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Limit groups and spaces in regions and open manifolds

by

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Introduction.

This paper is concerned with certain limit groups and spaces in bounded open manifolds and regions. The theory of regions and of open manifolds generally is one of the least developed parts of topology, despite the fact that the notion of a region is one of the simplest and most frequently used topological notions in various mathematical subjects. In particular the theory of prime ends, which is concerned with the structure of the boundaries, has so far evaded modern topological methods [1]). This theory has remained an isolated subject depending on „direct'' (and often complicated) methods. We give now an interpretation and an extension of its foundations in the light of modern topology.

§ 1 contains the definition of systems of boundary divisors or ends. The notion of a divisor is quite elementary; a divisor is essentially a decreasing sequence of part regions tending to the boundary and represented by its limit set. Divisors are denoted by $\mathfrak{E}_G$. In particular the boundary Γ of a region G can be considered as a divisor which is denoted by Γ_G.

In § 2 we define the limit groups of a region. These are groups of 0-dimensional infinite cycles tending to the boundary [2]). The most important of these is the group Σ of pure cycles which is defined as the direct sum $A(\Gamma_G) + B(\Gamma_G)$ of the groups of „convergent'' (or α-) and „divergent'' (or β-) cycles. The former are

[1]) See my thesis in Math. Annalen **103** (1930), also my papers in Math. Ann. **106** (1932) and Math. Zeitschrift **36** (1932). It is little known that this theory is valid for all bounded open manifolds embedded in Euclidean spaces (and not only for 3-dimensional regions). In this paper we can confine ourselves to (n-dimensional) regions, as the extension of its results to bounded manifolds is obvious.

[2]) Limit cycles are denoted by $z_f = (z_k) = z_1, z_2, \ldots, z_k, \ldots$ or similarly.

limit cycles lying on finite systems of paths with (accessible) endpoints on the boundary, the latter are cycles of the opposite type (which „carry" no α-cycles). Similarly we define two fundamentally different types of boundary relations, the α- and β-homologies.

In § 3 we consider the group Σ as an abstract space Σ^*. Closures in this group are defined by means of local homologies. The closure $\overline{S}$ of an aggregate S is defined as a sum of two aggregates $(S)^1$ and $(S)^2$, which we call closures of the first and second kind. While $(S)^1$ is essentially induced by the topology of the region, it is the introduction of $(S)^2$ which makes the whole theory effective [3]). The limit space Σ^* is a neighbourhood space (in the sense of H. Weyl). It satisfies all axioms of a topological space, except the distributive law of closures, which in our case however can be replaced by an equivalent condition.

Similarly, the fact that the „points" are not closed and the corresponding separation axiom is not fulfilled is practically unimportant, since the closures of points are shown to be very simple aggregates.

In § 4 we consider α- and β-homology groups. With the aid of these groups we define the prime ends in the space F of all α- and β-limit sequences of points, which forms a subspace of the space Σ^*. This definition is essentially on the lines of my thesis. We also state some theorems and mention some general problems. But by far the most interesting problems arise in connection with the theory of conformal representation (of regions of *arbitrary* connectivity) and the theory of automorphic functions, which we shall consider elsewhere.

§ 1. Divisors of the boundary.

1. Let G be a region in the n-dimensional Euclidean space R^n, and let Γ be its boundary. (We assume G to be bounded.) Let Ψ^n be an arbitrary subdivision of R^n into n-dimensional convex cells (simplices, cubes, etc.) forming a cell complex of some arbitrary mesh. The sum of all cells in Ψ^n we call an n-dimensional (infinite) polyedron, and denote it by $|\Psi^n|$. The complex of all $(n-1)$-dimensional faces of Ψ^n we call the $(n-1)$-dimen-

[3]) This shows that the topology of the limit space depends not merely on „distances" between infinite chains („points") z_f but also essentially on the „distances" between finite chains z and the infinite chains z_f.

sional skeleton of Ψ^n, and denote it by Ψ^{n-1}. The sum of all cells in Ψ^{n-1} we denote by $|\Psi^{n-1}|$.

A subset Q of $|\Psi^{n-1}|$ we call a *cut* of G, if each point of Q is a boundary point of at least two components of $G - Q$. A cut Q of G is called *irreducible* if the boundaries Γ' and Γ'' of any two components G' and G'' of $G - Q$ coincide inside G, i.e. if $\Gamma' G = \Gamma'' G$. An irreducible cut Q is called *regular*, if $G - Q$ consists of precisely two components. If Q is a cut of G, and G' is an arbitrary component of $G - Q$, we speak also of a cut Q *corresponding* to G', and of a region G' *defined* by Q.

2. Let

$$(G_n) = G_1 \supset G_2 \supset \ldots \supset G_n \supset \ldots$$

be a decreasing sequence of regions defined by a sequence

$$(Q_n) = Q_1, Q_2, \ldots, Q_n, \ldots$$

of cuts of G such that the closures $\overline{Q}_\lambda$ and $\overline{Q}_\mu$ of any two cuts Q_λ and Q_μ of the sequence have no common points. The product

$$\mathfrak{S}_G = \prod_{n=1}^{\infty} (\overline{G}_n)$$

is obviously a continuum or a point. It is obvious that the set $\mathfrak{S}_G$ is uniquely defined by the sequence of cuts (Q_n). We call $\mathfrak{S}_G$ a *divisor* of the boundary Γ or the *end* of the region G if $\mathfrak{S}_G$ is a part of Γ. A divisor $\mathfrak{S}_G$ is called *regular* if all cuts Q_n of the sequence (Q_n) are regular.

It is easy to see that *there exists a regular divisor*

$$\mathfrak{S}_G \equiv \Gamma_G$$

coinciding with the boundary Γ itself; in particular the divisor Γ_G can be defined by a sequence of regular cuts (Q_n) such that each cut Q_n lies entirely in G.

A divisor $\mathfrak{S}'_G$ is said to be *contained* in a divisor $\mathfrak{S}_G$,

$$\mathfrak{S}'_G \subset \mathfrak{S}_G,$$

if $\mathfrak{S}'_G$ and $\mathfrak{S}_G$ are defined by sequences (G'_m) and (G_n) such that *almost all* regions G'_m of the first sequence are contained in *each* region G_n of the second sequence. In particular the boundary Γ of G, considered as a divisor, must obviously contain all divisors of Γ.

If $\mathfrak{S}'_G$ and $\mathfrak{S}''_G$ are two divisors such that the relations

$$\mathfrak{S}'_G \subset \mathfrak{S}''_G,$$

and

$$\mathfrak{G}''_G \subset \mathfrak{G}'_G$$

are fulfilled simultaneously, we say that $\mathfrak{G}'_G$ and $\mathfrak{G}''_G$ are *equivalent* and we write

$$\mathfrak{G}'_G = \mathfrak{G}''_G.$$

The set of points in a divisor $\mathfrak{G}_G$ considered independently from the defining sequence [4]) (G_n) we denote by $|\mathfrak{G}_G|$. Thus from $\mathfrak{G}'_G = \mathfrak{G}''_G$ it follows that

$$|\mathfrak{G}'_G| = |\mathfrak{G}''_G|,$$

but not vice versa.

