Compositio Mathematica

J. von Neumann

On infinite direct products

Compositio Mathematica, tome 6 (1939), p. 1-77
<http://www.numdam.org/item?id=CM_1939__6__1_0>

On infinite direct products

by

J. v. Neumann

Princeton, N.J.

Introduction.

1. In the theory of *vector spaces* two important general operations on such spaces are these: Formation of *direct sums* and formation of *direct products.* It is convenient to recall the definitions of these notions.

A (complex) *vector space* $\mathfrak{V}$ is a set of elements $f, g, \ldots,$ in which the operations $f + g$ and (for every complex number a) af are defined, and possess the usual properties (commutativity and associativity for $f + g$, associativity for af, both distributivities, the existence of 0, and $1f = f$, $0f = 0$) [1]). If a finite subset $f_1, \ldots, f_n$ of $\mathfrak{V}$ is such, that every element f of $\mathfrak{V}$ can be written as

$$f = a_1 f_1 + \ldots + a_n f_n \quad (a_1, \ldots, a_n \text{ complex numbers})$$

in one and only one way, then $f_1, \ldots, f_n$ form a *finite basis of* $\mathfrak{V}$.

If $\mathfrak{V}_1, \mathfrak{V}_2$ are two vector spaces with finite bases $f_1^1, \ldots, f_n^1$ and $f_1^2, \ldots, f_m^2$, then it is well known, how two vector spaces $\mathfrak{W}'$ and $\mathfrak{W}''$ can be defined, which have — if proper notations are used — bases formed by the elements $f_1^1, \ldots, f_n^1, f_1^2, \ldots, f_m^2$ resp., by the symbolic expressions $f_i^1 \otimes f_j^2$, $i = 1, \ldots, n, j = 1, \ldots, m$. $\mathfrak{W}'$ is the *direct sum* $\mathfrak{V}_1 \oplus \mathfrak{V}_2$, $\mathfrak{W}''$ is the *direct product* $\mathfrak{V}_1 \otimes \mathfrak{V}_2$.

$\mathfrak{V}_1 \oplus \mathfrak{V}_2$ can be formed without reference to finite bases of $\mathfrak{V}_1, \mathfrak{V}_2$: As the set of all pairs $\{f^1, f^2\}$, f^1 in $\mathfrak{V}_1$, f^2 in $\mathfrak{V}_2$, with the definitions

$$\{f^1, f^2\} + \{g^1, g^2\} = \{f^1 + g^1, f^2 + g^2\}$$
$$a\{f^1, f^2\} = \{af^1, af^2\}.$$

For $\mathfrak{V}_1 \otimes \mathfrak{V}_2$ such a general procedure would be hampered by

---

[1]) In the abstract-algebraical terminology: $\mathfrak{V}$ is an Abelian group (a modulus) with the complex numbers as operators.

many difficulties; for Hilbert-spaces a „basisless" procedure has been given in (7), pp. 127—133 (cf. particularly § 2.2, loc. cit.).

Returning to the original $\mathfrak{V}_1 \oplus \mathfrak{V}_2$ and $\mathfrak{V}_1 \otimes \mathfrak{V}_2$ it is clear, that these operations are both commutative and associative, so they permit us to define arbitrary finite direct sums $\mathfrak{V}_1 \oplus \ldots \oplus \mathfrak{V}_k$ and direct products $\mathfrak{V}_1 \otimes \ldots \otimes \mathfrak{V}_k$ $(k = 1, 2, \ldots)$.

2. These operations may be studied for Hilbert spaces $\mathfrak{V}_1, \ldots, \mathfrak{V}_k$ in particular, or somewhat more generally, for unitary spaces (cf. § 1.1). Now the application of the operation $\oplus$ has turned out to be a powerful tool in dealing with Hilbert spaces. Two examples may be quoted: The theory of closed and adjoint operators, as dealt with in (10) [2]; and the theory of operator rings, (9), where the fundamental Theorem 5 (pp. 393—396, loc. cit.) is established with its help [3]. Indications of similar possibilities for $\otimes$ exist. It seems reasonable, therefore, to study the effects of $\oplus$ and $\otimes$ on unitary spaces. By restricting ourselves to unitary spaces, we avoid all difficulties connected with the possible non-existence of bases, which are extremely serious in general vector spaces.

But if such a detailed study is undertaken, then the generalization to *infinite* direct sums and products, $\mathfrak{V}_1 \oplus \mathfrak{V}_2 \oplus \ldots$ and $\mathfrak{V}_1 \otimes \mathfrak{V}_2 \otimes \ldots$, seems to be desirable, too.

3. We say first a few words about infinite direct sums, although they will not be the subject of this paper [4]). It turns out, that $\mathfrak{V}_1 \oplus \mathfrak{V}_2 \oplus \ldots$ is not the widest possible generalization. If x is a parameter which varies over a space S in which a Lebesgue-measure $\mu(T)$ is defined [5]), and if for every x of S a unitary space $\mathfrak{H}_x$ is given, then a *direct integral* $\int_S \oplus\, \mathfrak{H}_x dx$ can be defined, which is a unitary space again. (The first example of [5]) leads then back to $\mathfrak{H}_1 \oplus \mathfrak{H}_2 \oplus \ldots$.)

---

[2]) The space $\mathcal{H}$ defined on p. 299, loc. cit., which is the basis of the entire investigation, is clearly our $\mathfrak{H} \oplus \mathfrak{H}$.

[3]) The Hilbert space $\overline{\mathfrak{H}}$ used there is clearly our $\mathfrak{H} \oplus \ldots \oplus \mathfrak{H}$ (k addends).

[4]) They will be dealt with exhaustively in another publication, which is to appear soon.

[5]) For instance: S the set of all positive integers, $\mu(T) =$ Number of elements of T. Or: S the set of all real numbers, $\mu(T)$ some Lebesgue-Stieltjes-measure $\int_T d\varphi(x)$ ($\varphi(x)$ a monotonous function).

And this generalization seems to be a very natural and convenient one, because it permits various interesting applications. Thus, with its help, the author succeeded in characterising all operator rings by means of those, which F. J. Murray and the author called „factors", and for which an extensive quantitative theory exists. (Cf. (7) concerning the „factors".) These investigations permit us to extend the reduction theory of unitary group-representations to all unitary spaces (including Hilbert spaces), and to connect it with the above mentioned theory of „factors". (This will be carried out in the publication mentioned in footnote [4]) above.)

4. Let us now return to direct products. As mentioned in § 2, finite direct products $\mathfrak{H}_1 \otimes \ldots \otimes \mathfrak{H}_n$ (for unitary spaces $\mathfrak{H}_1, \ldots, \mathfrak{H}_n$) have been defined in (7), as a tool for the theory of „factors". We will extend this to infinite ones, $\mathfrak{H}_1 \otimes \mathfrak{H}_2 \otimes \ldots$, and it will appear, that again a further generalisation is possible, but in a totally different sense than for the infinite direct sums (resp. direct integrals) discussed in § 3.

This generalisation consists in permitting direct products with any number of factors: If I is an arbitrary set, and if for every $\alpha \epsilon I$ a unitary space $\mathfrak{H}_\alpha$ is given, then the direct product $\Pi\otimes_{\alpha \epsilon I} \mathfrak{H}_\alpha$ can be formed [6]). Our main reason for considering all these $\Pi\otimes_{\alpha \epsilon I} \mathfrak{H}_\alpha$ is, that while the theory of the enumerably infinite direct products $\mathfrak{H}_1 \otimes \mathfrak{H}_2 \otimes \ldots$ presents essentially new features, when compared with that of the finite $\mathfrak{H}_1 \otimes \ldots \otimes \mathfrak{H}_n$, the passage from $\mathfrak{H}_1 \otimes \mathfrak{H}_2 \otimes \ldots$ to the general $\Pi\otimes_{\alpha \epsilon I} \mathfrak{H}_\alpha$ presents no further difficulties.

It seems worth pointing out, that while the generalisations of the direct sum point toward the theory of Lebesgue-Stieltjes-measure, the generalisations of the direct product lead to higher set-theoretical powers (G. Cantor's „Alephs"), and to no measure-problems at all.

5. The discussion of infinite direct products $\Pi\otimes_{\alpha \epsilon I} \mathfrak{H}_\alpha$ necessitates a careful analysis of infinite numerical products $\Pi_{\alpha \epsilon I} z_\alpha$ (the z_α are complex numbers). As this is done in Chapter 2 in considerable detail, we need not speak about it now. Three remarks, however, seem to be appropriate (all of which will be discussed more fully in the paper):

---

[6]) In this notation $\mathfrak{H}_1 \otimes \mathfrak{H}_2 \otimes \ldots$ would be $\Pi\otimes_{n \epsilon (1, 2, \ldots)} \mathfrak{H}_n$.

First: Infinite direct products $\Pi\otimes_{\alpha\in I}\mathfrak{H}_\alpha$ differ essentially from the finite ones in this, that they „split up" into „incomplete" direct products $\Pi\otimes^{\mathfrak{C}}_{\alpha\in I}\mathfrak{H}_\alpha$. The importance of this phenomenon is particularly put in evidence by Theorems I, V, VI, and X.

Second: The generalised notion of convergence („quasi-convergence") of $\Pi_{\alpha\in I} z_\alpha$, as described in § 2.5, could be avoided if we restricted ourselves ab initio to the „incomplete" direct products $\Pi\otimes^{\mathfrak{C}}_{\alpha\in I}\mathfrak{H}_\alpha$ (cf. § 4.1). This would have another advantage, too: If all $\mathfrak{H}_\alpha$ are separable, and I finite or enumerably infinite, then the $\Pi\otimes^{\mathfrak{C}}_{\alpha\in I}\mathfrak{H}_\alpha$ are again separable, while $\Pi\otimes_{\alpha\in I}\mathfrak{H}_\alpha$ is not. (Cf. Theorem V and Lemma 6.4.1.) Thus $\Pi\otimes^{\mathfrak{C}}_{\alpha\in I}\mathfrak{H}_\alpha$ would permit us to restrict ourselves to (finite dimensional) Euclidean and to Hilbert spaces, while $\Pi\otimes_{\alpha\in I}\mathfrak{H}_\alpha$ necessitates the use of general unitary spaces.

