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# On the directions of Borel of meromorphic functions of finite order $> \frac{1}{2}$

by

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Paris

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The object of this paper is to prove the following:

THEOREM IV. *$f(z)$ is a meromorphic function of finite order $\varrho > \frac{1}{2}$. Let $V(r)$ be a continuous function satisfying the conditions* (E)[1]. *Suppose that, in an angle $A$ of vertex* 0 *and of measure $\frac{\pi}{k}$ $\left(\frac{1}{2} < k < \varrho\right)$, we have*

$$\overline{\lim_{r \to \infty}} \frac{N(r, a, A)}{V(r)} = \beta > 0,$$

*for a value of $a$.*

*There exists, in an arbitrary angle $A'$ containing $A$ and of vertex* 0, *at least one semi-infinite line $D$ such that for an arbitrary angle $\Omega$ of vertex* 0 *and of bisector $D$, we have*

$$\overline{\lim_{r \to \infty}} \frac{n(r, \pi, \Omega)}{V(r)} > 0,$$

*for all elements $\pi$, except at the most two, in the family $K(\eta, f)$, where $K(\eta, f)$ denotes the aggregate of all the distinct constants and the meromorphic functions $\pi(z)$ satisfying*

$$T(r, \pi) < \eta(r) V(r), \quad r > r_0(\pi), \quad \lim_{r \to \infty} \eta(r) V(r) = \infty,$$

*where $\eta(r)$ is an infinitesimal.*

In reality, the foregoing theorem is a complement to the theorem due to Valiron as follows:

THEOREM of Valiron[2]. *$f(z)$ is a meromorphic function of*

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*) Research fellow of the China Foundation for the Promotion of Education and Culture.

[1]) See p. [3] . . .

[2]) Acta Math. **47** (1926), 137—138.

*finite order* $\varrho > \frac{1}{2}$. *Let* $V(r)$ *be a continuous function satisfying the conditions* (E). *Suppose that, in an angle* $A$ *of vertex* 0 *and of measure* $\frac{\pi}{k}$ $\left(\frac{1}{2} < k < \varrho\right)$, *we have*

$$\overline{\lim_{r \to \infty}} \frac{N(r, a, A)}{V(r)} = \beta > 0$$

*for a value of* $a$, *and let* $A'$ *be an arbitrary angle containing* $A$ *and of vertex* 0.

*There exist three positive finite numbers* $h$, $h_1$, $h_2$ *and an infinite sequence of positive numbers* $(R_m)$, *such that*

$$\lim_{m \to \infty} \frac{\log T(R_m, f)}{\log R_m} = \varrho, \quad R_{m+1} > hR_m$$

*in relation with the following property: in the region* $\Delta_m$, *being the common region of* $A'$ *and the circular ring* $R_m < |z| < hR_m$, *the function* $f(z)$ *takes* $kT(R_m, f)$ *times all values* $c$ *except at the most two provided that* $m > m_\varrho$ *where* $h_1 < k < h_2$.

In the whole paper, $n(r, \varphi, \Omega)$ denotes the number of zeros of the function $f(z) - \varphi(z)$ in the common part of the region $\Omega$ and the circle $|z| \leqq r$; and $N(r, \varphi, \Omega)$ denotes the corresponding density

$$\int_0^r \frac{n(r, \varphi, \Omega) - n(0, \varphi, \Omega)}{r} dr + n(0, \varphi, \Omega) \log r.$$

**1.** The present work is based principally upon the following THEOREM of Rauch [3]. *Let* $f(z)$, $P(z)$, $Q(z)$, $R(z)$ *be four distinct meromorphic functions in a region* $(\Delta)$ *and* $(D)$ *a region contained in* $(\Delta)$. *Divide the region* $(D)$ *into* $p$ *partial regions* $(D_i)$ *and let* $(\Gamma_i)$ *be the circle concentric to the smallest circle containing* $(D_i)$ *and radius* 20 *times larger. Suppose that the following conditions are satisfied:*