Two divisors $\mathfrak{G}'_G$ and $\mathfrak{G}''_G$ defined by the sequences (G'_n) and (G''_m) are said to be *distinct* if there exist two integers $n = \lambda$ and $m = \mu$ such that

$$G'_\lambda G''_\mu = 0.$$

It is obvious that $\mathfrak{G}'_G$ and $\mathfrak{G}''_G$ are distinct if, and only if, there exists a pair of integers λ, μ such that

$$G'_{\lambda+\nu} G''_{\mu+\nu} = 0$$

for each $\nu = 1, 2, \ldots$.

3. *Limit sequences in a divisor.* A sequence of points of the region G

$$(P_\lambda) = P_1, P_2, \ldots, P_\lambda, \ldots,$$

such that each limit point of (P_λ) lies on the boundary Γ of G we call a *limit* sequence of points in G. A limit sequence of points with precisely one limit point we call a *convergent* sequence. A divisor $\mathfrak{G}_G$ defined by a sequence (G_n) *contains* a limit sequence (P_λ) if each region G_n $(n = 1, 2, \ldots)$ contains almost all points of the sequence (P_λ). The aggregate of all convergent limit sequences (P_λ) contained in $\mathfrak{G}_G$ we denote by $f(\mathfrak{G}_G)$ [5]).

It is easy to see that two divisors $\mathfrak{G}'_G$ and $\mathfrak{G}''_G$ are equivalent if, and only if, the aggregates $f(\mathfrak{G}'_G)$ and $f(\mathfrak{G}''_G)$ are identical. In other words, from $\mathfrak{G}'_G = \mathfrak{G}''_G$ it follows that

$$f(\mathfrak{G}'_G) = f(\mathfrak{G}''_G).$$

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[4]) From the above definition it is clear that a divisor $\mathfrak{G}_G$ is not merely a subset of Γ, but a subset of Γ defined by and associated with a sequence (G_n). One could indeed define the sequence (G_n) itself as a divisor of Γ, but it is more convenient to represent it by the product set $\mathfrak{G}_G$.

[5]) Instead of aggregates of convergent limit sequences in $\mathfrak{G}_G$ we could consider here the aggregates of arbitrary limit sequences in $\mathfrak{G}_G$, but this is not essential.

Thus the aggregate $f(\mathfrak{C}_G)$ of convergent limit sequences in $\mathfrak{C}_G$ is independent of the sequences (G_n) defining $\mathfrak{C}_G$.

On the other hand, two divisors $\mathfrak{C}'_G$ and $\mathfrak{C}''_G$ are distinct if, and only if, there exists a convergent sequence of points contained simultaneously in $\mathfrak{C}'_G$ and in $\mathfrak{C}''_G$, i.e. from $\mathfrak{C}'_G\mathfrak{C}''_G = 0$ it follows that

$$f(\mathfrak{C}'_G)f(\mathfrak{C}''_G) = 0.$$

If $\mathfrak{C}'_G$ and $\mathfrak{C}''_G$ are arbitrary, then the aggregate

$$f(\mathfrak{C}'_G)f(\mathfrak{C}''_G)$$

of all convergent limit sequences which are contained in both $\mathfrak{C}'_G$ and $\mathfrak{C}''_G$ is called the *common part* or the product of $\mathfrak{C}'_G$ and $\mathfrak{C}''_G$.

4. *Full systems of distinct divisors.* A system $M(\mathfrak{C}_G)$ of divisors of Γ we call a system of distinct divisors if any two divisors in $M(\mathfrak{C}_G)$ are distinct. A *full system* of distinct divisors of the boundary Γ is a system $M(\mathfrak{C}_G)$ of distinct divisors such that each convergent limit sequence (P_λ) contains at least one subsequence

$$(P'_\lambda) = P'_1, P'_2, \ldots, P'_\lambda, \ldots$$

contained in a divisor $\mathfrak{C}_G$ of the system $M(\mathfrak{C}_G)$.

A system $M(\mathfrak{C}_G)$ consisting of a single divisor $\mathfrak{C}_G = \Gamma_G$ is obviously a full system. It is not difficult to show that in any region G the totality of all full systems of distinct divisors is infinite, and that it has virtually the power $2^{\aleph}$.

§ 2. Limit groups of the region.

5. The aggregate of all inner points of the region G can be considered as a field of vertices in which abstract, geometrical („flat'') and topological complexes can be defined. Let $[x^1]$ be the system of all *closed* topological 1-cells (arcs) lying entirely in G. Each 1-cell of $[x^1]$ is defined as a topological transformation of a segment $0 \leqq x \leqq 1$. By $[x^0]$ we denote the aggregate of all 0-cells (points) in G.

A finite system K^1 of 1-cells in G we call a 1-dimensional complex (1-complex) in G. A 1-complex K^1 we call *regular* if any two 1-cells in K^1 have at most their endpoints in common. The set of all points contained in either of the 1-cells of K^1 we call the *corresponding set* of K^1 and denote it by $| K^1 |$. The corresponding set of a regular complex in G consists obviously of a finite system of (non-intersecting) arcs; the corresponding set of a regular geometrical (flat) 1-complex consists of a finite number of connected polygon lines.

A 0-complex is by definition a finite set of 0-cells (points) in G.

6. Let I_2 be the group of integers reduced mod 2. Its elements (classes) we denote by 0 and 1, and we apply to these the usual (algebraic) operations mod 2. We define now in G 0- and 1-dimensional chains mod 2 with respect to the systems $[x^0]$ and $[x^1]$ of all 0- and 1-cells as variables.

An r-chain C^r mod 2 ($r = 0$ or $= 1$) is a linear form in which each r-cell of $[x^r]$ is associated with a coefficient belonging to the group I_2 and such that at most a finite number of r-cells in C^r are taken with a coefficient $\neq 0$. Thus, omitting terms with the coefficient 0, we can write C^r in the form

$$C^r = \Sigma\, x_i^r \quad (r = 0, 1)$$

where i varies between 1 and some integer. An r-chain with all coefficients 0 we call an r-chain 0 or an empty chain. Since all variables of dimension r occur in a chain, we can define the summation of chains. The sum of two chains mod 2 is again a chain mod 2. The r-chains mod 2 of the region G form thus an additive group L^r in which each element coincides with its inverse, the element 0 being the empty r-chain [6]). It is obvious that r-chains mod 2 can also be considered as r-complexes for which addition mod 2 is defined. The notion of a corresponding set can thus be applied to a chain C^r mod 2; the corresponding set of C^r we denote by $|\, C^r\,|$.

7. The boundary (mod 2) of a 1-cell $x^1 = P_0P_1$ with the end-points P_0, P_1 is the linear form $P_0 + P_1$, and the boundary $\dot{C}^1$ of a chain C^1 is the sum of the boundaries of all 1-cells in C^1. The boundary $\dot{C}^0$ of a 0-chain C^0 is by definition 0.