But since no real new difficulties arise, and since $\Pi\otimes_{\alpha\in I}\mathfrak{H}_\alpha$ seems to be a more natural basis for our considerations than $\Pi\otimes^{\mathfrak{C}}_{\alpha\in I}\mathfrak{H}_\alpha$, particularly in the light of the results of Part IV, we choose the first alternative. And once $\Pi\otimes_{\alpha\in I}\mathfrak{H}_\alpha$ is used, there seems to be no reason to insist on I's enumerability.

Third: As I may be unenumerable, we must define unenumerably infinite products $\Pi_{\alpha\in I} z_\alpha$ (and sums $\Sigma_{\alpha\in I} z_\alpha$, too). This is done in Chapter 2, and causes no difficulties. In particular, the complication of „quasi-convergence" (cf. § 2.5) arises already for enumerably infinite I's.

6. An essential result of our theory is, that the ring $\mathcal{B}^{\#}$ of all those bounded operators of $\Pi\otimes_{\alpha\in I}\mathfrak{H}_\alpha$ which are generated (algebraically or by limiting-processes) by operators of the $\mathfrak{H}_\alpha$, $\alpha\in I$, does not contain all bounded operators of $\Pi\otimes_{\alpha\in I}\mathfrak{H}_\alpha$. Its structure is exactly determined in Theorems IX and X.

What happens could be described in the quantum-mechanical terminology as a „splitting up" of $\Pi\otimes_{\alpha\in I}\mathfrak{H}_\alpha$ into „non-intercombining systems of states", corresponding to the „incomplete" direct products $\Pi\otimes^{\mathfrak{C}}_{\alpha\in I}\mathfrak{H}_\alpha$. This viewpoint, as well as its connection with the theory of „hyperquantisation" will be discussed elsewhere.

Another application of our theory could be made to the theory of measure in infinite products of spaces, which is the basis for the modern theory of probabilities. (Cf. (2), (3), (5).) Here a certain „incomplete" direct product $\Pi\otimes^{\mathfrak{C}}_{\alpha\in I}\mathfrak{H}_\alpha$ is fundamental. This application too, will be discussed in another publication.

7. Part IV shows in a very characteristic way, how differently the various parts of a simply defined subring of $\mathcal{B}^{\#}$ may behave, when the $\Pi\otimes_{\alpha\in I}^{\mathfrak{C}}\mathfrak{H}_{\alpha}$-decomposition of $\Pi\otimes_{\alpha\in I}\mathfrak{H}_{\alpha}$ is applied to them. A special example of particular interest is discussed in detail. (Cf. in particular §§ 7.3—7.5.) It seems to be essentially connected with the theory of „factors'' of F. J. Murray and of the author, (7), and provides particularly simple examples of various sorts of such „factors'', particularly of the important type (II_1). („Finite-continuous'', cf. (7) pp. 172, 209—229.)

8. A detailed table of contents has been given, to facilitate orientation in the paper. All quotations refer to the bibliography, (1)—(15). The notations to be used are fully explained in § 1.1. The reader is supposed to be familiar with the general theory of Hilbert space, as contained in (8), (12) or (14) and its generalisation to unitary spaces, as given in (4), (12), (13), or (15) (cf. 1.1, (b)). For Part III at least familiarity with the general ideas of (7) or (9) is desirable. In § 7.5 only will results of (7) be used.

Contents.

Bibliography.

(1) G. Birkhoff: Moore-Smith convergence in general topology, Annals of Math. (2) 38 (1937), 39—56.

(2) E. Hopf: On causality, statistics, and probability, Journal of Math. and Phys. 13 (1934), 51—102.

(3) A. Kolmogoroff: Grundbegriffe der Wahrscheinlichkeitsrechnung, J. Springer, Berlin (1933).

(4) H. Löwig: Komplexe euclidische Räume von beliebiger endlicher oder transfiniter Dimensionszahl, Acta Szeged 7 (1934), 1—33.
(5) Z. Lomnicki and S. Ulam: Sur la théorie de la mesure dans les espaces combinatoires et son application au calcul des probabilités, I, Fund. Math. 23 (1934), 237—278.
(6) E. H. Moore and H. L. Smith: A general theory of limits, American Journal of Math. 44 (1922), 102—121.
(7) F. J. Murray and J. v. Neumann: On rings of operators, Annals of Math. (2) 37 (1936), 116—229.
(8) J. v. Neumann: Allgemeine Eigenwerttheorie Hermitescher Funktionaloperatoren, Math. Annalen 102 (1929), 49—131.
(9) J. v. Neumann: Zur Algebra der Funktionaloperatoren, Math. Annalen 102 (1929), 370—427.
(10) J. v. Neumann: Über adjungierte Funktionaloperatoren, Annals of Math. (2) 33 (1932), 294—310.
(11) J. v. Neumann: On a certain topology for rings of operators, Annals of Math. (2) 37 (1936), 111—115.
(12) J. v. Neumann: Lectures on operator theory [Princeton 1934, mimeographed].
(13) F. Rellich: Spektraltheorie in nichtseparablen Räumen, Math. Annalen 110 (1935), 342—356.
(14) M. H. Stone: Linear Transformations in Hilbert space, Amer. Math. Soc. Coll. Publications, Vol. XV (1932).
(15) Y. Y. Tseng: Thesis, Chicago (1932).

Part I: Preparatory considerations.

Chapter 1: Notations.

1.1. We will use the notations of (8), (9) in about the same way as in (7). It will be necessary, however, to include non-separable hyper-Hilbert-spaces ab initio in our discussions, thereby diverging from loc. cit. above. For this reason it seems appropriate to give an independent account of the notions and symbols to be used.

(a) $\alpha \in S$ means that α is an element of the set S, $S \subset T$ or $T \supset S$ that S is a subset of T (including the possibility of $S = T$). The set-theoretical sum of all sets S_α, α running over all elements possessing a certain property $\varepsilon(\alpha)$, will be denoted by $\mathfrak{S}(S_\alpha; \varepsilon(\alpha))$ [7]. If these S_α may be written as a finite or (enumerably) infinite sequence $S_1, S_2, \ldots$, we will write $\mathfrak{S}(S_1, S_2, \ldots)$ too. If S has a unique element x we may write x for S. The empty set will be denoted by Θ.

(b) A complex linear space with a (Hermitean and definite)

---

[7]) In particular: If α runs over all elements of a given set I, we write $\mathfrak{S}(S_\alpha; \alpha \in I)$. In (7) the letter $\mathfrak{S}$ was omitted.

linear inner product, which is complete, will be denoted by $\mathfrak{H}$. (We will make free use of affixes and suffixes, as many such spaces will occur.) In other words: $\mathfrak{H}$ is a space in which operations af, $f \pm g$, (f, g) satisfying the conditions **A**, **B**, **E** of (8) p. 64—66, are given. Conditions **C** and **D** (loc. cit.) are explicitly excepted. (They express the separability and the infinite-dimensionality of $\mathfrak{H}$.) It is known, that in spite of these omissions $\mathfrak{H}$ can be treated almost precisely along the same lines as in (8), (12) and (14) (where all conditions **A** — **E** are used). In particular: A system of elements $\varphi_\alpha \in \mathfrak{H}$, where α runs over an arbitrarily given set of indices I, is a *complete normalised orthogonal set*, if

$$\text{(I)}\quad (\varphi_\alpha, \varphi_\beta)\left\{ = \begin{matrix}1\\0\end{matrix} \text{ for } \begin{matrix}\alpha = \beta\\ \alpha \neq \beta\end{matrix}\right.$$

$$\text{(II)}\quad \text{if } f \in \mathfrak{H} \text{ and } (f, \varphi_\alpha) = 0 \text{ for all } \alpha \in I, \text{ then } f = 0.$$

Such systems φ_α, $\alpha \in I$ do exist, and for all of them I has the same power $\aleph = \aleph(\mathfrak{H})$, the *dimension of* $\mathfrak{H}$. (Cf. (15), also (4), (13) or (12).) Correspondingly $\mathfrak{H}$ will belong to one of the three following types:

(1) $\aleph(\mathfrak{H}) < \aleph_0$. Then $\aleph(\mathfrak{H})$ is finite; $\aleph(\mathfrak{H}) = N = 1, 2, \ldots$ and $\mathfrak{H}$ is an N-dimensional (complex, unitary) Euclidean space. (**C** fails, **D** holds.)

(2) $\aleph(\mathfrak{H}) = \aleph_0$. Then $\aleph(\mathfrak{H})$ is enumerably infinite, and $\mathfrak{H}$ is a Hilbert space. (Both **C**, **D** hold.)

(3) $\aleph(\mathfrak{H}) > \aleph_0$. Then $\aleph(\mathfrak{H})$ is unenumerably infinite, and $\mathfrak{H}$ is a (non-separable) hyper-Hilbert space. (**C** holds, **D** fails.) We exclude explicitly the case $\aleph(\mathfrak{H}) = N = 0$, where $\mathfrak{H} = (0)$.

Any such $\mathfrak{H}$ will be called, for the sake of brevity, a *unitary space.*

(c) Closed linear subsets of $\mathfrak{H}$ are denoted by $\mathfrak{M}$, $\mathfrak{N}$. As they are again unitary spaces (except when $= (0)$), their symbols sometimes replace $\mathfrak{H}$.

