EINAR HILLE

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by

Einar Hille

New Haven, Conn.

1. *Introduction.* The present note is an outgrowth of an attempt to present a discussion of abstract spaces of types (G) and (F) in Banach's terminology based upon a topology in which closure rather than distance is the fundamental notion. In (G)-spaces addition is the basic operation, in (F)-spaces we also have the operation of scalar multiplication, and in both cases it is desired to adjust the topology in such a manner that the basic operations become continuous. This can be done in various ways; in the present note we postulate that closure is invariant under the basic operations. The resulting topology is perhaps of some interest also in other connections. In order to show the power of the method I restate and prove some of the theorems in Chapters I and III of Banach's treatise „Théorie des Opérations Linéaires" on this new basis [1]).

2. *Group spaces.* We are concerned with a space S of points $x, y, \ldots$ In the space is defined a single-valued binary operation which associates with every ordered pair of points x, y a third point z, their *sum*. This operation of *addition* is to satisfy the usual postulates for additive groups, viz.

A_0. $x \in S$, $y \in S$ implies $x + y \in S$.

A_1. $(x+y) + z = x + (y+z)$.

A_2. There exists a zero element θ such that

$$x + \theta = \theta + x = x \text{ for every } x \in S.$$

A_3. To every x in S there is an element $-x$ such that

$$x + (-x) = \theta.$$

It is easily shown that the zero element is unique and the

[1]) My attention has been called to a paper by C. Kuratowski, Sur la propriété de Baire dans les groupes métriques [Studia Math. 4 (1933), 38–40], a footnote of which indicates that the author has considered similar ideas.

same applies to the negative of x. Further, $(-x) + x = \theta$. Finally, $x + y = x + z$ or $y + x = z + x$ implies $y = z$.

We suppose that S is a *topological space* in the sense that with every set $X \subset S$ there is associated another set $\overline{X}$, the closure of X, satisfying the usual postulates:

C_1. $\overline{X+Y} = \overline{X} + \overline{Y}$.

C_2. $\overline{X} = X$ if X is void or finite.

C_3. $\overline{\overline{X}} = \overline{X}$.

We can describe these postulates by saying that the forming of closures is an operation on sets of S to sets of S which is additive, idempotent, and reduces to the identity for finite sets.

With the space S we associate the group $\mathfrak{S}$ of rigid motions which leave S invariant. This group is generated by the *reflection* R, the *right-hand translations* T_y and the *left-hand translations* ${}_yT$, $y \in S$. R takes the set $X = \{x\}$ into the set $RX = -X = \{-x\}$. T_y takes X into $T_yX = X + y = \{x+y\}$, whereas ${}_yT$ takes X into ${}_yT\,X = y + X = \{y+x\}$.

We notice the following evident but useful relations:

$x \in X$ implies and is implied by either of the inclusions $-x \in -X$, $x + y \in X + y$, and $y + x \in y + X$.

We now bring in the continuity assumptions on addition by assuming that the closure is invariant under the group $\mathfrak{S}$, or, explicitly,

C_4. $\overline{-X} = -\overline{X}$.

C_5. $\overline{X+y} = \overline{X} + y$, $\overline{y+X} = y + \overline{X}$.

We can also say that the closure operation shall commute with the operations of $\mathfrak{S}$.

Any abstract space satisfying postulates $A_0 - A_3$, $C_4 - C_5$ shall be called a *group space*. It should be noted, however, that C_5 is unnecessarily restrictive for most purposes. Thus in the present note we shall use only the first half of C_5, i.e., the invariance of closure under right-hand translations. Spaces for which only the first half of C_5 is postulated will be referred to as *right-hand group spaces*. We shall give the proofs only for right-hand group spaces, but it will be obvious that the same considerations apply to left-hand group spaces for which the second but not the first half of C_5 is postulated.

This notion of a group space differs somewhat from the space of type (G) of Banach. Such a space is supposed to be metric and complete and continuity of addition is defined by the postulates:

C_4^1. $x_n \to x$ implies $-x_n \to -x$.