An r-chain C^r ($r = 0, 1$) such that $\dot{C}^r = 0$ is called an *r-cycle.* Thus all 0-chains are 0-cycles. If a 0-cycle z^0 is the boundary of a chain C^1, we write also $C^1 \to z^0$. The r-cycles mod 2 form obviously a subgroup Z^r of the group L^r.

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[6]) The group of r-chains defined above forms obviously a special case of the usual group of singular chains mod 2 on a generalized (infinite) polyedron. In the case of a region it is usual to assume (as above) that the corresponding sets of all complexes are contained in the region.

A cycle z^0 is called a *boundary cycle* and is said to be *homologous* 0 mod 2,

$$z^0 \sim 0 \pmod 2,$$

if there exists a complex C^1 such that $\dot{C}^1 = z^0$. This relation can also be written in the form $C^1 \to z^0$. Two cycles z_1^0 and z_2^0 are said to be homologous to each other (mod 2), if there exists a chain C^1 such that

$$C^1 \to z_1^0 + z_2^0.$$

A 0-cycle 0 is by definition ~ 0. Thus each cycle z^0 is homologous to itself,

$$z^0 \sim z^0.$$

Since any two 0-cells in G can be joined by a 1-cell in G, it is easy to see that a 0-cycle z^0 is ~ 0 if, and only if, the number of its points (0-cells) is $\equiv 0$ mod 2. The 0-cycles homologous 0 form a subgroup of the group Z^0, which we denote by H^0.

Throughout this paper all boundary and homology relations are understood mod 2.

8. Let $\mathfrak{E}_G$ be a divisor of Γ defined by a sequence

$$(G_m) = G_1 \supset G_2 \supset \ldots \supset G_m \supset \ldots$$

of decreasing regions. A sequence

$$C^r = (C_k^r) = C_1^r, \ldots, C_k^r, \ldots$$

of r-dimensional chains ($r = 0$ or $= 1$) we call an r-dimensional *limit chain* in $\mathfrak{E}_G$ if for each m *almost all* corresponding sets $|C_k^r|$, $k = 1, 2, \ldots,$ are contained in G_m. The sum

$$|C^r| = \sum_{k=1}^{\infty} |C_k^r|$$

we call the *corresponding set* of the limit chain C^r. By t we denote the topological limit of the set C^r. The set t which lies on Γ we call the *boundary limit* of the chain C^r.

If C^r is a limit chain in $\mathfrak{E}_G$, then any subsequence of chains

$$C^{r\prime} = (C_{\mu_k}^r) = C_{\mu_1}^{r\prime}, C_{\mu_2}^{r\prime}, \ldots, C_{\mu_k}^{r\prime}, \ldots \quad (\mu_k > \mu_{k-1})$$

of $(C_{\mu_k}^r)$ forms a limit chain in $\mathfrak{E}_G$, which we call a *subordinate* chain of C^r in $\mathfrak{E}_G$. If $C^{r\prime}$ is a subordinate chain of C^r we write

$$C^{r\prime} \prec C^r.$$

The system of all subordinate chains of C^r we denote by $\{C^r\}$ [7]). The limit chains $C^r = (C^r_k)$ and $C^{r\prime} = (C^{r\prime}_k)$ we consider as being *identical*,

$$C^{r\prime} = C^r,$$

if almost all chains C^r_k and $C^{r\prime}_k$ are identical, i.e. if there exists an integer k_0 such that

$$C^{r\prime}_{k_0+k} = C^r_{k_0+k}$$

for each $k = 1, 2, \ldots$.

A 0-dimensional limit chain

$$z_f = z_1, z_2, \ldots, z_k, \ldots$$

in $\mathfrak{G}_G$ we call a *limit* 0-*cycle* [8]) in $\mathfrak{G}_G$, if for each m we can define an integer k_m such that

$$z_{k_m+k} \sim z_{k_m+k+1} \text{ in } G_m \text{ [9])}$$

for each $k = 1, 2, \ldots$. A limit 0-cycle z_f in $\mathfrak{G}_G$ is by definition 0 if almost all cycles z_k are 0. The sum of two limit cycles $z_f = (z_k)$ and $z'_f = (z'_k)$ is by definition the limit 0-cycle

$$z_f + z'_f = (z_k + z'_k).$$

It is obvious that the sum of two limit cycles in $\mathfrak{G}_G$ is a limit cycle in $\mathfrak{G}_G$. Thus the limit 0-cycles in $\mathfrak{G}_G$ form a group which we denote by $Z(\mathfrak{G}_G)$.

A limit cycle $z_f = (z_k)$ is homologous 0 in $\mathfrak{G}_G$,

$$z_f \sim 0 \quad \text{in } \mathfrak{G}_G,$$

if there exists a limit chain $C^1 = (C^1_k)$ in $\mathfrak{G}_G$ such that

$$C^1_k \to z_k \quad \text{for } k = 1, 2, \ldots$$

Two limit cycles z'_f and z''_f are homologous to each other,

$$z'_f \sim z''_f \quad \text{in } \mathfrak{G}_G,$$

if $z'_f + z''_f \sim 0$ in $\mathfrak{G}_G$. The limit cycle 0 is by definition ~ 0 in $\mathfrak{G}_G$.

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[7]) More generally, if $M(C^r)$ is an arbitrary aggregate of a limit chain, we denote by $\{M(C^r)\}$ the sum $\sum_{C^r \in M(C^r)} \{C^r\}$; of all systems for all C^r 's in $M(C^r)$.

[8]) Each chain z_k of the limit chain z_f is obviously a (0-dimensional) cycle. We write z for z^0, omitting the dimensional index number 0.

[9]) It is sufficient to assume this homology to be valid in G. In general, if $G' \subset G$ and $|z| \subset G'$, then from $z \sim 0$ in G it follows that $z \sim 0$ in G'. If $C' \to z$ in G, we can easily construct a chain $'C^1$ isomorphic with C^1 such that $'C^1 \to z$ in G'. (This construction is obviously possible, because C is 1-dimensional.)

We get thus

$$z_f \sim z_f \qquad \text{in } \mathfrak{G}_G.$$

The limit 0-cycles homologous 0 in $\mathfrak{G}_G$ form a subgroup $H(\mathfrak{G}_G)$ of $Z(\mathfrak{G}_G)$. It is easy to see that the factor group $Z(\mathfrak{G}_G) - H(\mathfrak{G}_G)$ is isomorphic with the group of integers reduced mod 2.

9. *The groups of α- and β-cycles.* We define now subgroups of „convergent'' and „divergent'' limit cycles in a region. The following definition of α- and β-cycles makes this distinction clear.

A topological transformation J of the interval $0 \leqq x < 1$ contained in G we call a *path* in G, if the closure $\overline{J}$ of J is a simple arc consisting of J and (precisely) one single point t lying on the boundary Γ.

A linear combination

$$x_f = x_f^1 + \ldots + x_f^i + \ldots + x_f^\lambda$$

of a finite number of limit 0-cycles $x_f^i = (x_k^i)$ we call an *α-cycle* if for each i the corresponding set $|\, x_f^i \,|$ of the cycle x_f^i lies on a path J_i in G.