The smallest linear or the smallest closed linear set containing certain sets and elements are denoted by $\mathfrak{S}\{\ldots\}$ resp. $\mathfrak{S}[\ldots]$. (The details of this notation are as in (a), where the smallest set containing them — that is their set-theoretical sum — was denoted by $\mathfrak{S}(\ldots)$.) [8]) The set of all elements of $\mathfrak{M}$ which are orthogonal to $\mathfrak{N}$ is a closed linear set, to be denoted by $\mathfrak{M} - \mathfrak{N}$.

(d) For operators, rings of operators, etc., we use the same notations as in (7), p. 127.

---

[8]) In (7) the letter $\mathfrak{S}$ was omitted in all these symbols.

(e) The topologies to be used in $\mathfrak{H}$ and in the space $\mathcal{B} = \mathcal{B}(\mathfrak{H})$ of all bounded operators of $\mathfrak{H}$, are those discussed in (7), p. 127. Considering the cases (1)—(3) in (b) above, we see: In cases (1), (2) (Euclidean spaces and Hilbert space) these topologies behave as described loc. cit., and (9), (11). In case (3) (hyper-Hilbert spaces) one verifies easily, that the conditions are identical with those of case (2), with one exception: The second countability axiom of Hausdorff holds for none of our topologies, not even in the unit-sphere of $\mathfrak{H}$ or $\mathcal{B}$ (defined by $\|\varphi\| \leqq 1$ resp. $|||A||| \leqq 1$).

Chapter 2: Convergence.

2.1. Let I be a set of indices of arbitrary size, and let for each $\alpha \in I$ a unitary space $\mathfrak{H}_\alpha$ be given. We wish to define a *direct product* of these $\mathfrak{H}_\alpha$, $\alpha \in I$, which will be denoted by $\Pi\otimes_{\alpha \in I} \mathfrak{H}_\alpha$, under the guidance of the following heuristic principles:

We desire that $\Pi\otimes_{\alpha \in I} \mathfrak{H}_\alpha$ be again a unitary space. For any given sequence of elements $f_\alpha \in \mathfrak{H}_\alpha$, α runs over I, this $\Pi\otimes_{\alpha \in I} \mathfrak{H}_\alpha$ shall contain a (symbolic) element $\Pi\otimes_{\alpha \in I} f_\alpha$. For these elements we require

$$(*) \qquad (\Pi\otimes_{\alpha \in I} f_\alpha, \Pi\otimes_{\alpha \in I} g_\alpha) = \Pi_{\alpha \in I} (f_\alpha, g_\alpha)\text{[9]}.$$

The $\Pi_{\alpha \in I} (f_\alpha, g_\alpha)$ on the right side of (*) is a numerical product, which may have infinitely, perhaps even unenumerably infinitely, many factors. Therefore its convergence is a serious question, which must be dealt with by appropriate definitions, before a notion of $\Pi\otimes_{\alpha \in I} \mathfrak{H}_\alpha$ fulfilling our heuristic requirements can be satisfactorily described.

Specialise (*) with $f_\alpha = g_\alpha$, then this results:

$$(**) \qquad \|\Pi\otimes_{\alpha \in I} f_\alpha\| = \Pi_{\alpha \in I} \|f_\alpha\|.$$

This formula shows, that we cannot insist on forming $\Pi\otimes_{\alpha \in I} f_\alpha$ for all sequences $f_\alpha \in \mathfrak{H}_\alpha$, $\alpha \in I$:

(1) Only sequences f_α, $\alpha \in I$, with a convergent $\Pi_{\alpha \in I} \|f_\alpha\|$ can be permitted [10]).

---

[9]) We denote the inner product and the absolute value by (Φ, Ψ) and $\|\Phi\|$ if $\Phi, \Psi \in \Pi\otimes_{\alpha \in I} \mathfrak{H}_\alpha$ and also by (f_α, g_α) and $\|f_\alpha\|$ if $f_\alpha, g_\alpha \in \mathfrak{H}_\alpha$.

[10]) For a finite I the problem does not arise; for an unenumerably infinite one $\Pi_{\alpha \in I}$ has not yet been defined. But if I is enumerably infinite, it is obvious, that $\Pi_{\alpha \in I}$ can diverge in the usual sense.

Another observation:

(2) In the definition of convergence to be given, convergence of $\prod_{\alpha\in I}\|f_\alpha\|$ to 0 [11]) may be considered as convergence. But sequences f_α, $\alpha\in I$, with $\prod_{\alpha\in I}\|f_\alpha\| = 0$ are of no importance for our purpose, because (**) forces us to define for them $\prod\otimes_{\alpha\in I} f_\alpha = 0$.

(*) is a *relation* between two sequences f_α, $\alpha\in I$ and g_α, $\alpha\in I$ and not a *property* of one. This is apt to be a source of complications, except if we manage to secure this:

(3) If $\prod_{\alpha\in I}\|f_\alpha\|$ and $\prod_{\alpha\in I}\|g_\alpha\|$ converge, then $\prod_{\alpha\in I}(f_\alpha, g_\alpha)$ converges too.

Finally we wish, that our direct products $\prod\otimes_{\alpha\in I}\mathfrak{H}_\alpha$ fulfill the *commutative rule* of multiplication unrestrictedly. This makes it plausible to require:

(4) The definitions of convergence for $\prod_{\alpha\in I}\|f_\alpha\|$ and for $\prod_{\alpha\in I}(f_\alpha, g_\alpha)$ shall depend on no ordering of the set I.

2.2. We proceed now to define the notion of convergence for $\prod_{\alpha\in I} z_\alpha$, the z_α being arbitrary complex numbers, so that the desiderata (1)—(4) of § 2.1 are fulfilled as far as possible. It is convenient, to define at the same time $\sum_{\alpha\in I} z_\alpha$ too.

(4) forbids us to introduce any ordering of I. Therefore the following definition seems natural [12]):

Definition 2.2.1. $\sum_{\alpha\in I} z_\alpha$ resp. $\prod_{\alpha\in I} z_\alpha$ is *convergent*, and a is its *value* (the z_α as well as a are complex numbers), if there exists for every $\delta > 0$ a finite set $I_0 = I_0(\delta) \subset I$, such that for every finite set $J = \mathfrak{S}(\alpha_1, \ldots, \alpha_n)$ (the $\alpha_1, \ldots, \alpha_n$ being mutually different) with $I_0 \subset J \subset I$

$$|z_{\alpha_1} + \ldots + z_{\alpha_n} - a| \leqq \delta \text{ resp. } |z_{\alpha_1} \cdot \ldots \cdot z_{\alpha_n} - a| \leqq \delta.$$

Corollary: The value a of $\sum_{\alpha\in I} z_\alpha$ resp. $\prod_{\alpha\in I} z_\alpha$ is unique, if it exists at all (that is: if we have convergence).

Proof: Let a', a'' be two values. If $\delta > 0$, choose the corresponding finite sets $I_0' = I_0'(\delta)$, $I_0'' = I_0''(\delta)$. Put $J = \mathfrak{S}(I_0', I_0'') = \mathfrak{S}(\alpha_1, \ldots, \alpha_n)$. J is finite, $I_0' \subset J \subset I$, $I_0'' \subset J \subset I$ so

$$|z_{\alpha_1} + \ldots + z_{\alpha_n} - a'| \leqq \delta, \quad |z_{\alpha_1} + \ldots + z_{\alpha_n} - a''| \leqq \delta \text{ resp.}$$
$$|z_{\alpha_1} \cdot \ldots \cdot z_{\alpha_n} - a'| \leqq \delta, \quad |z_{\alpha_1} \cdot \ldots \cdot z_{\alpha_n} - a''| \leqq \delta$$

and thus $|a' - a''| \leqq 2\delta$. As $\delta > 0$ was arbitrary, we have $a' = a''$.

---

[11]) Which, if all $\|f_\alpha\| \neq 0$, is usually called „divergence to 0".

[12]) It is a special case of a notion of limit in „directed sets", due to E. H. Moore, H. L. Smith, and G. Birkhoff. Cf. (1), (6).

2.3. We now derive the basic properties of $\sum_{\alpha \in I} z_\alpha$.

LEMMA 2.3.1. If all z_α are real and $\geqq 0$, then $\sum_{\alpha \in I} z_\alpha$ converges if and only if the set $\mathfrak{S}(z_{\alpha_1} + \ldots + z_{\alpha_n};\ \alpha_1, \ldots, \alpha_n$ mutually different, and all $\in I)$ is bounded. Its value is then the l.u.b. [13]) of this set.

Proof: Necessity: If $\sum_{\alpha \in I} z_\alpha$ converges, then let a be its value, and put $I_0 = I_0(1)$. If $\alpha_1, \ldots, \alpha_n$ are mutually different and all $\in I$, then let $\alpha_{n+1}, \ldots, \alpha_m$ be the different elements of I_0, which are $\neq \alpha_1, \ldots, \alpha_n$. Now $J = \mathfrak{S}(\alpha_1, \ldots, \alpha_m)$ satisfies $I_0 \subset J \subset I$ and so (as all $z_\alpha \geqq 0$)

$$0 \leqq z_{\alpha_1} + \ldots + z_{\alpha_n} \leqq z_{\alpha_1} + \ldots + z_{\alpha_m} \leqq a + 1.$$

Thus the set in question is bounded.