$(C_1)$ *The number of the zeros of* $f(z) - c$ *in* $(D)$ *is superior to* $M$ *for all values of* $c$ *in a certain circle of radius* $\frac{1}{2}$ *on the Riemann sphere.*

$$(C_2) \quad \frac{1}{\text{area of } (\Gamma_i)} \iint_{(\Gamma_i)} \overset{+}{\log} \left( |P(z) + Q(z)| + \frac{1}{|P(z) - Q(z)|} + \frac{1}{|Q(z) - R(z)|} + \frac{1}{|R(z) - P(z)|} \right) d\sigma < \frac{M}{p}.$$

[3] Journ. de Math. (9) 12 (1933), 133

$$(C_3)\quad \left\{ \begin{array}{l} \textit{The number of the zeros and poles of the functions } P(z), \\ Q(z),\ R(z),\ P(z)-Q(z),\ Q(z)-R(z),\ R(z)-P(z) \textit{ in} \\ \textit{in } (\Delta) \textit{ is inferior to } \varepsilon\frac{M}{p}, \textit{ where } \varepsilon \textit{ is a numerical constant} \\ <\alpha<1. \end{array} \right.$$

*Then there exists at least one circle* $(\Gamma_i)$ *in which the number of the zeros of the function* $f(z)-\pi(z)$ *is superior to* $c_1\frac{M}{p}$ *for at least one function* $\pi(z)$ *out of the three* $P(z)$, $Q(z)$, $R(z)$, *where* $c_1$ *is a numerical constant.*

*This theorem is valid when* $\frac{M}{p}$ *is superior to a numerical constant.*

It is to be remarked that, for the condition $(C_3)$, the numbers of the zeros and the poles of the constans 0 and $\infty$ are counted as zero.

2. Let $f(z)$ be a meromorphic function of positive finite order $\varrho$. We are going to show that there exist continuous functions $V(r)$ adjoined to the characteristic function $T(r,f)$ of $f(z)$, satisfying the following conditions:

$$(E)\quad \left\{ \begin{array}{l} \lim\limits_{r\to\infty} \dfrac{V(hr)}{V(r)} = h^{\varrho}, \quad \text{for every } h>0, \\ \lim\limits_{r\to\infty} \dfrac{\log V(r)}{\log r} = \varrho, \\ T(r,f) \leqq V(r), \quad r > r_0, \\ \overline{\lim\limits_{r\to\infty}} \dfrac{T(r,f)}{V(r)} = 1. \end{array} \right.$$

In fact, after Valiron [4]), there exist continuous functions $\varrho(r)$ differentiable in adjacent closed intervals of which the end points are finite in number at finite distance, satisfying the following conditions:

$$(1)\quad \begin{array}{l} \lim\limits_{r\to\infty} \varrho(r) = \varrho, \quad \lim\limits_{r\to\infty} (r\varrho'(r)\log r) = 0, \\ T(r,f) \leqq r^{\varrho(r)}, \quad \overline{\lim\limits_{r\to\infty}} \dfrac{T(r,f)}{r^{\varrho(r)}} = 1. \end{array}$$

$\varrho(r)$ is known as a proximate order of $f(z)$. If we put $V(r) = r^{\varrho(r)}$, evidently, the last three conditions in (E) are satisfied. Let us show that it satisfies also the first condition.

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[4]) C. R. 194 (1932), 1305—1306.

The case $h = 1$ is trivial, and the case $h < 1$ follows immediately from the case $h > 1$. Consider this case. Put

$$r = e^x, \quad V(r) = e^{x\omega(x)}, \quad \omega(x) = \varrho(e^x),$$

then

$$\lim_{x \to \infty} \omega(x) = \varrho,$$

and from (1) the first condition in (E) is equivalent to

$$\lim_{x \to \infty} [(H + x)\omega(H + x) - x\omega(x)] = \varrho H, \quad H = \log h. \tag{2}$$

But

$$\lim_{x \to \infty} [(H + x)\omega(H + x) - x\omega(x)] =$$

$$= \lim_{x \to \infty} \int_x^{x+H} (x\omega(x))' dx = \lim_{x \to \infty} \int_x^{x+H} x\omega'(x)dx + H\varrho,$$

and from (1)

$$\lim_{x \to \infty} \int_x^{x+H} x\omega'(x)dx = 0,$$

hence we have (2) and we see that the function $V(r) = r^{\varrho(r)}$ satisfies all the conditions (E).