From this definition it follows easily that for each i there exists a limit chain C^{1i} *lying entirely on* J and such that

$$C_k^{1i} \to z_k^i + z_{k+1}^i.$$

A limit 0-cycle $y_f = (y_k)$ we call a *β-cycle* [10]) in G if there exists no α-cycle lying on the corresponding set $|\, y_f \,|$ of y_f.

If z_f is an arbitrary limit cycle, then it can easily be proved that there exists always an α- or a β-cycle lying on the corresponding set $|\, z_f \,|$ of z_f. A limit cycle z_f, which is either an α- or a β-cycle, we call briefly a *pure* cycle.

A limit 0-cycle 0 is by definition both an α- and a β-cycle. With addition defined in the usual sense, the totality of all α-cycles in G forms a group which we denote by $A(\Gamma_G)$; this group obviously cannot be empty. Similarly the system of all β-cycles forms a group which we denote by $B(\Gamma_G)$.

If $\mathfrak{G}_G$ is an arbitrary divisor of Γ, then the systems of all α- and β-cycles in $\mathfrak{G}_G$ form also groups, which we denote by $A(\mathfrak{G}_G)$ and $B(\mathfrak{G}_G)$ correspondingly. It is clear that for an arbitrary $\mathfrak{G}_G \neq \Gamma_G$ either of these groups can be empty. The common part of the

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[10]) We could define α- and β-cycles with respect to an arbitrary divisor $\mathfrak{G}_G$ of Γ. But this is not essential, as it can easily be shown that the definition of α- and β-cycles is independent of the choice of divisors.

groups $A(\mathfrak{G}_G)$ and $B(\mathfrak{G}_G)$ is 0. The totality of cycles which are either in $A(\mathfrak{G}_G)$ or in $B(\mathfrak{G}_G)$ is a group which can be considered as the direct sum

$$\Sigma = A(\mathfrak{G}_G) + B(\mathfrak{G}_G)$$

of the groups $A(\mathfrak{G}_G)$ and $B(\mathfrak{G}_G)$.

α- and β-*homologies*. We define now homology relations with respect to the groups $A(\Gamma_G)$ and $B(\Gamma_G)$. These relations are fundamental for our theory, and are very similar in both cases.

An α-cycle x_f is said to be α-*homologous* 0,

$$x_f \underset{\alpha}{\sim} 0 \qquad \text{in } G,$$

if x_f bounds a limit chain C^1 in G such that each pure cycle on the corresponding set $|C^1|$ is an α-cycle.

Two α-cycles x'_f and x''_f are said to be α-*homologous to each other*

$$x'_f \underset{\alpha}{\sim} x''_f,$$

if the cycle $x'_f + x''_f$ is α-homologous 0 in G.

A β-cycle y_f is said to be β-*homologous* 0,

$$y_f \underset{\beta}{\sim} 0 \qquad \text{in } G,$$

if y_f bounds a limit chain C^1 such that each pure cycle on $|C^1|$ is a β-cycle.

Two β-cycles y'_f and y''_f are said to be β-*homologous to each other* in G,

$$y'_f \underset{\beta}{\sim} y''_f,$$

if the chain $y'_f + y''_f$ is β-homologous 0 in G.

The limit 0-cycle 0 is by definition α- and β-homologous 0. If the limit cycles z'_f and z''_f are α- or β-homologous 0 in G, then the sum $z'_f + z''_f$ is α- or β-homologous 0 as well. Thus the α-cycles α-homologous 0 and the β-cycles β-homologous 0 form subgroups of the groups $A(\Gamma_G)$ and $B(\Gamma_G)$, which we denote by $H_\alpha(\Gamma_G)$ and $H_\beta(\Gamma_G)$.

It is obvious that the notions of α- and β-homologies can be extended to an arbitrary divisor $\mathfrak{G}_G$ of Γ, and we can define similarly the groups $H_\alpha(\mathfrak{G}_G)$ and $H_\beta(\mathfrak{G}_G)$.

We define further a simple but important subgroup of the group $H_\alpha(\Gamma_G)$.

Let x_f be an α-cycle, and let t be the (finite) set of its limit points on Γ. We say, x_f is *locally* homologous 0 if x_f bounds a limit chain C^1 such that the limit set of $|C^1|$ on Γ is contained in t. It is easy to see that in this case all pure cycles on $|C^1|$ are α-cycles.

Two α-cycles x'_f and x''_f are said to be locally homologous to each other if $x'_f + x''_f$ is locally ~ 0. The α-cycles 0 are locally ~ 0 by definition. Thus the α-cycles locally ~ 0 form a subgroup of the group $H_\alpha(\Gamma_G)$, which we denote by $H^l_\alpha(\Gamma_G)$.

Similarly we can define the subgroup $H^l_\alpha(\mathit{6}_G)$ of $H(\mathit{6}_G)$, where $\mathit{6}_G$ is an arbitrary divisor of Γ. We shall be mainly concerned with the groups $H_\beta(\Gamma_G)$, and $H^l_\alpha(\Gamma_G)$, making no use of the group $H_\alpha(\Gamma_G)$.

§ 3. Limit spaces of the region.

10. We shall now consider the group Σ of all pure (α- and β-) cycles as an abstract space [11]). We shall define in Σ a (generalized) topology, which is fundamental in the theory of prime ends.

Let S be an arbitrary aggregate of limit cycles of Σ. The *closure* $\overline{S}$ is defined as a set of limit cycles of Σ

$$\overline{S} = (S)^1 + (S)^2$$

consisting of two parts: the closure $(S)^1$ of the *first kind*, and the closure S^2 of the *second kind*. If S is empty, we write $S = 0$ and set $\overline{S} = \overline{0} = 0$. If $S \neq 0$, we define $(S)^1$ and $(S)^2$ as follows:

Let $z_f = z_1, z_2, \ldots, z_k, \ldots$ be a limit cycle of the group Σ, and let t be its boundary limit (§ 2, 8). Consider a spherical neighbourhood [12]) $U(t, \varepsilon)$ of t, i.e. the sum of all spherical regions (with a diameter $\leq \varepsilon$) containing points of t. The open set

$$U(t, \varepsilon)G = U(z_f, \varepsilon)$$

which obviously contains almost all cycles z_k, we call an *ε-neighbourhood* of z_f in G. If we omit a finite number of cycles z_k in z_f, the limit cycle z_f will be replaced by an *identical* cycle, which we denote again by z_f. We can therefore assume that all cycles z_k of z_f lie in $U(z_f, \varepsilon)$, ε being arbitrarily small.

Now let $\zeta_f = \zeta_1, \zeta_2, \ldots, \zeta_k, \ldots$ be an arbitrary limit cycle of the group Σ.

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[11]) More generally we could consider the group $Z(\Gamma_G)$ of all limit cycles as an abstract space. The results of this paragraph can be extended to this group, which is of some interest from the abstract point of view. In this paper we shall however make no further use of this group.

[12]) This notion is purely auxiliary; the space defined by means of these neighbourhoods is of no importance, and must not be confused with the abstract space Σ^* defined below.