Sufficiency and value: If the set $\mathfrak{S}(z_{\alpha_1} + \ldots + z_{\alpha_n};\ \alpha_1, \ldots, \alpha_n$ mutually different and all $\in I)$ is bounded, then let a be its l.u.b. For every $\delta > 0$ choose $\bar{\alpha}_1, \ldots, \bar{\alpha}_n$ mutually different and $\in I$, with $z_{\bar{\alpha}_1} + \ldots + z_{\bar{\alpha}_n} \geqq a - \delta$. Put $I_0 = I_0(\delta) = \mathfrak{S}(\bar{\alpha}_1, \ldots, \bar{\alpha}_n)$. Now if $J = \mathfrak{S}(\alpha_1, \ldots, \alpha_m)$ is finite and $I_0 \subset J \subset I$ (the $\alpha_1, \ldots, \alpha_m$ mutually different), then the $\bar{\alpha}_1, \ldots, \bar{\alpha}_n$ occur among the $\alpha_1, \ldots, \alpha_m$ and so (as all $z_\alpha \geqq 0$)

$$a - \delta \leqq z_{\bar{\alpha}_1} + \ldots + z_{\bar{\alpha}_n} \leqq z_{\bar{\alpha}_1} + \ldots + z_{\bar{\alpha}_m} \leqq a,$$

$$| z_1 + \ldots + z_{\alpha_m} - a | \leqq \delta.$$

As $\delta > 0$ was arbitrary, $\sum_{\alpha \in I} z_\alpha$ is convergent, and its value is a.

LEMMA 2.3.2. If all z_α are real and $\geqq 0$, then $\sum_{\alpha \in I} z_\alpha$ converges if and only if

(I) $z_\alpha \neq 0$ occurs for a finite or enumerably infinite number of $\alpha \in I$ only, say for the (mutually different) $\alpha_1, \alpha_2, \ldots$ [14]),

(II) $z_{\alpha_1} + z_{\alpha_2} + \ldots$ (in the usual sense) is finite. Its value is then the $z_{\alpha_1} + z_{\alpha_2} + \ldots$ of (II).

Proof: Necessity: Denote the l.u.b. of $\mathfrak{S}(z_{\beta_1} + \ldots + z_{\beta_n};\ \beta_1, \ldots, \beta_n$ mutually different and $\in I)$ by a. If we had $z_{\beta_1}, \ldots, z_{\beta_n} \geqq \delta$ for some fixed $\delta > 0$, then $a \geqq z_{\beta_1} + \ldots + z_{\beta_n} \geqq n\delta$, $n \leqq \frac{a}{\delta}$ would ensue. So only a finite number of $\alpha \in I$ with $z_\alpha \geqq \delta$ exists. Put $\delta = 1, \frac{1}{2}, \frac{1}{3}, \ldots$ successively; this proves (I).

Form the $\alpha_1, \alpha_2, \ldots$ of (I). Then $z_{\alpha_1} + \ldots + z_{\alpha_m} \leqq a$ and as all $z_{\alpha_1}, z_{\alpha_2}, \ldots \geqq 0$; this implies the finiteness of $z_{\alpha_1} + z_{\alpha_2} + \ldots$.

---

13) l.u.b. = least upper bound.

14) The length of this sequence may be $0, 1, 2, \ldots$ or ∞.

Sufficiency and value: If (I), (II) hold, then $z_{\alpha_1} + z_{\alpha_2} + \dots$ is clearly the l.u.b. described in Lemma 2.3.1.

LEMMA 2.3.3. If the z_α are arbitrary complex numbers, then $\sum_{\alpha \in I} z_\alpha$ converges if and only if $\sum_{\alpha \in I} |z_\alpha|$ converges.

Proof: The convergence of $\sum_{\alpha \in I} z_\alpha$ is clearly equivalent to the combined convergences of $\sum_{\alpha \in I} \Re z_\alpha$, $\sum_{\alpha \in I} \Im z_\alpha$ [15]. The same is true for $\sum_{\alpha \in I} |z_\alpha|$ and $\sum_{\alpha \in I} |\Re z_\alpha|$, $\sum_{\alpha \in I} |\Im z_\alpha|$ owing to

$$|\Re z_\alpha| \text{ and } |\Im z_\alpha| \leqq |z_\alpha| \leqq |\Re z_\alpha| + |\Im z_\alpha|.$$

(Use Lemma 2.3.1.) So we may consider $\Re z_\alpha$, $\Im z_\alpha$ instead of z_α. That is: We may assume that z_α is real.

Necessity: If $\sum_{\alpha \in I} z_\alpha$ converges, then let a be its value, and $I_0 = I_0(1) = \mathfrak{S}(\bar\alpha_1, \dots, \bar\alpha_n)$, the $\bar\alpha_1, \dots, \bar\alpha_n$ mutually different. If $\alpha_1, \dots, \alpha_m$ are mutually different and $\neq \bar\alpha_1, \dots, \bar\alpha_n$ then

$$|z_{\bar\alpha_1} + \dots + z_{\bar\alpha_n} - a| \leqq 1, \quad |z_{\bar\alpha_1} + \dots + z_{\bar\alpha_n} + z_{\alpha_1} + \dots + z_{\alpha_m} - a| \leqq 1,$$

so

$$|z_{\alpha_1} + \dots + z_{\alpha_m}| \leqq 2.$$

Now denote the $\alpha_1, \dots, \alpha_m$ with $z_{\alpha_i} > 0$ by $\alpha_1', \dots, \alpha_s'$ and those with $z_{\alpha_i} \leqq 0$ by $\alpha_1'', \dots, \alpha_{m-s}''$. Then we have similarly

$|z_{\alpha_1'} + \dots + z_{\alpha_s'}|$ and $|z_{\alpha_1''} + \dots + z_{\alpha_{m-s}''}| \leqq 2$. But

$$|z_{\alpha_1'} + \dots + z_{\alpha_s'}| = z_{\alpha_1'} + \dots + z_{\alpha_s'} = |z_{\alpha_1'}| + \dots + |z_{\alpha_s'}|,$$

$$|z_{\alpha_1''} + \dots + z_{\alpha_{m-s}''}| = - z_{\alpha_1''} - \dots - z_{\alpha_{m-s}''} = |z_{\alpha_1''}| + \dots + |z_{\alpha_{m-s}''}|$$

and so

$$|z_{\alpha_1}| + \dots + |z_{\alpha_m}| \leqq 4.$$

Now if $\beta_1, \dots, \beta_p$ are mutually different, but otherwise arbitrary, then let $\alpha_1, \dots, \alpha_m$ be those $\beta_1, \dots, \beta_p$ which are $\neq \bar\alpha_1, \dots, \bar\alpha_n$. Then

$$|z_{\beta_1}| + \dots + |z_{\beta_p}| \leqq |z_{\bar\alpha_1}| + \dots + |z_{\bar\alpha_n}| + |z_{\alpha_1}| + \dots + |z_{\alpha_m}| \leqq$$
$$\leqq |z_{\bar\alpha_1}| + \dots + |z_{\bar\alpha_n}| + 4 = a_0$$

(say). So $\sum_{\alpha \in I} |z_\alpha|$ converges by Lemma 2.3.1.

Sufficiency: Let I' be the set of all $\alpha \in I$ with $z_\alpha > 0$; then $I - I'$ consists of those with $z_\alpha \leqq 0$. If $\sum_{\alpha \in I} |z_\alpha|$ converges, then $\sum_{\alpha \in I'} |z_\alpha|$, $\sum_{\alpha \in I - I'} |z_\alpha|$ converge too, by Lemma 2.3.1. But for all $\alpha \in I'$, $z_\alpha = |z_\alpha|$, and for all $\alpha \in I - I'$, $z_\alpha = -|z_\alpha|$. So $\sum_{\alpha \in I'} z_\alpha$,

---

[15]) If $z = u + iv$, u, v real, then $\Re z = u$, $\Im z = v$.

$\Sigma_{\alpha \in I - I'} z_\alpha$ converge too, and this clearly implies the convergence of $\Sigma_{\alpha \in I} z_\alpha$.

LEMMA 2.3.4. If the z_α are arbitrary complex numbers, then $\Sigma_{\alpha \in I} z_\alpha$ converges if and only if

(I) $z_\alpha \neq 0$ occurs for a finite or enumerably infinite number of $\alpha \in I$ only, say for the (mutually different) $\alpha_1, \alpha_2, \ldots$ [13]).

(II) $|z_{\alpha_1}| + |z_{\alpha_2}| + \ldots$ (in the usual sense) is finite. Its value is then $z_{\alpha_1} + z_{\alpha_2} + \ldots$ (in the usual sense).

Proof: Necessity and sufficiency: Immediate by Lemmata 2.3.2 and 2.3.3.

Value: As we may consider $\Sigma_{\alpha \in I} \Re z_\alpha$, $\Sigma_{\alpha \in I} \Im z_\alpha$ instead of $\Sigma_{\alpha \in I} z_\alpha$, we may assume that all z_α are real. Let I' again be the set of all $\alpha \in I$ with $z_\alpha > 0$, so that $I - I'$ consists of those with $z_\alpha \leqq 0$. Our statement holds for $\Sigma_{\alpha \in I'} z_\alpha$ because here $z_\alpha = |z_\alpha|$, as well as for $\Sigma_{\alpha \in I - I'} z_\alpha$ because there $z_\alpha = -|z_\alpha|$. (In both cases use the last statement of Lemma 2.3.2.) So it holds for $\Sigma_{\alpha \in I} z_\alpha$ too.

COROLLARY: If I is finite, that is $I = \mathfrak{S}(\alpha_1, \ldots, \alpha_n)$ (the $\alpha_1, \ldots, \alpha_n$ mutually different), then $\Sigma_{\alpha \in I} z_\alpha$ is always convergent and its value is $z_{\alpha_1} + \ldots + z_{\alpha_n}$. If I is enumerably infinite, that is $I = \mathfrak{S}(\alpha_1, \alpha_2, \ldots)$ (the $\alpha_1, \alpha_2, \ldots$ mutually different), then $\Sigma_{\alpha \in I} z_\alpha$ is convergent if and only if $z_{\alpha_1} + z_{\alpha_2} + \ldots$ is *absolutely* convergent in the usual sense, and then its value is $z_{\alpha_1} + z_{\alpha_2} + \ldots$

Proof: Clear by Lemma 2.3.4.

Our notion of convergence is thus an extension of the usual notion of absolute convergence. At any rate $\Sigma_{\alpha \in I} z_\alpha$ conserves its usual meaning for finite sets I.