**3.** In this section, we shall prove the theorem as follows which is analogous to the foregoing theorem of Valiron and also of fundamental importance:

THEOREM I. *$f(z)$ is a meromorphic function of finite order $\varrho > \frac{1}{2}$. Let $V(r)$ be a continuous function satisfying the conditions* (E). *Suppose that for an angle $A$ of vertex* 0 *and of measure $\frac{\pi}{k}$ $\left(\frac{1}{2} < k < \varrho\right)$, we have*

$$\overline{\lim_{r \to \infty}} \frac{N(r, a, A)}{V(r)} = \beta > 0 \tag{3}$$

*for a value of $a$.*

*There exists at least one sequence $(R_n)$ of values of $r$, $\lim_{n \to \infty} R_n = \infty$, such that*

$$n[S(n, B), f = c] > K(\varrho, k')V(R_n) \tag{4}$$

*for all values of $c$ in a certain circle $(C_n)$ of radius $\frac{1}{2}$ on the Riemann sphere, where $n[S(n, B), f = c]$ denotes the number of the zeros of $f(z) - c$ in the common part $S(n, B)$ of the ring $\frac{R_n}{1+S} \leqq |z| \leqq R_n$*

*and an arbitrary angle $B$ containing $A$, of vertex 0 and of measure* $\frac{\pi}{k'}$ $\left(\frac{1}{2} < k' < k\right)$, *and $s$ is a suitably chosen positive constant.*

This theorem is established in modifying a method due to Valiron as follows.

From the second fundamental theorem of R. Nevanlinna, we see easily that for every value of a except at the most two, there exists at least one angle $A$ of vertex 0 and of measure $\frac{\pi}{k}$ $\left(\frac{1}{2} < k < \varrho\right)$, such that (3) is satisfied. Hence, that hypothesis is possible.

It follows from (3),

$$n(r, a, A) < (1 + \varepsilon)\varrho e V(r) \tag{5}$$

for $r > r_0$, and

$$n(r, a, A) > \beta H(\varrho)V(r) \tag{6}$$

for a sequence of values of $r$ tending to infinity.

Let $B$ be an arbitrary angle containing $A$, of vertex 0 and of measure $\frac{\pi}{k}$ $\left(\frac{1}{2} < k < \varrho\right)$. Without loss of generality, we may suppose that the bisectors of $A$ and $B$ coincide with the positive real axis. Make the transformation $Z = z^{-k'}$, where $Z$ is real when $z$ is so, and then the transformation $Z = 1 - z$, so that the function $f(z)$ in the angle $B$ corresponds to a meromorphic function $F(z)$ in the unit circle $|z| < 1$. From (6) we have

$$n(r, F = a) > \beta H_1(\varrho)V(r) \tag{7}$$

for a sequence of values of $r$ tending to 1. Hence

$$\begin{aligned} T\left(r, \frac{1}{F-a}\right) \geqq N(r, a) > (1-r)n(2r-1, F=a) > \\ > \beta H_2(\varrho)(1-r)V\left[\left(\frac{1}{1-r}\right)^{\frac{1}{k'}}\right] \end{aligned} \tag{8}$$

for a sequence of values of $r$ tending to 1. But

$$T\left(r, \frac{1}{F-a}\right) = T(r, F) + h(r, a)$$

$h(r, a)$ being bounded when $a$ is fixed, therefore

$$T(r, F) > \beta H_3(\varrho)(1-r)V\left[\left(\frac{1}{1-r}\right)^{\frac{1}{k'}}\right] \tag{9}$$

for a sequence of values of $r$ tending to 1. On the other hand, it is known that

$$(10) \qquad T(r, F) < H_4(\varrho)\,(1-r)V\left[\left(\frac{1}{1-r}\right)^{\frac{1}{k'}}\right]$$ [5]

for $r > r_0$, which shows that the hypothesis is essential and that $F(z)$ is of finite order.

The foregoing calculations are due to Valiron.

