty)$ whose φ-series is given by (3.9) with $\Sigma_1^\infty |a_n|^2 < \infty$, we see immediately that its Laplace transform is the ψ-series (3.8) which is absolutely convergent for $\Re(z) > 0$ and, by the above argument, represents a function in $H_2(0)$.

We have consequently proved the

Theorem. The following three classes of analytical functions are identical (I) $H_2(0)$, (II) *the set of all ψ-series*

$$\Sigma_1^\infty a_n \psi_n(z) \text{ with } \Sigma_1^\infty |a_n|^2 < \infty \text{ and } x > 0,$$

and (III) *the set of Laplace transforms of all φ-series*

$$\Sigma_1^\infty a_n \varphi_n(u) \text{ with } \Sigma_1^\infty |a_n|^2 < \infty .$$

Remark. Class (III) is of course identical with the class of Laplace transforms of $L_2(0, \infty)$. That the latter class is identical with $H_2(0)$ was proved by Paley and Wiener [5]).

4. *Special cases.* The best known orthogonal system in $L_2(0, \infty)$ is perhaps

$$\varphi_n(u) = e^{-\frac{u}{2}} L_n(u), \tag{4.1}$$

where $L_n(u)$ is the nth polynomial of Laguerre. Here

$$\psi_n(z) = \frac{(z-\frac{1}{2})^n}{(z+\frac{1}{2})^{n+1}}. \tag{4.2}$$

In this case it is easy to see directly that the system Ψ is closed in $L_2(-\infty, \infty)$. Indeed, the transformation $z = \frac{i}{2} \tan \frac{t}{2}$ carries the imaginary axis of the z-plane into the interval $(-\pi, \pi)$ on the real axis of the t-plane and takes the system Ψ into the system

$$e^{-(k+\frac{1}{2})it}, \quad k = 0, \pm 1, \pm 2, \ldots$$

The latter is evidently closed in $L_2(-\pi, \pi)$. Hence Ψ is closed in $L_2(-\infty, \infty)$ so that Φ_+ is closed in $L_2(0, \infty)$. This is offered as an alternative proof of the closure of the system of Laguerre functions.

To this system Φ_+ corresponds the following pair of associated expansions

$$f(z) = \Sigma_{n=0}^\infty a_n \frac{(z-\frac{1}{2})^n}{(z+\frac{1}{2})^{n+1}}, \tag{4.3}$$

$$F(u) \sim \Sigma_{n=0}^\infty a_n L_n(u) e^{-\frac{u}{2}}. \tag{4.4}$$

---

[5]) Fourier transforms in the complex domain [Amer. Math. Soc. Colloquium Publ. XIX (New York, 1934)], p. 8. See also Hille & Tamarkin, On moment functions [Proc. Nat. Acad. Sci. 19 (1933), 902–908].

While these formulas figure already in a note by S. Wigert, interest in them has been revived by the recent work of D. V. Widder, F. Tricomi, M. Picone and others.[6]) The credit of having first called attention to the importance of formula (4.4) as an inversion formula for the Laplace integral is due to Tricomi. Picone settled the case $F(u) \in L_2(0, \infty)$, whereas Widder showed that the formally integrated series inverted the Laplace-Stieltjes integral when $f(z)$ is completely monotonic on the positive real axis. It should be noted that the system $\{\psi_n(z)\}$ is closed not merely with respect to the class $H_2(0)$ but with respect to all functions analytic in the right half-plane. It is obvious that every such function can be expanded in one and only one way in a ψ-series which is absolutely convergent for $\Re(z) > 0$. The coefficients are not necessarily given by formula (3.2) which may cease to have a sense, but a_n is obviously a linear form in $f(\frac{1}{2}), f'(\frac{1}{2}), \ldots, f^{(n)}(\frac{1}{2})$ with rational coefficients.