2.4. We next discuss $\Pi_{\alpha \in I} z_\alpha$, again beginning with the special case, where all z_α are real and $\geqq 0$.

LEMMA 2.4.1. If all z_α are real and $\geqq 0$, then

(I) $\Pi_{\alpha \in I} z_\alpha$ converges if and only if either $\Sigma_{\alpha \in I} \mathrm{Max}\,(z_\alpha - 1, 0)$ converges, or some $z_\alpha = 0$,

(II) $\Pi_{\alpha \in I} z_\alpha$ converges and is $\neq 0$ if and only if $\Sigma_{\alpha \in I} |z_\alpha - 1|$ converges and all $z_\alpha \neq 0$.

Proof: If any $z_\beta = 0$, then $\Pi_{\alpha \in I} z_\alpha$ is convergent and has the value 0: $I_0 = I_0(\delta) = \mathfrak{S}(\beta)$ will do for any $\delta > 0$. So $z_\beta = 0$ has the desired effect in both (I), (II), and therefore we may assume that all $z_\beta \neq 0$, and discuss (I), (II) under this assumption.

Necessity of (I): Assume that $\Pi_{\alpha \in I} z_\alpha$ converges, and that its value is a. Put $I_0 = I_0(1) = \mathfrak{S}(\bar{\alpha}_1, \ldots, \bar{\alpha}_n)$, the $\bar{\alpha}_1, \ldots, \bar{\alpha}_n$ mutually different. Let $\alpha_1, \ldots, \alpha_m$ be mutually different and

ϵI. Some $\bar{\alpha}_1, \ldots, \bar{\alpha}_n$ may occur among the $\alpha_1, \ldots, \alpha_m$, say $\bar{\alpha}_1, \ldots, \bar{\alpha}_p$. Now $J = \mathfrak{S}(\alpha_1, \ldots, \alpha_m, \bar{\alpha}_{p+1}, \ldots, \bar{\alpha}_n)$ is finite and $I_0 \subset J \subset I$, so we have $z_{\alpha_1} \cdot \ldots \cdot z_{\alpha_m} \cdot z_{\bar{\alpha}_{p+1}} \cdot \ldots \cdot z_{\bar{\alpha}} \leqq a + 1$. Therefore

$$z_{\alpha_1} \cdot \ldots \cdot z_{\alpha_m} \leqq \frac{a+1}{z_{\bar{\alpha}_{p+1}} \cdot \ldots \cdot z_{\bar{\alpha}_n}} \leqq \frac{a+1}{\operatorname{Min}(z_{\bar{\alpha}_{p+1}}, 1) \cdot \ldots \cdot \operatorname{Min}(z_{\bar{\alpha}_n}, 1)} \leqq$$
$$\leqq \frac{a+1}{\operatorname{Min}(z_{\bar{\alpha}_1}, 1) \cdot \ldots \cdot \operatorname{Min}(z_{\bar{\alpha}_n}, 1)}.$$

But if $z_{\alpha_i} - 1 > 0$ for all z_{α_i}, then

$$z_{\alpha_i} \cdot \ldots \cdot z_{\alpha_m} = (1 + (z_{\alpha_1} - 1)) \cdot \ldots \cdot (1 + (z_{\alpha_m} - 1)) \geqq$$
$$\geqq 1 + (z_{\alpha_1} - 1) + \ldots + (z_{\alpha_m} - 1),$$

and so

$$(z_{\alpha_1} - 1) + \ldots + (z_{\alpha_m} - 1) \leqq \frac{a+1}{\operatorname{Min}(z_{\bar{\alpha}_1}, 1) \cdot \ldots \cdot \operatorname{Min}(z_{\bar{\alpha}_n}, 1)} - 1.$$

Now denote the $\alpha_1, \ldots, \alpha_m$ with $z_{\alpha_i} - 1 > 0$ by $\alpha_1', \ldots, \alpha_s'$. Then the above evaluation does hold for $(z_{\alpha_1'} - 1) + \ldots + (z_{\alpha_s'} - 1)$. In other words:

$$\operatorname{Max}(z_{\alpha_1} - 1, 0) + \ldots +$$
$$+ \operatorname{Max}(z_{\alpha_m} - 1, 0) \leqq \frac{a+1}{\operatorname{Min}(z_{\bar{\alpha}_1}, 1) \cdot \ldots \cdot \operatorname{Min}(z_{\bar{\alpha}_n}, 1)} - 1.$$

By Lemma 2.3.1 this establishes the convergence of $\Sigma_{\alpha \epsilon I} \operatorname{Max}(z_\alpha - 1, 0)$.

Sufficiency of (I): We will prove below that $\Pi_{\alpha \epsilon I} z_\alpha$ converges, if $\Sigma_{\alpha \epsilon I} |z_\alpha - 1|$ converges. So we need only consider the case where $\Sigma_{\alpha \epsilon I} \operatorname{Max}(z_\alpha - 1, 0)$ converges and $\Sigma_{\alpha \epsilon I} |z_\alpha - 1|$ does not, that is [16] $\Sigma_{\alpha \epsilon I} \operatorname{Max}(1 - z_\alpha, 0)$ does not.

By Lemma 2.3.1 the first statement implies

$$\operatorname{Max}(z_{\alpha_1} - 1, 0) + \ldots + \operatorname{Max}(z_{\alpha_m} - 1, 0) \leqq a_0$$

for some fixed a_0, whenever the $\alpha_1, \ldots, \alpha_m$ are mutually different. Hence

$$z_{\alpha_1} \cdot \ldots \cdot z_{\alpha_m} \leqq e^{\operatorname{Max}(z_{\alpha_1} - 1, 0)} \cdot \ldots \cdot e^{\operatorname{Max}(z_{\alpha_m} - 1, 0)} \leqq e^{a_0}.$$

For the same reason the second statement implies the existence of a set of mutually different $\bar{\alpha}_1, \ldots, \bar{\alpha}_n$ for any given $A > 0$ such that $\operatorname{Max}(1 - z_{\bar{\alpha}_1}, 0) + \ldots + \operatorname{Max}(1 - z_{\bar{\alpha}_n}, 0) > A$. Clearly

---

[16]) Observe that $|u| = \operatorname{Max}(u, 0) + \operatorname{Max}(-u, 0)$ for all real u.

every $\bar{\alpha}_i$ with $z_{\bar{\alpha}_i} \geqq 1$ could be omitted from this set, so we may assume that all $z_{\bar{\alpha}_i} < 1$. Now we have

$$z_{\bar{\alpha}_1} \cdot \ldots \cdot z_{\bar{\alpha}_n} \leqq e^{-\mathrm{Max}\,(1-z_{\bar{\alpha}_1},\,0)} \cdot \ldots \cdot e^{-\mathrm{Max}\,(1-z_{\bar{\alpha}_n},\,0)} < e^{-A}.$$

Any $\delta > 0$ being given, choose A with $e^{a_0 - A} \leqq \delta$, and put $I_0 = I_0(\delta) = \mathfrak{S}(\bar{\alpha}_1, \ldots, \bar{\alpha}_n)$. Then any finite J with $I_0 \subset J \subset I$ has the form $J = \mathfrak{S}(\bar{\alpha}_1, \ldots, \bar{\alpha}_n, \alpha_1, \ldots, \alpha_m)$ the $\bar{\alpha}_1, \ldots, \bar{\alpha}_n, \alpha_1, \ldots, \alpha_m$ being mutually different, and

$$0 \leqq z_{\bar{\alpha}_1} \cdot \ldots \cdot z_{\bar{\alpha}_n} \cdot z_{\alpha_1} \cdot \ldots \cdot z_{\alpha_m} < e^{a_0 - A} \leqq \delta.$$

Thus $\prod_{\alpha \in I} z_\alpha$ converges (its value is 0).

Necessity and sufficiency of (II): That $\prod_{\alpha \in I} z_\alpha$ be convergent with the value a, all $z_\alpha \neq 0$ and $a \neq 0$ is clearly equivalent to $\sum_{\alpha \in I} \ln z_\alpha$ being convergent with the value $\ln a$. By Lemma 2.3.1 this means, that $\sum_{\alpha \in I} |\ln z_\alpha|$ be convergent.

Compare this with the convergence of $\sum_{\alpha \in I} |z_\alpha - 1|$. If either expression converges, Lemma 2.3.2 requires, that $|\ln z_\alpha| > \frac{1}{3}$ resp. $|z_\alpha - 1| > \frac{1}{2}$ occur only a finite number of times. The second inequality implies the first one, so we have $|z_\alpha - 1| \leqq \frac{1}{2}$, with a finite number of exceptions. Now $|z_\alpha - 1| \leqq \frac{1}{2}$ implies $\frac{2}{3}|z_\alpha - 1| \leqq |\ln z_\alpha| \leqq 2|z_\alpha - 1|$ [17]), and so the two convergences are equivalent by Lemma 2.3.1.

Corollary: Explicit criteria for the convergence of $\prod_{\alpha \in I} z_\alpha$ resp. for its convergence with a value $\neq 0$ (all z_α real and $\geqq 0$) can now be obtained by applying Lemmata 2.3.1 and 2.3.2.

Lemma 2.4.2. If the z_α are arbitrary complex numbers, then $\prod_{\alpha \in I} z_\alpha$ converges if and only if

(I) either $\prod_{\alpha \in I} |z_\alpha|$ converges and its value is 0,

(II) or $\prod_{\alpha \in I} |z_\alpha|$ converges and its value is $\neq 0$, and $\sum_{\alpha \in I} |\mathrm{arcus}\, z_\alpha|$ [18]) converges.

In case (I) the value of $\prod_{\alpha \in I} z_\alpha$ is 0, in case (II) it is $\prod_{\alpha \in I} |z_\alpha| \cdot e^{i \sum_{\alpha \in I} \mathrm{arcus}\, z_\alpha}$, and thus $\neq 0$.