Now, from another well known theorem of Valiron [6], we deduce that there exists at least one circle $C(r)$ of radius $\frac{1}{2}$ on the Riemann sphere, such that for all values $c$ in that circle, we have

$$(11) \qquad N(r, F = c) > \frac{\beta}{4} H_3(\varrho)(1-r)V\left[\left(\frac{1}{1-r}\right)^{\frac{1}{k'}}\right]$$

for all values of $r > r_0^1$ in the sequence for which (9) is satisfied, and that for all values of $R > R_0$,

$$(12) \qquad T\left(R, \frac{1}{F-c}\right) < T(R, F) + C(F),$$

$$(13) \qquad T\left(R, \frac{1}{f-c}\right) < T(R, f) + C_1(f)$$

$C(F)$, $C_1(f)$ being constants depending only upon $F(z)$ and $f(z)$ respectively.

From (9), (10), (12), we have for all values of $c$ in $C(r)$

$$(14) \qquad n(r, F = c) > H(\varrho, k')V(r)$$

for at least one sequence of values of $r$ tending toward $\infty$. In passing back to $f(z)$, it follows from (14), for all values of $c$ in $(C_n)$,

$$(15) \qquad n(R_n, c, B) > H_1(\varrho, k')V(R_n)$$

for at least one sequence $(R_n)$ tending toward infinity with $n$.

Moreover, from (13) we have

$$(16) \qquad n\left(\frac{R_n}{1+S}, c, B\right) < \mu\left(\frac{2}{1+S}\right)^{\varrho} V(R_n) \quad \left(\frac{R_n}{1+S} > R_0\right)$$

for all values of $c$ in $(C)$.

It follows from (15) and (16), by choosing $s$ such that $H_1(\varrho, k') - \mu\left(\frac{2}{1+S}\right)^{\varrho} > K(\varrho, k')$,

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[5] Valiron, l. c. [2]), 136.

[6] Valiron, l. c. [2]), 123—124.

$$n(S(n, B), f = c) > K(\varrho, k')V(R_n),$$

where $n(S(n, B), f - c)$ denotes the number of zeros of the function $f(z) - c$ in the region $S(n, B)$ common to $B$ and the ring $\frac{R_n}{1+S} < |z| \leqq R_n$, and for all values of $c$ in $(C_n)$.

Thus the foregoing theorem is proved.

**4.** For the sake of convenience, we shall employ the following notation. Let $T^*(r, \varphi)$ be $T(r, \varphi)$ if $\varphi(z) \not\equiv \infty$, and be 0 if $\varphi(z) \equiv \infty$. Let $C^*(\varphi)$ be $C(\varphi) = T\left(r, \frac{1}{\varphi}\right) - T(r, \varphi)$ if $\varphi(z) \not\equiv 0, \infty$, and be 0 if $\varphi(z) \equiv 0, \infty$. $T(r, \varphi)$ is the characteristic function of $\varphi(z)$.

THEOREM II. *$f(z)$ is a meromorphic function of finite order $\varrho > \frac{1}{2}$. Let $V(r)$ be a continuous function satisfying the conditions* (E). *Suppose that for an angle $A$ of vertex 0 and of measure $\frac{\pi}{k}\left(\frac{1}{2} < k < \varrho\right)$, we have*

$$\overline{\lim_{r\to\infty}} \frac{N(r, a, A)}{V(r)} = \beta > 0$$

*for a value of $a$, and let $(R_n)$ be the sequence of values of $r$ in Theorem I.*

*There exists, in an arbitrary angle $A'$ containing $A$ and of vertex 0, at least one sequence of circles $\Gamma(n)$*

$$|z - x(n)| = \alpha |x(n)|, \quad \frac{R_n}{1+S} < |x(n)| < R_n, \tag{17}$$

*such that if $P(z)$, $Q(z)$, $R(z)$ are any three distinct meromorphic functions satisfying the conditions*

$$\begin{cases} T^*[(1+\alpha)R_n, \varphi] < \alpha^4 V(R_n), \quad \varphi(z) \equiv P(z),\ Q(z),\ R(z); \\ C^*(\psi) > -\alpha^4 V(R_n), \\ \psi(z) \equiv P(z), Q(z), R(z), P(z)-Q(z), Q(z)-R(z), R(z)-P(z), \end{cases} \tag{18}$$

*then we have*

$$n(\Gamma(n), f - \pi) > a^3 V(R_n) \tag{19}$$

*for at least one function out of the three $P(z)$, $Q(z)$, $R(z)$, where $n(\Gamma(n), f - \pi)$ denotes the number of the zeros of $f(z) - \pi(z)$ in $\Gamma(n)$.*

*This theorem is valid when* $\frac{1}{a}$ *and* $\alpha^4 V(r)$ *are greater than a certain constant.*

Let $B$ be an angle having the properties given in Theorem I, and contained in $A'$.