C_5^1. $x_n \to x$ and $y_n \to y$ implies $x_n + y_n \to x + y$.

In both cases, however, the continuity postulates express that the topological properties of two sets X and Y are the same if one set can be carried into the other by a transformation of the group $\mathfrak{S}$.

3. *Sub-groups.* S_1 is a sub-group space if $x \epsilon S_1$ and $y \epsilon S_1$ imply that $-x \epsilon S_1$ and $x + y \epsilon S_1$. To S_1 corresponds the sub-group $\mathfrak{S}_1$ of $\mathfrak{S}$ generated by R and by the translations T_y and ${}_yT$ with $y \epsilon S_1$. The following theorem is due to Banach [2]).

THEOREM I. *If a right-hand (left-hand) group space contains a sub-group space which is of the second category and possesses the Baire property, then the sub-group space is both open and closed and if the space is connected the sub-space is the space itself.*

Banach's proof uses the properties of metric spaces. I shall outline a proof valid for group spaces as here defined.

Let $D(X)$ be the set of points in S where a given set X is of the second category locally. In other words, $p \epsilon D(X)$ if there is no neighborhood of p in which X is of the first category. Among the properties of $D(X)$ we shall need the following. [3])

$D(X) = 0$ if and only if X is of the first category. $X - D(X)$ is of the first category. If $D(Y) = 0$ then

$$D(X+Y) = D(X-Y) = D(X).$$

Finally

$$D(X) = \overline{\text{Int}\,[D(X)]} \subset \overline{X}.$$

Consider now a sub-group space H of the second category in S, and the corresponding sub-group $\mathfrak{H}$, and form the set $D(H)$. This set is not void since H is of the second category, and, $D(H)$ being the closure of its own interior, Int $[D(H)]$ is not void. Since H is of the second category at every point of the open set Int $[D(H)]$, we conclude that $H \cdot \text{Int}\,[D(H)] \neq 0$.

Let us now consider how these sets are transformed by the operations of $\mathfrak{H}$. Let $h \epsilon H$, then $-H = H$ and $H + h = H$,

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[2]) loc. cit., pp. 21—22.

[3]) See KURATOWSKI, Topologie I, pp. 43—49 for proofs of these properties.

since H is invariant under $\mathfrak{H}$. But if the group space is right-handed the two sets X and $X+h$ have the same topological properties, whence it follows that $\text{Int}\,(X+h) = \text{Int}\,X + h$. Thus

$$\text{Int}\,[D(H)] = \text{Int}\,[D(-H)] = \text{Int}\,[-D(H)] = -\text{Int}\,[D(H)],$$
$$\text{Int}\,[D(H)] = \text{Int}\,[D(H+h)] = \text{Int}\,[D(H)+h] = \text{Int}\,[D(H)]+h,$$

or $\text{Int}\,[D(H)]$ and, a fortiori, $D(H)$ itself are invariant under the operations R and T_h of $\mathfrak{H}$. But this implies that if one point of H belongs to $\text{Int}\,[D(H)]$, so do all points of H. Hence

$$H \subset \text{Int}\,[D(H)] \subset D(H) \subset \overline{H},$$

and the same conclusion is valid if the space is left-handed.

Suppose now that $p \in \overline{H}$. This implies that every neighborhood of p contains elements of H. Let h be such a point in the open set $\text{Int}\,[D(H)] + p$ which is clearly a neighborhood of p. It follows that $h - p \in \text{Int}\,[D(H)]$. Hence also $p - h \in \text{Int}\,[D(H)]$ and finally $p \in \text{Int}\,[D(H)]$. In other words, $\overline{H} \subset \text{Int}\,[D(H)]$, and consequently

$$\text{Int}\,[D(H)] = \overline{H}.$$

The set on the left is open, the one on the right is closed. Hence $\overline{H}$ is both open and closed. If the space S is supposed to be connected, i.e., not the union of two disjoint closed sets, we conclude that $\overline{H} = S$.