Proof: Necessity: As

$$\big||z_{\alpha_1}| \cdot \ldots \cdot |z_{\alpha_n}| - |a|\big| \leqq |z_{\alpha_1} \cdot \ldots \cdot z_{\alpha_n} - a|$$

---

[17]) $\dfrac{d(\ln z_\alpha)}{d(z_\alpha - 1)} = \dfrac{1}{z_\alpha}$ lies between $\dfrac{1}{1+\frac{1}{2}} = \dfrac{2}{3}$ and $\dfrac{1}{1-\frac{1}{2}} = 2$.

[18]) Is $z \neq 0$, $z = |z| e^{i\theta}$ with $-\pi < \theta \leqq \pi$, then arcus $z = \theta$.

the convergence of $\prod_{\alpha \in I} z_\alpha$ clearly implies the convergence of $\prod_{\alpha \in I} |z_\alpha|$. It remains to be shown, that when $\prod_{\alpha \in I} |z_\alpha| \neq 0$, then it implies the convergence of $\sum_{\alpha \in I} |\text{arcus } z_\alpha|$ too.

As $\prod_{\alpha \in I} z_\alpha$, $\prod_{\alpha \in I} |z_\alpha|$ converge, the latter with a value $\neq 0$ so $\prod_{\alpha \in I} \frac{z_\alpha}{|z_\alpha|} = \prod_{\alpha \in I} e^{i \cdot \text{arcus } z_\alpha}$ (all $z_\alpha \neq 0$ necessarily!) converges too. Let the limit be ϱ, and put $I_0 = I_0\left(\frac{1}{2\sqrt{2}}\right) = \mathfrak{S}(\bar{\alpha}_1, \ldots, \bar{\alpha}_n)$ (for $\prod_{\alpha \in I} e^{i \cdot \text{arcus } z_\alpha}$), the $\bar{\alpha}_1, \ldots, \bar{\alpha}_n$ mutually different. Consider any mutually different $\alpha_1, \ldots, \alpha_m$, all $\neq \bar{\alpha}_1, \ldots, \bar{\alpha}_n$. Then we have

$$\left| e^{i \cdot \text{arcus } z_{\bar{\alpha}_1}} \cdot \ldots \cdot e^{i \cdot \text{arcus } z_{\bar{\alpha}_n}} - \varrho \right| < \frac{1}{2\sqrt{2}}$$

$$\left| e^{i \cdot \text{arcus } z_{\bar{\alpha}_1}} \cdot \ldots \cdot e^{i \cdot \text{arcus } z_{\bar{\alpha}_n}} \cdot e^{i \cdot \text{arcus } z_{\alpha_1}} \cdot \ldots \cdot e^{i \cdot \text{arcus } z_{\alpha_m}} - \varrho \right| < \frac{1}{2\sqrt{2}}$$

and so

$$\left| e^{i \cdot \text{arcus } z_{\bar{\alpha}_1}} \cdot \ldots \cdot e^{i \cdot \text{arcus } z_{\bar{\alpha}_n}} \left(e^{i \cdot \text{arcus } z_{\alpha_1}} \cdot \ldots \cdot e^{i \cdot \text{arcus } z_{\alpha_m}} - 1 \right) \right| < \frac{1}{\sqrt{2}}.$$

As $\left| e^{i \cdot \text{arcus } z_{\bar{\alpha}_k}} \right| = 1$, this means

$$\left| e^{i \cdot \text{arcus } z_{\alpha_1}} \cdot \ldots \cdot e^{i \cdot \text{arcus } z_{\alpha_m}} - 1 \right| < \frac{1}{\sqrt{2}}$$

and therefore excludes

$$\frac{\pi}{2} \leqq |\text{arcus } z_{\alpha_1} + \ldots + \text{arcus } z_{\alpha_m}| \leqq \frac{3\pi}{2}.$$

Considering α_j alone instead of $\alpha_1, \ldots, \alpha_m$ excludes $\frac{\pi}{2} \leqq |\alpha_j| \leqq \frac{3\pi}{2}$ and considering $|\alpha_j| \leqq \pi$, we obtain $|\alpha_j| < \frac{\pi}{2}$. Consider now $\alpha_1, \ldots, \alpha_l$ instead of $\alpha_1, \ldots, \alpha_m$ for all $l = 0, 1, 2, \ldots, m$. $|\text{arcus } z_{\alpha_1} + \ldots + \text{arcus } z_{\alpha_l}|$ is 0 for $l = 0$, it changes by $\leqq |\text{arcus } z_{\alpha_{l+1}}| \leqq \frac{\pi}{2}$ when l is replaced by $l+1$, and it never enters the interval $\frac{\pi}{2} \leqq u \leqq \frac{3\pi}{2}$. Therefore it remains always $< \frac{\pi}{2}$. In particular $l = m$ gives:

$$|\text{arcus } z_{\alpha_1} + \ldots + \text{arcus } z_{\alpha_m}| < \frac{\pi}{2}.$$

Denote now those $\alpha_1, \ldots, \alpha_m$ for which $\text{arcus } z_{\alpha_j} > 0$ by $\alpha'_1, \ldots, \alpha'_s$, and the others, for which $\text{arcus } z_{\alpha_j} \leqq 0$, by $\alpha''_1, \ldots, \alpha''_{m-s}$.

Then again

$$|\text{arcus}\, z_{\alpha_1'}| + \ldots + |\text{arcus}\, z_{\alpha_s'}| = \text{arcus}\, z_{\alpha_1'} + \cdots + \text{arcus}\, z_{\alpha_s'} =$$
$$= |\text{arcus}\, z_{\alpha_1'} + \ldots + \text{arcus}\, z_{\alpha_s'}| < \frac{\pi}{2}$$

$$|\text{arcus}\, z_{\alpha_1''}| + \ldots + |\text{arcus}\, z_{\alpha_{m-s}''}| = -\text{arcus}\, z_{\alpha_1''} - \cdots - \text{arcus}\, z_{\alpha_{m-s}''} =$$
$$= |\text{arcus}\, z_{\alpha_1''} + \ldots + \text{arcus}\, z_{\alpha_{m-s}''}| < \frac{\pi}{2}.$$

Adding gives

$$|\text{arcus}\, z_{\alpha_1}| + \cdots + |\text{arcus}\, z_{\alpha_m}| < \pi$$

and so

$$|\text{arcus}\, z_{\bar{\alpha}_1}| + \ldots + |\text{arcus}\, z_{\bar{\alpha}_n}| + |\text{arcus}\, z_{\alpha_1}| + \cdots + |\text{arcus}\, z_{\alpha_m}| <$$
$$< |\text{arcus}\, z_{\bar{\alpha}_1}| + \cdots + |\text{arcus}\, z_{\bar{\alpha}_n}| + \pi = b_0$$

(say).

Thus if $\beta_1, \ldots, \beta_p$ are mutually different, but otherwise arbitrary, then

$$|\text{arcus}\, z_{\beta_1}| + \cdots + |\text{arcus}\, z_{\beta_p}| < b_0.$$

Thus Lemma 2.3.1 secures the convergence of $\Sigma_{\alpha\in I}|\text{arcus}\, z_\alpha|$.

Sufficiency and value: Case (I): As

$$|z_{\alpha_1}| \cdot \ldots \cdot |z_{\alpha_n}| = |z_{\alpha_1} \cdot \ldots \cdot z_{\alpha_n}|,$$

the convergence of $\Pi_{\alpha\in I}|z_\alpha|$ with the value 0 implies the same for $\Pi_{\alpha\in I} z_\alpha$.

Case (II): $\Sigma_{\alpha\in I}|\text{arcus}\, z_\alpha|$ converges, so $\Sigma_{\alpha\in I}\,\text{arcus}\, z_\alpha$ converges too (by Lemma 2.3.3), and with it $\Pi_{\alpha\in I} e^{i\cdot\text{arcus}\, z_\alpha}$. The latter's value is $e^{i\theta}$, where θ is the value of $\Sigma_{\alpha\in I}\,\text{arcus}\, z_\alpha$. Now $\Pi_{\alpha\in I}|z_\alpha|$ converges and its value is an $a \neq 0$ so $\Pi_{\alpha\in I} z_\alpha = \Pi_{\alpha\in I}|z_\alpha|\, e^{i\cdot\text{arcus}\, z_\alpha}$ converges too, and its value is $ae^{i\theta} \neq 0$.

Corollary: Explicit criteria for the convergence of $\Pi_{\alpha\in I} z_\alpha$, resp. for its convergence with a value $\neq 0$ (the z_α are arbitrary complex numbers), can again be obtained by applying Lemmata 2.3.1 and 2.3.2. For a finite set $I = \mathfrak{S}(\alpha_1, \ldots, \alpha_n)$ (the $\alpha_1, \ldots, \alpha_n$ mutually different) in particular, $\Pi_{\alpha\in I} z_\alpha$ is always convergent and its value is $z_{\alpha_1} \cdot \ldots \cdot z_{\alpha_n}$.

2.5. We see (from Lemma 2.3.3 resp. 2.4.2): While the convergence of $\Sigma_{\alpha\in I}|z_\alpha|$ is necessary and sufficient for that one of $\Sigma_{\alpha\in I} z_\alpha$, the convergence of $\Pi_{\alpha\in I}|z_\alpha|$ is necessary but not sufficient for that of $\prod_{\alpha \in I} z_\alpha$. This is very inconvenient, because it violates our desideratum (3) in 2.1: Choose for each $\alpha \in I$ a $\varphi_\alpha \in \mathfrak{H}_\alpha$ with $\|\varphi_\alpha\| = 1$ and put $f_\alpha = |z_\alpha|^{\frac{1}{2}} e^{i \cdot \text{arcus}\, z_\alpha} \varphi_\alpha$, $g_\alpha = |z_\alpha|^{\frac{1}{2}} \varphi_\alpha$. Then $\prod_{\alpha \in I} \|f_\alpha\| = \prod_{\alpha \in I} \|g_\alpha\| = \prod_{\alpha \in I} |z_\alpha|^{\frac{1}{2}}$ converge (along with $\prod_{\alpha \in I} |z_\alpha|$), but $\prod_{\alpha \in I} (f_\alpha, g_\alpha) = \prod_{\alpha \in I} z_\alpha$ does not converge.