We are going to apply the theorem of Rauch. Let the region $S(n, B)$ defined in theorem I be the region $(D)$ and let $M = K(\varrho, k')V(R_n)$. Then by Theorem I, the condition $(C_1)$ in the Theorem of Rauch is satisfied. Divide the angle $B$ into equal sectors of measure $\frac{\pi}{\alpha_1 k'}$ by semi-infinite lines issued from the origin and describe the circles

$$|z| = \frac{R_n}{1+S}(1+\alpha_1)^i \qquad \left(\frac{R_n}{1+S}(1+\alpha_1)^q = R_n;\ i = 1,2,\ldots,q\right).$$

The region $(D)$ is thus divided into

$$p = c(k')\frac{\log(1+S)}{\alpha_1^2}$$

similar curvilinear rectangles $D_i(n)$, where $\alpha_1$ is sufficiently small and $c(k')$ is a constant depending only upon $k'$. Let $D_i(n)$ be the partial regions $(D_i)$ in the theorem of Rauch and $\Gamma_i(n)$ the corresponding circles $\Gamma_i$. Then the circles $\Gamma_i(n)$ are contained in the ring

$$(1 - 15\alpha_i)\frac{R_n}{1+S} < |z| < (1 + 15\alpha_1)R_n$$

which is taken for each $n$ as the region $(\Delta)$.

Let $\varrho_i(n)$ be the modulus of the center of $\Gamma_i(n)$, then its radius is $k_i(n)\alpha_1\varrho_i(n)$, $k_i(n)$ lying between two numerical constants $h_1$ and $h_2$.

In modifying suitably a method due to Rauch [7]), we see that the conditions $(C_2)$, $(C_3)$ in his theorem given above are satisfied if the conditions (18) have been imposed, where $\alpha$ is taken to be the largest of $16\alpha_1$ and $h_2\alpha_1$, and when $\frac{1}{\alpha}$ and $\alpha^4 V(R_n)$ are greater than a certain constant.

Hence, by the theorem of Rauch, for each large integer $n$, there exists at least one circle defined by (17), such that (19) is satisfied.

It is evident that when $\alpha$ is less than a certain constant, all the circles $\Gamma(n)$ are contained in $A'$.

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[7]) RAUCH, l. c., 133—138.

**5.** Let $H(\alpha, f)$ be the family of meromorphic functions $\pi(z)$ satisfying the condition

$$T^*[(1+\alpha)r, \pi] < \alpha^4 V(r), \qquad r > r_0(\pi).$$

Consider a certain infinite sequence of circles $\Gamma(n)$ in theorem II. We are going to show that for all functions $\pi(z)$ of the family $H(\alpha, f)$, except at the most two, we have

$$n(\Gamma(n'), f - \pi) > \alpha^3 V(R_{n'}), \qquad n' > n_0(\pi).$$

Suppose that there are two functions $P(z)$, $Q(z)$ in the family $H(\alpha, f)$, such that

$$n(\Gamma(n'), f - P) \leqq \alpha^3 V(R_{n'}),$$
$$n(\Gamma(n'), f - Q) \leqq \alpha^3 V(R_{n'})$$

for an infinite sequence of values of $n'$. Let $R(z)$ be a function in the family $H(\alpha, f)$ distinct from $P(z)$ and $Q(z)$. Evidently when $n' > n_0(R)$ the conditions (18) are satisfied, hence by theorem II, we have

$$n(\Gamma(n'), f = R) > \alpha^3 V(R_{n'}), \qquad n' > n_0(R).$$

The statement is therefore proved and we have the following

THEOREM III. *$f(z)$ is a meromorphic function of finite order $\varrho > \frac{1}{2}$. Let $V(r)$ be a continuous function satisfying the conditions* (E). *Suppose that in an angle $A$ of vertex* 0 *and of measure $\frac{\pi}{k}$ $\left(\frac{1}{2} < k < \varrho\right)$, we have*