H having the Baire property, every open set contains a point at which either H or $S - H$ is of the first category. $\overline{H} = D(H)$ is an open set and H is of the second category at all of its points. Hence $S - H$ is of the first category at all points of $\overline{H}$. Suppose that $p \in \overline{H} - H \neq 0$, and form the co-set $H + p$. It has no points in common with H, i.e., $H + p \subset S - H$; it is of the second category and has the Baire property since the topological properties of H are unchanged by translations to the right. It follows that $H + p$ is of the second category at all points of $\overline{H+p} = \overline{H} + p$. But $p \in \overline{H}$ and $\overline{H}$ is clearly a sub-group space of S. Hence $\overline{H} + p = \overline{H}$ and $H + p$ is of the second category at all points of $\overline{H}$. But we have just seen that $H + p$, being a sub-set of $S - H$, is of the first category at all points of $\overline{H}$. It follows that $\overline{H} - H = 0$, or H is both open and closed and if S is connected $H = S$.

4. *Continuous transformations.* Let S_1 and S_2 be two group spaces. For the sake of simplicity we use the same notation for

addition, the zero element, and the negative in both spaces. Let $y = U(x)$ be a transformation on S_1 to S_2 whose domain is a sub-group space of S_1 and whose range is a sub-group space of S_2. We suppose $U(x)$ to be *additive*

$$U(x_1+x_2) = U(x_1) + U(x_2).$$

It follows that $U(\theta) = \theta$ and $U(-x) = -U(x)$.

$U(x)$ is said to be *continuous* at $x = x_0$ if $x_0 \in \overline{X}$ implies $U(x_0) \in \overline{U(X)}$. We have the following theorem. [4])

THEOREM 2. *If $U(x)$ is an additive transformation on one group space to another which is continuous at $x = x_0$, then $U(x)$ is continuous throughout its domain.*

PROOF. Suppose that $x \in \overline{X}$. This implies that

$$x_0 \in \overline{X} - x + x_0 = \overline{X - x + x_0}.$$

Hence

$$U(x_0) \in \overline{U(X - x + x_0)} = \overline{U(X) - U(x) + U(x_0)}$$
$$= \overline{U(X)} - U(x) + U(x_0),$$

and

$$U(x) \in \overline{U(X)}.$$

Thus $U(x)$ is continuous everywhere in its domain.

5. *Continuous vector spaces.* In his treatise Banach introduces a topology in linear vector spaces in two different ways obtaining the spaces of types (F) and $(B.)$ These spaces are supposed to be metric and complete. It is possible to introduce a weaker and still useful topology using the methods of this note.

We assume that the space S satisfies all the postulates of § 2. In addition a notion of *scalar multiplication* shall be defined in S. Let Σ be a set of scalars α which form a field. [5])

For every $\alpha \in \Sigma$ and every $x \in S$ the scalar product αx shall have a meaning and be an element of S. The operations are subject to the following additional postulates.

A_4. $x + y = y + x$.

M_1. $\alpha(x+y) = \alpha x + \alpha y$.

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[4]) Banach, loc. cit., p. 23 for spaces of type (G).

[5]) We can get along with weaker assumptions. Σ has to be a ring with unit element and without divisors of zero, but it is not necessary that multiplication be commutative.

M_2. $(\alpha+\beta)x = \alpha x + \beta x$.

M_3. $\alpha(\beta x)=(\alpha\beta)x$.

M_4. $1 \cdot x = x$.

In the last postulate 1 denotes the unit-element of Σ. We note that $(-1)\,x = -x$ and that $0\,x = \theta$ where 0 is the zero element of Σ.

The closure definition must now be so adjusted that scalar multiplication becomes a continuous operation with respect to both α and x. We write Ax for the set of all elements $\{ax\}$ where $\alpha \epsilon A$ and $x \epsilon S$ is fixed. Similarly, αX stands for the set $\{\alpha x\}$ where $\alpha \epsilon \Sigma$ is fixed and $x \epsilon X$. We suppose that a notion of closure is defined in Σ satisfying C_1, C_2, and C_3. Then the additional postulates are:

C_6. $\overline{\alpha X} = \alpha \overline{X}$,

C_7. $\overline{A\,x} = \overline{A}\,x$.