We remove this difficulty by defining:

Definition 2.5.1. $\prod_{\alpha \in I} z_\alpha$ is *quasi-convergent*, if and only if $\prod_{\alpha \in I} |z_\alpha|$ is convergent. Its *value* is

(I) the value of $\prod_{\alpha \in I} z_\alpha$ (in the sense of Definition 2.2.1), if it is even convergent,

(II) 0, if it is not convergent.

Corollary: The value of $\prod_{\alpha \in I} z_\alpha$ is again unique, if it exists at all (that is: if we have quasi-convergence).

For $z_\alpha \geqq 0$ we have $z_\alpha = |z_\alpha|$, and so convergence and quasi-convergence are then identical.

Thus we have introduced this convention: If $\prod_{\alpha \in I} |z_\alpha|$ converges, but if $\prod_{\alpha \in I} z_\alpha$ does not, owing to a too vehement oscillation of the arcus z_α (cf. Lemma 2.4.2, (II)), then we *attribute* to $\prod_{\alpha \in I} z_\alpha$ the value 0. This convention is somewhat arbitrary, but probably simpler and more plausible than any alternative one would be. Besides it secures (3) in § 2.1 (cf. Lemma 2.5.2), and leads to a workable theory of direct products $\prod \otimes_{\alpha \in I} \mathfrak{H}_\alpha$, as will appear in the subsequent parts of this paper.

Lemma 2.5.1. Quasi-convergence of $\prod_{\alpha \in I} z_\alpha$ with a value $\neq 0$ is equivalent to convergence with such a value. It holds if and only if all $z_\alpha \neq 0$, and $\sum_{\alpha \in I} |z_\alpha - 1|$ converges.

Proof: The first statement is immediate by Definition 2.5.1. Now Lemmata 2.4.2 and 2.4.1, (II), give this necessary and sufficient condition: All $z_\alpha \neq 0$, and $\sum_{\alpha \in I} \big||z_\alpha| - 1\big|$, $\prod_{\alpha \in I} |\text{arcus}\, z_\alpha|$ converge. But clearly

$$\big||z_\alpha| - 1\big| \text{ and } \frac{1}{\pi} |\text{arcus}\, z_\alpha| \leqq |z_\alpha - 1| \leqq \big||z_\alpha| - 1\big| + |\text{arcus}\, z_\alpha|$$

and so these convergences are equivalent to that one of $\sum_{\alpha \in I} |z_\alpha - 1|$ (use Lemma 2.3.1).

Corollary: Explicit criteria can again be obtained by applying Lemmata 2.3.1 and 2.3.2.

In what follows, the values of expressions $\prod_{\alpha \in I} z_\alpha$ will always be understood in the sense of quasi-convergence, except where the opposite is stated.

We are now able to prove (3) in 2.1:

Lemma 2.5.2. If $f_\alpha, g_\alpha \in \mathfrak{H}_\alpha$ for all $\alpha \in I$, and if $\prod_{\alpha \in I} \|f_\alpha\|$, $\prod_{\alpha \in I} \|g_\alpha\|$ are (quasi-)convergent, then $\prod_{\alpha \in I} (f_\alpha, g_\alpha)$ is quasi-convergent too.

Proof: $\prod_{\alpha \in I} (\|f_\alpha\|)^2$, $\prod_{\alpha \in I} (\|g_\alpha\|)^2$ converge along with $\prod_{\alpha \in I} \|f_\alpha\|$, $\prod_{\alpha \in I} \|g_\alpha\|$. From these we wish to derive the convergence of $\prod_{\alpha \in I} |(f_\alpha, g_\alpha)|$. Since an $\|f_\alpha\| = 0$ or $\|g_\alpha\| = 0$ implies $(f_\alpha, g_\alpha) = 0$, Lemma 2.4.1, (I), shows that we need only to derive the convergence of $\sum_{\alpha \in I} \mathrm{Max}\,(|(f_\alpha, g_\alpha)| - 1, 0)$ from those of $\sum_{\alpha \in I} \mathrm{Max}\,((\|f_\alpha\|)^2 - 1, 0)$, $\sum_{\alpha \in I} \mathrm{Max}\,((\|g_\alpha\|)^2 - 1, 0)$.

Now $|(f_\alpha, g_\alpha)| \leqq \frac{1}{2}(\|f_\alpha\|)^2 + \frac{1}{2}(\|g_\alpha\|)^2$, so

$$|(f_\alpha, g_\alpha)| - 1 \leqq \frac{1}{2}((\|f_\alpha\|)^2 - 1) + \frac{1}{2}((\|g_\alpha\|)^2 - 1)$$

and hence

$$\mathrm{Max}\,(|(f_\alpha, g_\alpha)| - 1, 0) \leqq \frac{1}{2}\mathrm{Max}\,((\|f_\alpha\|)^2 - 1, 0) + \frac{1}{2}\mathrm{Max}\,((\|g_\alpha\|)^2 - 1, 0).$$

So Lemma 2.3.1 gives the desired result.

Part II: The direct product.

Chapter 3: Construction of the complete direct product.

3.1. As in 2.1, let I be a set of indices of arbitrary size, and let for each $\alpha \in I$ a unitary space $\mathfrak{H}_\alpha$ be given.

We are now able to live up to (1) in § 2.1, and define those sequences f_α, $\alpha \in I$, for which $\prod \otimes_{\alpha \in I} f_\alpha$ will be later on (in Definition 3.1.3) defined.

After having obtained these $\prod \otimes_{\alpha \in I} f_\alpha$ we will form all their finite linear aggregates (in Definition 3.1.3) and then „complete" their space. This could be done entirely abstractly, in the manner of G. Cantor, but we prefer to use a specific representation by means of „conjugate-linear functionals" (cf. below, and later in Definition 3.5.1 and Theorem III).

Definition 3.1.1. A sequence f_α, $\alpha \in I$, is a C-*sequence* if and only if $f_\alpha \in \mathfrak{H}_\alpha$ for all $\alpha \in I$, and $\prod_{\alpha \in I} \|f_\alpha\|$ converges.

Lemma 3.1.1. If f_α, $\alpha \in I$, and g_α, $\alpha \in I$, are two C-sequences, then $\prod_{\alpha \in I} (f_\alpha, g_\alpha)$ is quasi-convergent.

Proof: Immediate by Lemma 2.5.2.

Now we define the *conjugate-linear functionals*, on which our construction of $\prod \otimes_{\alpha \in I} \mathfrak{H}_\alpha$ will be based.

We will consider functionals Φ, which have complex numerical values, and an argument f_α for each $\alpha \in I$, the domain of f_α being

$\mathfrak{H}_\alpha$. We will denote such functionals by $\Phi(f_\alpha; \alpha \in I)$. Another possible aspect would be this: The argument of Φ is the sequence f_α, $\alpha \in I$, as a whole. The Φ we shall consider, will however, be defined for C-sequences only.

Whenever it is necessary to consider a particular argument f_{α_0} (for some fixed $\alpha_0 \in I$) separately, we will write $\Phi(f_{\alpha_0} | f_\alpha; \alpha \in I, \alpha \neq \alpha_0)$ instead of $\Phi(f_\alpha; \alpha \in I)$.

Definition 3.1.2. Consider those functionals $\Phi(f_\alpha; \alpha \in I)$ which have complex numerical values, are defined for all C-sequences f_α, $\alpha \in I$ and for those only and which are conjugate-linear in each f_{α_0}, $\alpha_0 \in I$:

$$\text{(I)} \quad \Phi(zf_{\alpha_0} | f_\alpha; \alpha \in I, \alpha \neq \alpha_0) = \bar{z}\Phi(f_{\alpha_0} | f_\alpha; \alpha \in I, \alpha \neq \alpha_0),$$

$$\text{(II)} \quad \Phi(f_{\alpha_0} + g_{\alpha_0} | f_\alpha; \alpha \in I, \alpha \neq \alpha_0) = \\ = \Phi(f_{\alpha_0} | f_\alpha; \alpha \in I, \alpha \neq \alpha_0) + \Phi(g_{\alpha_0} | f_\alpha; \alpha \in I, \alpha \neq \alpha_0).$$

Denote the set of these Φ by $\Pi\otimes_{\alpha \in I} \mathfrak{H}_\alpha$.

$\Pi\otimes_{\alpha \in I} \mathfrak{H}_\alpha$ is a set of complex-valued functionals, therefore the operations $u\Phi$ (u any complex number) and $\Phi + \Psi$ have an immediate meaning for its elements Φ, Ψ. Clearly $u\Phi$, $\Phi + \Psi$ belong to $\Pi\otimes_{\alpha \in I} \mathfrak{H}_\alpha$ again (that is, they are conjugate-linear), as well as the identically vanishing functional 0. So we see: $\Pi\otimes_{\alpha \in I} \mathfrak{H}_\alpha$ is a linear space with complex coefficients.

We will now define certain special elements $\Pi\otimes_{\alpha \in I} f^0_\alpha$.