The cycle ζ_f is called a *closure cycle of the first kind* with respect to S if each (arbitrarily small) neighbourhood $U(\zeta_f, \varepsilon)$ of ζ_f contains a cycle $z_f = z_1, z_2, \ldots, z_k, \ldots$ such that

$$\zeta_f \sim z_f \qquad \text{in } U(\zeta_f, \varepsilon).$$

The aggregate of all closure cycles of S of the first kind we denote by $(S)^1$, and call it the *closure of the first kind* of S.

We make use now of the following notation. Let $U(z_f)$ be a neighbourhood of $z_f = z_1, z_2, \ldots, z_k, \ldots$, and let z be a cycle (mod 2) of the group Z. If for each $k = 1, 2, \ldots$

$$z \sim z_k \qquad \text{in } U(z_f),$$

we write briefly

$$z \sim z_f \qquad \text{in } U(z_f).$$

The cycle $\zeta_f = \zeta_1, \zeta_2, \ldots, \zeta_\lambda$ is called a *closure cycle of the second kind* with respect to S, if each ε-neighbourhood $U(\zeta_f, \varepsilon)$ contains a limit cycle $z_f^\lambda = z_{1\lambda}, z_{2\lambda}, \ldots, z_{k\lambda}, \ldots$ of S for each λ such that

$$\zeta_\lambda \sim z_f^\lambda \qquad \text{in } U(\zeta_f, \varepsilon),$$

or, more precisely, if

$$\zeta_\lambda \sim z_{k\lambda} \qquad \text{in } U(\zeta_f, \varepsilon) \text{ for each}$$

$\lambda = 1, 2, \ldots$.

Here, as always, we must bear in mind the convention by which the limit cycles ζ_f and z_f can be replaced by identical [13]) cycles with a finite number of cycles ζ_λ and $z_{k\lambda}$ $(\lambda=1, 2, \ldots)$ omitted. The aggregate of all closure cycles of the second kind we denote by $(S)^2$, and call it the closure of the second kind of S.

The closure $\overline{S} = (S)^1 + (S)^2$ of S is now defined. The group Σ can therefore be considered as a general topological space [14]), which we denote by Σ^*.

[13]) See § 2, 7. The limit cycles $z_f = z_1, z_2, \ldots, z_k, \ldots$ and $z_f' = z_1', z_2', \ldots, z_k', \ldots$ are identical if $z_k = z_k'$ for almost all k. In particular, if we omit in z_f a finite number of cycles z_k, we obtain a limit cycle identical with z_f, which we denote again by z_f. We make use of this definition throughout this paper without special references.

[14]) A set E of elements („points") is called a *general topological space* if to each $S \subset E$ there corresponds an $\overline{S} \subset E$ (the closure of S). E is called a *topological space* if the above correspondence satisfies the four axioms of Kuratowski. (See Alexandroff-Hopf, Topologie I.) The equivalence of Kuratowski's axioms with the usual Hausdorff axioms of a topological neighbourhood space can easily be proved. Sufficient and necessary conditions for E to be a neighbourhood space (in the sense of H. Weyl) were given by A. Markoff (Alexandroff-Hopf, p. 42).

We introduce now the following notations: by $L(S)$ we denote the part of $\overline{S}$ which is not in S; correspondingly we denote by $L^1(S)$ and $L^2(S)$ the parts of $(S)^1$ and $(S)^2$ not in S. Further we use the notations $L_\alpha(S)$, $L_\beta(S)$, $L^1_\alpha(S)$, $L^1_\beta(S)$ etc. to represent the part of L, $L^1(S)$, $L^2(S)$ etc. consisting of all α- and all β-cycles of these aggregates respectively.

11. We shall establish now some elementary properties of the space Σ^*.

I. If z_f is an arbitrary limit cycle, then $z_f + z_f = 0$, and thus $z_f \sim z_f$ in Γ_G; obviously we can consider a limit cycle $\mathbf{0}$ to be $\sim \mathbf{0}$ in any arbitrarily small neighbourhood $U(z_f, \varepsilon)$. Thus

$$z_f \sim z_f \qquad \text{in } U(z_f, \varepsilon),$$

and by definition of the closure of the first kind we get

$$S \subset (S)^1 \subset \overline{S}.$$

II. *The group Σ^* is a neighbourhood space,* i.e. the general topological correspondence in Σ^* can be induced by means of a full system of neighbourhoods of all elements in Σ^*.

For if $\mathbf{0}$ is the empty set in Σ^*, and S' and S'' are two arbitrary sets in Σ^*, then

1. $\overline{\mathbf{0}} = \mathbf{0}$.

2. from $S' \subset S''$ it follows that $(S')^1 \subset (S')^2 \subset (S'')^2$, and therefore $\overline{S'} \subset \overline{S''}$.

Thus the necessary and sufficient (A. Markoff's) conditions for Σ^* to be a neighbourhood space [15]) are fulfilled.

If z_f is an arbitrary limit cycle, and E an arbitrary set of limit cycles such that $\overline{E} \not\ni z_f$ then the set

$$V(z_f) = \Sigma^* - E$$

is defined as a neighbourhood of z_f. The system of all sets such as $\Sigma^* - E$ forms the full system of neighbourhoods of z_f.

III. Σ^* satisfies the following relations,

$$\begin{aligned}\overline{S} = (S)^1 + (S)^2 &= S + ((S)^1)^2 \\ &= S + ((S)^2)^1.\end{aligned}$$

---

[15]) It may be noted here that the neighbourhoods in Σ^* are not necessarily open.

According to I the set S can be omitted in the first of these two relations. It is sufficient to verify

$$(1) \quad ((S)^1)^2 \subset \overline{S}$$
$$(2) \quad ((S)^2)^1 \subset \overline{S}.$$

The inverse relations follow at once from I and from Markoff's conditions in II.

Suppose (1) is not fulfilled. Then there exists a limit cycle $\zeta_f = \zeta_1, \zeta_2 \ldots, \zeta_k, \ldots$ such that

$$\zeta_f \not\subset (S)^2 \text{ and } \zeta_f \subset ((S)^1)^2.$$

We can therefore define an arbitrarily small neighbourhood $U(\zeta_f, \varepsilon)$ such that each cycle ζ_k is $\sim$ to a limit cycle ζ_f^k of $(S)^1$. For a given k let $U(\zeta_f^k, \delta)$ be an arbitrarily small neighbourhood of ζ_f^k This neighbourhood must contain a cycle z_f^k of S such that

$$\zeta_f^k \sim z_f^k \qquad \text{in } U(\zeta_f^k, \varepsilon),$$

k being arbitrary, and ε arbitrarily small. Thus (1) is proved.