$$\overline{\lim_{r\to\infty}} \frac{N(r, a, A)}{V(r)} = \beta > 0$$

*for a value of $a$. Let $H(\alpha, f)$ be the family of meromorphic functions satisfying the condition*

$$T^*[(1+\alpha)r, \pi] < \alpha^4 V(r), \qquad r > r_0(\pi).$$

*There exists, in an arbitrary angle $A'$ containing $A$ and of vertex* 0, *at least one sequence of circles $\Gamma(n)$ defined by* (17) *such that*

$$n(\Gamma(n), f - \pi) > \alpha^3 V(r), \qquad n > n_0(\pi), \tag{20}$$

*for all functions $\pi(z)$ of the family $H(\alpha, f)$ except at the most two.*

*This theorem is valid when $\frac{1}{\alpha}$ and $\alpha^4 V(r)$ are greater than a certain constant.*

**6.** Let $A_\alpha$ be the smallest of the angles contained in $A'$ and of vertex 0 in which there are an infinite number of the circles $\Gamma(n)$ in theorem III. Then

$$\overline{\lim_{r\to\infty}} \frac{n(r, \pi, A_\alpha)}{V(r)} \geqq \frac{\alpha^3}{2(1+\alpha)^\varrho} \tag{21}$$

from (20).

Let $(\alpha_n)$ be a sequence of values of $\alpha$, tending to 0 with $\frac{1}{n}$. Let $(D_{\alpha_n})$ be the bisector of $A_{\alpha_n}$ and $D$ a limit-line of the semi-lines $D_{\alpha_n}$. An arbitrary angle $\Omega$ of vertex 0 and of bisector $D$ contains then an infinity of the angles $A_{\alpha_n}$.

Let $K(r, f)$ be the family of meromorphic functions $\pi(z)$ satisfying the condition

$$T^*(r, \pi) \leqq \eta(r)V(r), \qquad r > r_0(\pi), \quad \lim \eta(r)V(r) = \infty,$$

where $\eta(r)$ is an infinitesimal. It is evident that the family $K(\eta, f)$ is contained in the family $H(\alpha, f)$ for every fixed value of $\alpha$. Hence from (21) we have

$$\overline{\lim_{r\to\infty}} \frac{n(r, \pi, \Omega)}{V(r)} > 0$$

for all elements $\pi(z)$ in the family $K(\eta, f)$ except at the most two.

It is also evident that the family $K(\eta, f)$ is the aggregate of all the distinct constants, and the meromorphic functions (non-degenerated to constants) $\pi(z)$ satisfying

$$T(r, \pi) \leqq \eta(r)V(r), \qquad r \geqq r_0(\pi).$$

We have therefore the following

THEOREM IV.[8]) *$f(z)$ is a meromorphic function of finite order $\varrho > \frac{1}{2}$. Let $V(r)$ be a continuous function satisfying the conditions* (E). *Suppose that, in an angle $A$ of vertex 0 and of measure $\frac{\pi}{k}$ $\left(\frac{1}{2} < k < \varrho\right)$, we have*

$$\overline{\lim_{r\to\infty}} \frac{N(r, a, A)}{V(r)} = \beta > 0,$$

*for a value of $a$.*

*There exists, in an arbitrary angle $A'$ containing $A$ and of vertex 0 at least one semi-line $(D)$ issued from 0, such that for an arbitrary angle $\Omega$ of vertex 0 and of bisector $D$, we have*

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[8]) This theorem has been stated in a Note in Comptes Rendus **206** (1938), 811—812.

$$\overline{\lim_{r\to\infty}} \frac{n(r,\pi,\Omega)}{V(r)} > 0$$

*for all elements* $\pi$ *of the family* $K(r,f)$ *excepi at the most two.*

It is to be remarked that the foregoing theorem and the theorem IX [9]) in the Thèse of Rauch do not contain each other and that the family $K(\alpha,f)$ in the latter must be stated analogously to the family $K(\eta,f)$ in our theorem IV.

Finally, the writer wishes to thank Prof. Valiron for his useful criticisms.

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[9]) Rauch, l. c. [3]), 157.