A space satisfying $A_0 - A_4$, $M_1 - M_4$, $C_1 - C_7$ will be called a *continuous vector space.* These spaces may be regarded as generalizations of the (F)-spaces of Banach. An (F)-space is a linear vector space, satisfying $A_1 - A_4$, $M_1 - M_4$, which is a complete metric space, the distance between x and y being subject to the condition $(x, y) = (x-y, \theta)$. Further

$$x_n \to \theta \text{ implies } \alpha x_n \to \theta \text{ for all } \alpha \epsilon \Sigma,$$

$$\alpha_n \to 0 \text{ implies } \alpha_n x \to \theta \text{ for all } x \epsilon S.$$

Finally Σ is the field of real numbers.

THEOREM 3. *If S is a continuous vector space and Σ is connected, and if $H \subset S$ is any linear vector space which is of the second category and has the Baire property, then $H = S$.*

PROOF. Since H obviously is a sub-group space of the group space S, it is sufficient to prove that the connectedness of Σ implies that of S.

Suppose that $S = S_1 + S_2$ is a disjunction of S and let $x \epsilon S_1$, $y \epsilon S_2$, then the set of points $\{\alpha x + (1-\alpha)y\}$, where α ranges over Σ, is a subset L of S containing x and y. Put $L_1 = S_1 L$, $L_2 = S_2 L$. Then $L = L_1 + L_2$ is a disjunction of L. To this corresponds a disjunction of $\Sigma = \Sigma_1 + \Sigma_2$ where Σ_1 contains the values of α which give rise to points in L_1 and Σ_2 those of L_2. But Σ being connected, at least one of these sets is not closed. Suppose Σ_1 is not closed. Then there exists an $\alpha_0 \epsilon \overline{\Sigma}_1 \cdot \Sigma_2$, and the corresponding point $x_0 = \alpha_0 x + (1-\alpha_0)y$ is in L_2. But if A is any open set in Σ containing α_0, then $x=\{\alpha x+(1-\alpha)y\}$, $\alpha \in A$, is a set containing x_0 and open relative to L. But this set contains points both of L_1 and of L_2, so that L_1 is not closed relative to L and a fortiori not in S. In other words, L is a connected set and consequently also S. This completes the proof of the theorem.

We shall finally consider an elementary theorem on continuous transformations from one continuous vector space S_1 to another S_2. Additive and continuous transformations are defined as in § 4. We say that $U(x)$ is *homogeneous* if

$$U(\alpha x) = \alpha\, U(x)$$

for every $\alpha \in \Sigma$.

THEOREM 4. *An additive and continuous transformation from one continuous vector space to another, having the same scalar field Σ, is homogeneous if the sub-field of rational numbers is dense in Σ, and Σ is a regular space.*

PROOF. It is easily seen that $U(rx) = r\,U(x)$ for every rational r. Let R denote the sub-field of rational numbers, and let $\alpha \in \Sigma - R$. Every open set A in Σ which contains α contains points of R. By assumption $\alpha \in \overline{AR}$. $U(x)$ being continuous, we conclude that $U(\alpha x) \in \overline{U(ARx)}$. But $U(ARx) = AR\,U(x)$, and the image space S_2 satisfies C_7. Hence

$$\overline{AR\,U(x)} = \overline{AR}\,U(x).$$

Thus

$$U(\alpha x) \in \overline{AR}\,U(x) \subset \overline{A}\,U(x)$$

for all open sets A containing α. But if Σ is a regular space and $\beta \neq \alpha$, we can find a neighborhood A of α such that β is not in $\overline{A}$. It follows that the only element of Σ which belongs to all sets $\overline{A}$ such that $\alpha \in A$ is α itself. Thus $U(\alpha x) = \alpha\, U(x)$.

Theorem 4 can be generalized further. We may assume, for instance, that Σ is an algebra of finite order over the field of real numbers and that the homogeneity relation holds for the basal units of Σ. If the transformation is additive and continuous, it will then also be homogeneous.

Yale University.

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