Suppose now that (2) is not fulfilled. Then there exists a limit cycle $\zeta_f = \zeta_1, \zeta_2, \ldots, \zeta_k, \ldots$ such that

$$\zeta_f \not\subset (S)^1 \text{ and } \zeta_f \subset ((S)^2)^1.$$

There exists therefore a sufficiently small neighbourhood $U(\zeta, \varepsilon)$ in which no cycle of S is $\sim \zeta_f$. Therefore we can choose a neighbourhood $U(\zeta_f, \frac{1}{2}\varepsilon)$ in which a cycle $\zeta_f = \zeta_1, \zeta_2, \ldots, \zeta_k, \ldots$ satisfies the relations

$$\zeta_f \sim \xi_f, \quad \xi_k \sim z_f^k, \qquad k = 1, 2, \ldots,$$

where z_f^k denotes some cycle of S. Thus

$$\xi_k \sim z_f^k \text{ in } U(\zeta_f, \varepsilon), \qquad k = 1, 2, \ldots,$$

and since ε is arbitrarily small, we get $\zeta_f \subset (S)^2 \subset \overline{S}$.

IV. *The group Σ^* satisfies the axiom* (c) *of a topological space.*

$$\overline{\overline{S}} = \overline{S}.$$

We have [16])

$$\overline{\overline{S}} = [(S)^1 + (S)^2]^1 + [(S)^1 + (S)^2]^2 =$$
$$= [(S)^1]^1 + [(S)^2]^1 + [(S)^1 + (S)^2]^2.$$

---

[16]) It is easy to see that the closures of the first kind satisfy the distributive law $(S' + S'')^1 = (S')^1 + (S'')^1$ and also the relation $[(S)^1]^1 = (S)^1$.

But

$$[(S)^1]^1 = (S)^1 \subseteq \overline{S}$$

and (according to III)

$$((S)^2) \subset \overline{S}.$$

It is thus sufficient to consider the last term in the above expression of $\overline{\overline{S}}$. If $\zeta_f = \zeta_1, \zeta_2, \ldots, \zeta_k, \ldots$ is an arbitrary cycle of $[(S)^1 + (S)^2]$, than in an (arbitrarily small) $\frac{1}{2}\varepsilon$-neighbourhood $U\left(\zeta_f, \frac{1}{2}\varepsilon\right)$ of ζ_f we have for each k)

$$\zeta_k \sim \xi_f^k \qquad \text{in } U\left(\zeta_f, \frac{1}{2}\varepsilon\right)$$

where ξ_f^k is a cycle of S, or of $(S)^1$, or of $(S)^2$. It is easy to see that in each of these cases there exists a cycle z_f^k of S such that

$$\zeta_k \sim z_f^k \qquad \text{in } U(\zeta_f, \varepsilon)$$

where $U(\zeta_f, \varepsilon)$ is an ε-neighbourhood of ζ_f. It follows that

$$(\overline{S})^2 \subset (S)^2 \subset \overline{S}.$$

V. We have seen above that the group Σ^* satisfies the axioms

$$\text{(a)} \quad \overline{0} = 0,$$
$$\text{(b)} \quad S \subset \overline{S},$$
$$\text{(c)} \quad \overline{\overline{S}} = \overline{S}$$

of a topological space as well as Markoff's conditions and the relations in III, the latter referring to the twofold character of closures. But the axiom

$$\text{(d)} \quad \overline{S'} + \overline{S''} = \overline{S' + S''}$$

expressing the distributive law of closures is not fulfilled, since the inverse of the relation

$$\overline{S'} + \overline{S''} \subset \overline{S' + S''}.$$

is not valid. But this disadvantage is sufficiently outweighed by the following property of the space Σ^*:

If ζ_f is a cycle of the closure $\overline{S' + S''}$ of the sum $S' + S''$, then there exists always a subordinate cycle $\zeta'_f \prec \zeta_f$ of ζ_f contained in the sum of closures $\overline{S}' + \overline{S}''$.

VI. The elements („points") in Σ^* are not closed, and the corresponding separation axiom is not fulfilled. But this again is not a serious disadvantage; the closures of the limit cycles are very simple units, of which all aggregates to be considered henceforth are built.

If $z_f = z_1, z_2, \ldots, z_k, \ldots$ is an arbitrary limit cycle, and t its boundary limit on Γ, then the boundary limit of an arbitrary cycle ζ_f of the closure $\bar{z}_f$ of z_f is contained in t.

If x_f is an α-cycle $\not\sim 0$, then the following relations are valid:

$$(1) \quad L_\beta(x_f) = 0,$$

and

$$(2) \quad (x_f)^1 = \bar{x}_f.$$

Further, if x_f is arbitrary, and $'x_f \prec x_f$ a subordinate cycle of x_f, then

$$(3) \quad '\bar{x}_f = \bar{x}_f,$$

and finally,

$$(4) \quad \bar{x}_f \supset \{\bar{x}_f\}\,[17]).$$

The proof of these relations depends on the following property of α-cycles, which can easily be verified:

We represent x_f in the form $x_f = \sum_{i=1}^{\lambda} x_f^i$, where $x_f^i = (x_k^i)$ is an α-cycle lying on a path J_i. There exists then a limit chain $C^{1(i)} = (C_k^{1(i)})$ on J_i for each i such that

$$C_k^{1(i)} \to x_k^i + x_{k+1}^i \quad (k = 1, 2, \ldots).$$

The proof of (1) is quite elementary; (2) follows easily from (1), and the proofs of (3) and (4) are similar. The proof of (4) is as follows:

Let $U(x_f, \varepsilon)$ be an arbitrary ε-neighbourhood of x_f, and let $\mu_1, \mu_2, \ldots, \mu_k, \ldots$ be an arbitrary sequence of increasing integers $(\mu_k > \mu_{k-1})$. We can define a limit chain $(C_{\mu_k}^1)$,

$$C_{\mu_k}^1 \to x_k + x_{\mu_k} \qquad (k=1, 2, \ldots)$$

such that almost all chains $C_{\mu_k}^1$ are in $U(x_f, \varepsilon)$. The limit cycle (x_{μ_k}) is obviously contained in $(x_f)^1$. If $\zeta_f = \zeta_1, \zeta_2, \ldots, \zeta_k, \ldots$ is an arbitrary cycle of $\bar{x}_f$, we have

$$\zeta_{\mu_k} \sim x_{\mu_k} \sim x_k \qquad \text{in } U(z_f, \varepsilon).$$

---

[17]) If ζ_f is an arbitrary limit cycle of $\bar{x}_f$, then $\{\zeta_f\}$ denotes the system of all subordinate cycles of ζ_f. $\{\bar{x}_f\}$ denotes then the sum $\sum_{\zeta_f \subset \bar{x}_f} \{\zeta_f\}$.

Any subordinate cycle $\zeta'_f \prec \zeta_f$ is obviously among the chains (ζ_{μ_k}), and we get

$$\zeta'_f \subset (x_f)^1 \subset x_f.$$

VII. From the definition of closures in Σ^* it follows at once that

$$(S)^2 \supset \{(S)^2\},$$

i.e. *all subordinate cycles of a cycle of* $(S)^2$ *are contained in* $(S)^2$.