As a second example let us take

$$\varphi_n(u) = e^{-\frac{u}{2}} X_n(e^{-u}), \tag{4.5}$$

where $X_n(t)$ denotes the normalized Legendre polynomial for the interval (0, 1). Here

$$\psi_n(z) = \sqrt{2n+1}\,\frac{(z-\frac{1}{2})\cdots(z-n+\frac{1}{2})}{(z+\frac{1}{2})(z+\frac{3}{2})\cdots(z+n+\frac{1}{2})}\,. \tag{4.6}$$

Expansions in terms of the latter system are a special case of the interpolation series studied by R. Lagrange.[7]) It follows from his investigations that any function bounded and holomorphic in the right half-plane can be represented there by a convergent ψ-series. In particular, one concludes that every function belonging to a class $H_p(0)$, $p \geqq 1$, admits of such a representation.[8]) On the other hand, fairly simple calculations show that a function representable by a ψ-series, convergent in some half-plane, has to be of finite order with respect to z

---

[6]) S. Wigert, Contributions à la théorie des polynomes d'Abel-Laguerre [Arkiv f. Mat. **15** (1921), No. 25]. D. V. Widder, An application of Laguerre polynomials [Duke Math. Journal **1** (1935), 126—136]. F. Tricomi, Trasformazione di Laplace e polinomi di Laguerre [Rendiconti Atti Accad. Naz. Lincei (6) **21** (1935), 232—239]. M. Picone, Sulla trasformazione di Laplace [ibid., 306—313].

[7]) Mémoire sur les séries d'interpolation [Acta Math. **64** (1935), 1—80], Ch. VI.

[8]) Such a function is bounded in every half-plane $x \geqq \delta > 0$.

in such a half-plane, so that the class of representable functions is fairly limited in the present case.

Let $A(u)$ be of bounded variation in $(0, \infty)$, $A(0) = 0$, and consider

$$f(z) = \int_0^\infty e^{-zu}\, dA(u), \tag{4.7}$$

which is absolutely convergent in $\Re(z) \geqq 0$. A simple calculation shows that

$$f(z) = \sum_{n=0}^{\infty} a_n \sqrt{2n+1}\, \frac{(z-\frac{1}{2}) \cdots (z-n+\frac{1}{2})}{(z+\frac{1}{2})(z+\frac{3}{2}) \cdots (z+n+\frac{1}{2})}, \tag{4.8}$$

where

$$a_n = \int_0^\infty e^{-\frac{u}{2}} X_n(e^{-u})\, dA(u). \tag{4.9}$$

A more delicate analysis proves that

$$A(u) = \sum_{n=0}^{\infty} a_n \int_0^u e^{-\frac{t}{2}} X_n(e^{-t})\, dt, \tag{4.10}$$

the series being convergent for all values of u. This is essentially one of Hausdorff's solutions of the moment problem for a finite interval. [9])

There are numerous other possibilities. We could replace the Laguerre polynomials by general Laguerre polynomials in (4.1) and the Legendre polynomials by Jacobi polynomials in (4.5). The transformation $t = e^{-u}$ applied to the system $\{e^{2\pi i n t}\}$ leads to interesting ψ-functions expressible in terms of the incomplete gamma function. Further examples can be had from the theory of Bessel functions, Hermitian polynomials etc.

5. *Generalizations.* In the general discussions of § 3 we restricted ourselves to functions of the class $H_2(0)$. But it is clear that interesting expansion problems are associated with functions of other classes, e.g., the classes $H_p(0)$, the class of bounded functions, the class of convergent Laplace integrals etc. These problems can be attacked when suitable restrictions are imposed on the system Φ.

The method would seem to be capable of further applications to other transformations than that of Laplace. As an example

[9]) F. Hausdorff, Momentprobleme für ein endliches Intervall [Math. Zeitschrift **16** (1923), 220—248], especially 227—231.

we may mention the transformation of Gauß-Weierstraß depending upon the kernel $\exp\left[-(z-u)^2\right]$. Here we have the bilinear formula

$$e^{2zu-z^2} = \sum_{n=0}^{\infty} \frac{1}{n!} H_n(u)\, z^n\,,$$

which links the Hermitian polynomials with the powers of z and gives rise to associated expansions. [10])

(Received November 19th, 1937.)

[10]) See E. Hille, A class of reciprocal functions [Annals of Math. (2) **27** (1926), 427–464].