With regard to the closure of the first kind, we have

$$(S)^1_\alpha \subset (S)^2$$

If $\zeta_f = \zeta_1, \zeta_2, \ldots, \zeta_k, \ldots$ is in $(S)^1_\alpha$, and if $U(\zeta_f, \frac{1}{2}\varepsilon)$ is an $\frac{1}{2}\varepsilon$-neighbourhood of ζ_f, then

$$\zeta_k \sim z_k \text{ in } U(\zeta_f, \tfrac{1}{2}\varepsilon)$$

where $z_f = z_1, z_2, \ldots, z_k, \ldots$ is a cycle of S. But for any $k = k_0$ we have

$$\zeta_{k_0} \sim \zeta_{k_0+\nu} \text{ in } U(\zeta_f, \varepsilon), \qquad \nu = 1, 2, \ldots$$

Thus we have

$$\zeta_{k_0} \sim z_{k_0+\nu} \qquad (\nu = 1, 2, \ldots),$$

i.e

$$\zeta_{k_0} \sim z_f$$

and therefore

$$\zeta_f \subset (S)^2.$$

12. A slightly different, but practically equivalent, topological relation in the group Σ^* can be obtained by a more restricted definition of closures. For each aggregate S of limit cycles we set

$$\overline{S} = S + L^1_\alpha(S) + L^2_\beta(S),$$

thus omitting in the closure β-cycles of the first kind and α-cycles of the second kind not in S. The space obtained by means of this definition we denote by $'\Sigma^*$. In $'\Sigma^*$ each β-cycle $\nsim 0$ locally (i.e. in an arbitrary neighbourhood of its boundary limit) is closed.

Let $z_f = z_1, z_2, \ldots, z_\lambda, \ldots$ be a β-cycle $\nsim 0$ in the neighbourhoods $U(z_f, \varepsilon_\nu)$; $\varepsilon_\nu \to 0$ with $\nu \to \infty$. Any cycle $\zeta_f = \zeta_1, \zeta_2, \ldots, \zeta_k, \ldots$ of $\bar{z}_f$ must obviously be a β-cycle, and such that the boundary sets of z_f and ζ_f coincide. According to the above definition of $\overline{S}$ it is sufficient to show that ζ_f is not in $L^2_\beta(S)$. Otherwise we have

$$\zeta_k \sim z_f \qquad \text{for each } k \text{ and each } \nu.$$

Since $z_f \not\sim 0$ in $U(z_f, \varepsilon_\nu)$, there exists for each ν a component h_ν of $U(z_f, \varepsilon_\nu)$ containing points of a set $|z_\lambda|$ for arbitrarily large values of λ. It is therefore possible to construct a decreasing sequence of components

$$h_1 \supset h_2 \supset \ldots \supset h_\nu \supset \ldots$$

containing points of the set $|z_f|$. It follows that a path J in G meets points of $|z_f|$ arbitrarily near Γ, and this contradicts the definition of β-cycles.

§ 4. The prime ends.

13. *α- and β-homology groups.* We consider now the factor groups

$$\mathfrak{A} = A(\Gamma_G) - H^l_\alpha(\Gamma_G)$$

and

$$\mathfrak{B} = B(\Gamma_G) - H_\beta(\Gamma_G).$$

If ζ_f is an α- or β-cycle $\neq 0$ of the group $A(\Gamma_G)$ or $B(\Gamma_G)$, we denote by $C(\zeta_f)$ the class of the group $\mathfrak{A}$ or $\mathfrak{B}$ containing ζ_f. By $\overline{C}(\zeta_f)$ we denote the closure of $C(\zeta_f)$ in Σ^*. The cycle ζ is called *soluble* if any two subordinate cycles $\zeta'_f < \zeta_f$ and $\zeta''_f < \zeta_f$ satisfy the relation

$$\overline{C}(\zeta'_f)\overline{C}(\zeta''_f) \neq 0.$$

If ζ_f is soluble, then all cycles of $C(\zeta_f)$ are soluble. A class of soluble cycles is called a *soluble class*.

A class $C(\zeta_f)$ of $\mathfrak{A}$ or $\mathfrak{B}$ such that for any two cycles $\zeta'_f < \zeta_f$ and $\zeta''_f < \zeta_f$

$$C(\zeta'_f) = C(\zeta''_f)$$

we call an (α- or β-) *oscillator.*

Let x_f be an arbitrary α-cycle. From the definition of local homologies it follows that

$$C(x_f) = (x_f)^1.$$

According to VI, § 3, we have

$$(x_f)^1 = (x_f)^2 = \bar{x}_f.$$

Thus we get the identities

$$(\alpha) \qquad (x_f)^1 = (x_f)^2 = \bar{x}_f = C(x_f) = \overline{C}(x_f).$$

Again according to VI, § 3, we have

$$\bar{x}'_f = \bar{x}''_f$$

where $x'_f \prec x_f$ and $x''_f \prec x_f$ are subordinate cycles of x_f. Thus

$$C(x'_f) = C(x''_f)$$

which implies that x_f is soluble, and that all elements of $\mathfrak{A}$ are oscillators.

14. *The space of convergent limit sequences.* Let

$$P_f = (P_k) = P_1, P_2, \ldots, P_k, \ldots$$

be a convergent limit sequence (see § 1) in Γ_G. It is clear that P_f can be considered as a limit cycle $z_f = (z_k) \bmod 2$, in which each cycle $z_k = P_k$ consists of a single point P_k taken with the coefficient 1. Sequences P_f which are in this sense pure (α- or β-) cycles we call *pure* (α- or β-) *sequences.* All subordinate sequences $P'_f \prec P_f$ satisfy the relation $P'_f \not\sim 0$ in Γ_G.

Let F_α and F_β be the aggregates of all α- and β-sequences in Γ_G. The aggregate

$$F = F_\alpha + F_\beta$$

is contained in the group space Σ^*. Therefore we can consider F as a subspace of Σ^* in which a general topological correspondence is induced by Σ^*; if $S \subset F$ and $\bar{S}$ is the closure of S in Σ^*, then the closure of S in F is $\bar{S}F$. Obviously all topological relations of § 3 apply also to the space F.

Let $C(P_f)$ be the class of the factor group $\mathfrak{A}$ or $\mathfrak{B}$ containing P_f. The aggregate

$$\Delta(P_f) = C(P_f)F$$

we call a *conjugate* (α- or β-) *class* in F. Since the conjugate classes are contained in the classes of the groups $\mathfrak{A}$ or $\mathfrak{B}$, it is clear that the classes Δ are distinct, i.e. if $\Delta' \neq \Delta''$ then $\Delta'\Delta'' = 0$.

A limit sequence P_f is called *soluble* if for any two sub-ordinate sequences $P'_f \prec P_f$ and $P''_f \prec P_f$

$$\bar{\Delta}(P'_f)\,\bar{\Delta}(P''_f) \neq 0.$$

In the case of conjugate α-classes all identities (α) in section **13** are valid in the space F as well. In particular all α-sequences are soluble. In addition we can give a simple interpretation of all aggregates in (α). *For we can always construct* [18]) *a divisor* $\mathfrak{G}_G = P_G$ consisting of a single (accessible) point P such that

$$(\alpha') \qquad f(P_G) = \Delta = \overline{\Delta} = \overline{P}_f.$$

15. *Complexes of conjugate classes in* F. Let P_f be a soluble sequence, and let Δ be the conjugate class containing P_f. As usual we denote by $\{P_f\}$ the system of all subordinate sequences of P_f. Let $\{\Delta\}$ be the system of all conjugate classes which contain at least one sequence in $\{P_f\}$. Let $\{\overline{\Delta}\}$ be the system of all closures of the classes Δ in $\{\Delta\}$. The sum

$$\Delta_1 = \sum_{\Delta \in \{\overline{\Delta}\}} (\overline{\Delta})$$

of all classes $\overline{\Delta}$ in $\{\overline{\Delta}\}$ we call a *complex of the order* 1 [19]).

We state now (without proof) the first fundamental theorem in the theory of prime ends.

Theorem I. *Each limit sequence* P_f *in* G *contains a soluble subordinate sequence* $P'_f \prec P_f$. (Primendentheorie, § 10.) *In particular there exists a subordinate sequence* $P'_f \prec P_f$ *such that the conjugate class* $\Delta \supset P'_f$ *satisfies the relation* $L^1_\alpha(\Delta) \neq 0$ [20]).

We define now complexes Δ_ϱ in F of the order ϱ, where ϱ is an arbitrary finite or transfinite ordinal number.

Let

$$1, 2, \ldots, \ldots, \eta, \ldots$$

be the ordered system of all ordinals $< \varrho$. Assume all complexes Δ_η of all orders $\eta < \varrho$ to have been defined. Let $M(\Delta_\eta)$ be a system of complexes ($\eta < \varrho$ but otherwise arbitrary) and let $M(\overline{\Delta}_\eta)$ be the system of closures of all complexes in $M(\overline{\Delta}_\eta)$. The sum

$$\Delta_\varrho = \sum_{\overline{\Delta}_\eta \in M(\overline{\Delta}_\eta)} (\overline{\Delta}_\eta)$$

of all closures $\overline{\Delta}_\eta$ in $M(\overline{\Delta}_\eta)$ we call *a complex of the order* ϱ if

[18]) We consider a sequence $U(P, \varepsilon_\nu)$ of decreasing spherical neighbourhoods of P. It is easy to show that (for each ν) a component h_ν of $U(P_f, \varepsilon)$ contains P_f. The decreasing sequence $h_1 \supset h_2 \supset \ldots \supset h_\nu \supset \ldots$ defines the divisor P_G, which coincides with P and is, in general, not regular.

[19]) If Δ is a conjugate α-class, the complex Δ_1 coincides with Δ and thus with all aggregates in (α) or (α').

[20]) The second part of Theorem I is proved in my paper „Über die Struktur der Komplexe beliebiger Ordnung in der Theorie der Primenden" [Math. Annalen **106** (1932)].

1. for each decomposition of the system $M(\overline{\Delta}_\eta)$ into two proper parts

$$M(\Delta_\eta) = M'(\Delta_\eta) + M''(\Delta_\eta)$$

there exists a complex $\Delta'_\eta \in M'(\Delta_\eta)$ and a complex $\Delta''_\eta \in M''(\Delta)$ such that

$$\overline{\Delta}_{\eta_1}\overline{\Delta}_{\eta_2} \neq 0.$$

2. there exists no system $N(\Delta_\eta) \supset M(\Delta_\eta)$ of which $M(\Delta_\eta)$ forms a proper part, i.e. if $N(\Delta_\eta)$ satisfies condition 1, then $N(\Delta_\eta) = M(\Delta_\eta)$.

A complex Δ_ϱ of the order ϱ is said to be *saturated* if there exists no complex of an order $> \varrho$ containing Δ_ϱ.

A regular divisor $\mathfrak{E}_G$ is called a *prime end of the order* ϱ if there exists a complex Δ_ϱ of the order ϱ such that

1. $\Delta_\varrho \subset f(\mathfrak{E}_G)$
2. Δ_ϱ contains a subordinate sequence $P'_f \prec P_f$ of each sequence P_f of $f(\mathfrak{E}_G)$.

We can now state the second and third fundamental theorems in the theory of prime ends.

THEOREM II. *There exists a full system of distinct regular divisors $M(E_G)$ in G such that each divisor E_G^ϱ is a prime end (of finite or transfinite order ϱ).* (Primendentheorie, § 11.)

THEOREM III. *The order ϱ of each prime end E_G^ϱ is enumerable,* i.e. *ϱ is an ordinal of the second Cantor class.* (Primendentheorie, § 12.)

16. *Problems.* We have considered above the groups of 0-dimensional cycles. There exists however no simple generalization of these groups to higher dimensions, and it is clear that our theory is essentially dependent on their 0-dimensionality. Further, if we examine the definition of the ends we can make an interesting observation:

This definition depends on the 0-dimensional Betti groups of the region G. For a divisor $\mathfrak{E}_G$ of G is defined by a decreasing sequence $G_1 \supset G_2 \supset \ldots \supset G_k \supset \ldots$ of part regions (§ 1, 2). But if G_k is a part region, its 0-dimensional Betti group $B(G_k)$ is isomorphic with the Betti group $B(G)$. Now we can consider decreasing sequences of part regions

$$G_1^{(r)} \supset G_2^{(r)} \supset \ldots \supset G_k^{(r)} \supset \ldots$$

such that

$$B^r(G_k) \approx B^r(G), \qquad k = 1, 2, \ldots,$$

where B^r denotes the r-th Betti group. The sequence $(G_k^{(r)})$ defines a divisor $\mathfrak{E}_G^{(r)}$ depending on the r-dimensional Betti group of G. More generally we can define divisors of the type $\mathfrak{E}_G^{(0,1,\ldots,r)}$. If we consider now full systems of distinct ends (§ 1, 2) of the type $\mathfrak{E}_G^{(r)}$ or $\mathfrak{E}_G^{(0,1,\ldots,r)}$, then certain of these systems, namely, the full systems of indivisible ends [21]) seem to have a special significance [22]).

Another problem is that of *decomposition* of the group $Z(\mathfrak{E}_G)$ of limit cycles. If $M(E_\varrho^r)$ is the system of prime ends, then $Z(E_G^\varrho)$ is a subgroup of $Z(\Gamma_G)$ for each E_G^ϱ. In what sense can the system of all $Z(E_G^\varrho)$ be considered as a decomposition of $Z(\Gamma_G)$?

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[21]) A full system of distinct ends $M(\mathfrak{E}_G)$ we call *indivisible* if there exists no full system $N(\mathfrak{E}'_G) \neq M(\mathfrak{E}_G)$ such that each $\mathfrak{E}'_G \subset \mathfrak{E}_G$.

[22]) Even in the case where the boundary Γ of G is a complex; the reader may consider the example of a region G bounded by the sum of a sphere and a segment in G with one endpoint on the boundary. From the point of view of connectivity groups the region and its boundary are indistinguishable from a spherical region and a sphere. But the segment is obviously an ideal element of the type $\mathfrak{E}_G^{(r)}$.