Definition 3.1.3. Given a C-sequence f^0_α, $\alpha \in I$, Lemma 2.5.2 permits us to form the functional

$$\Phi(f_\alpha; \alpha \in I) = \Pi_{\alpha \in I} (f^0_\alpha, f_\alpha),$$

where f_α, $\alpha \in I$, runs over all C-sequences, and those only. Clearly $\Phi \in \Pi\otimes_{\alpha \in I} \mathfrak{H}_\alpha$. Define

$$\Phi = \Pi\otimes_{\alpha \in I} f^0_\alpha.$$

Definition 3.1.4. Consider all finite linear aggregates of the above elements:

$$\Phi = \Sigma_{\nu=1}^{p} \Pi\otimes_{\alpha \in I} f^0_{\alpha,\nu}$$

where $p = 0, 1, 2, \ldots$ and $f^0_{\alpha,\nu}$, $\alpha \in I$, is a C-sequence for each $\nu = 1, \ldots, p$ [19]). Denote the set of these Φ by $\Pi'\otimes_{\alpha \in I} \mathfrak{H}_\alpha$.

---

[19]) Observe that $z\,\Pi\otimes_{\alpha \in I} f^0_\alpha$ (z any complex number) is again a $\Pi\otimes_{\alpha \in I} g^0_\alpha$: It suffices to put $g^0_{\alpha_0} = zf^0_{\alpha_0}$ and $g^0_\alpha = f^0_\alpha$ if $\alpha \neq \alpha_0$ for some $\alpha_0 \in I$. Thus it was unnecessary to include complex numerical coefficients in the above formula.

Clearly $\Pi'\otimes_{\alpha\in I}\mathfrak{H}_\alpha \subset \Pi\circledast_{\alpha\in I}\mathfrak{H}_\alpha$, and both sets are linear spaces with complex coefficients.

3.2. In $\Pi'\otimes_{\alpha\in I}\mathfrak{H}_\alpha$ (but not in $\Pi\circledast_{\alpha\in I}\mathfrak{H}_\alpha$!), an inner product can be defined.

LEMMA 3.2.1. If $\Phi, \Psi \in \Pi'\otimes_{\alpha\in I}\mathfrak{H}_\alpha$, that is if

$$\Phi = \Sigma_{\nu=1}^{p}\Pi\otimes_{\alpha\in I} f^0_{\alpha,\nu} \qquad \Psi = \Sigma_{\mu=1}^{q}\Pi\otimes_{\alpha\in I} g^0_{\alpha,\mu}$$

then Lemma 2.5.2 permits us to form

$$(\Phi, \Psi) = \Sigma_{\nu=1}^{p}\Sigma_{\mu=1}^{q}\Pi_{\alpha\in I}(f^0_{\alpha,\nu}, g^0_{\alpha,\mu}).$$

This expression depends on Φ, Ψ only, but not on the particular decompositions used for Φ, Ψ.

Proof: It suffices to prove that (Φ, Ψ) is unchanged, if only Φ's decomposition is changed, or if only Ψ's is changed. As $(\Phi, \Psi) = \overline{(\Psi, \Phi)}$ (for the same decompositions!), we need to consider the first case only. Instead of comparing two decompositions of Φ, we might as well compare their (formal) difference with 0. In other words: We must only prove $(\Phi, \Psi) = 0$ for $\Phi = 0$ (that is, identically $\Phi(f_\alpha; \alpha\in I) = 0$), for every possible decomposition of this Φ.

Now in this case

$$\begin{aligned}(\Phi, \Psi) &= \Sigma_{\mu=1}^{q}\{\Sigma_{\nu=1}^{p}\Pi_{\alpha\in I}(f^0_{\alpha,\nu}, g^0_{\alpha,\mu})\} = \\ &= \Sigma_{\mu=1}^{q}\{\Sigma_{\nu=1}^{p}(\Pi\otimes_{\alpha\in I} f^0_{\alpha,\nu})(g^0_{\alpha,\mu}; \alpha\in I)\} = \\ &= \Sigma_{\mu=1}^{q}\Phi(g^0_{\alpha,\mu}; \alpha\in I) = 0.\end{aligned}$$

LEMMA 3.2.2. (Φ, Ψ) is linear in Φ, conjugate-linear in Ψ, and of Hermitean symmetry in Φ, Ψ:

(1) $(u\Phi, \Psi) = u(\Phi, \Psi)$ (2) $(\Phi_1+\Phi_2, \Psi) = (\Phi_1, \Psi) + (\Phi_2, \Psi)$

(3) $(\Phi, u\Psi) = \bar{u}(\Phi, \Psi)$ (4) $(\Phi, \Psi_1+\Psi_2) = (\Phi, \Psi_1) + (\Phi, \Psi_2)$

(5) $(\Phi, \Psi) = \overline{(\Psi, \Phi)}$.

Proof: The uniqueness of (Φ, Ψ) being established, all these formulae are obvious.

LEMMA 3.2.3. $(\Phi, \Pi\otimes_{\alpha\in I} f^0_\alpha) = \Phi(f^0_\alpha; \alpha\in I)$.

Proof: Immediately by our definition of $(\ldots, \ldots)$.

3.3. Before we can continue the discussion of (Φ, Ψ) we must introduce and analyse a notion of equivalence for C-sequences.

DEFINITION 3.3.1. A sequence f_α, $\alpha\in I$, is a C_0-*sequence*, if and only if $f_\alpha \in \mathfrak{H}_\alpha$ for all $\alpha\in I$, and $\Sigma_{\alpha\in I}\big|\|f_\alpha\| - 1\big|$ converges.

LEMMA 3.3.1. Every C_0-sequence is a C-sequence, too; every C-sequence with $\Pi\otimes_{\alpha\in I} f_\alpha \neq 0$ is a C_0-sequence, too.

Proof: The first statement follows from Lemma 2.4.1, (I), the second one from Lemma 2.4.1, (II), if we replace $\Pi\otimes_{\alpha\in I} f_\alpha \neq 0$ by $\Pi_{\alpha\in I}\|f_\alpha\| \neq 0$. But the first inequality implies the second one. We argue a contrario: $\Pi_{\alpha\in I}\|f_\alpha\| = 0$ implies for every C-sequence g_α, $\alpha\in I$, that $\Pi_{\alpha\in I}\|f_\alpha\|\cdot\|g_\alpha\| = 0$. Now $|(f_\alpha, g_\alpha)| \leqq \|f_\alpha\|\cdot\|g_\alpha\|$ hence $\Pi_{\alpha\in I}|(f_\alpha, g_\alpha)| = 0$ and $\Pi_{\alpha\in I}(f_\alpha, g_\alpha) = 0$. But this means $(\Pi\otimes_{\alpha\in I} f_\alpha)(g_\alpha; \alpha\in I) = 0$ for all C-sequences g_α, $\alpha\in I$, hence $\Pi\otimes_{\alpha\in I} f_\alpha = 0$.

LEMMA 3.3.2. $\Sigma_{\alpha\in I}\big|\|f_\alpha\| - 1\big|$ converges if and only if $\Sigma_{\alpha\in I}\big|(\|f_\alpha\|)^2 - 1\big|$ converges.

Proof: In either case, $\big|\|f_\alpha\| - 1\big|$ resp. $\big|(\|f_\alpha\|)^2 - 1\big|$ being bounded, $\|f_\alpha\|$ must be bounded, say $\leqq C$. So

$$\big|\|f_\alpha\| - 1\big| \leqq \big|(\|f_\alpha\|)^2 - 1\big| \leqq (1+C)\big|\|f_\alpha\| - 1\big|$$

and thus the two convergences are equivalent. (Use Lemma 2.3.1.)

DEFINITION 3.3.2. Two C_0-sequences f_α, $\alpha\in I$ and g_α, $\alpha\in I$ are *equivalent*, in symbols $(f_\alpha; \alpha\in I) \approx (g_\alpha; \alpha\in I)$, if and only if $\Sigma_{\alpha\in I}|(f_\alpha, g_\alpha) - 1|$ converges.

LEMMA 3.3.3. The equivalence $\approx$ for C_0-sequences is reflexive, symmetric, and transitive:

(I) $(f_\alpha; \alpha\in I) \approx (f_\alpha; \alpha\in I)$,

(II) $(f_\alpha; \alpha\in I) \approx (g_\alpha; \alpha\in I)$ implies $(g_\alpha; \alpha\in I) \approx (f_\alpha; \alpha\in I)$,

(III) $(f_\alpha; \alpha\in I) \approx (g_\alpha; \alpha\in I)$, $(g_\alpha; \alpha\in I) \approx (h_\alpha; \alpha\in I)$ imply $(f_\alpha; \alpha\in I) \approx (h_\alpha; \alpha\in I)$.

Proof: Ad (I): Obvious by Lemma 3.3.2.

Ad (II): Obvious as

$$|(g_\alpha, f_\alpha) - 1| = |\overline{(f_\alpha, g_\alpha)} - 1| = |\overline{(f_\alpha, g_\alpha) - 1}| = |(f_\alpha, g_\alpha) - 1|.$$

Ad (III): We know that

$$\Sigma_{\alpha\in I}\big|\|f_\alpha\| - 1\big|,\ \Sigma_{\alpha\in I}\big|\|g_\alpha\| - 1\big|,\ \Sigma_{\alpha\in I}\big|\|h_\alpha\| - 1\big|,$$
$$\Sigma_{\alpha\in I}|(f_\alpha, g_\alpha) - 1|,\ \Sigma_{\alpha\in I}|(g_\alpha, h_\alpha) - 1|$$

are convergent, we must prove, that $\Sigma_{\alpha\in I}|(f_\alpha, h_\alpha) - 1|$ converges too.

Thus $\big|\|f_\alpha\| - 1\big|$, $\big|\|g_\alpha\| - 1\big|$, $\big|\|h_\alpha\| - 1\big|$, $|(f_\alpha, g_\alpha) - 1|$, $|(g_\alpha, h_\alpha) - 1|$ are all bounded, say $\leqq C$. We have reached our goal, if we can prove

$$\left|(f_\alpha, h_\alpha)-1\right| \leqq D\left\{\left|\|f_\alpha\|-1\right| + \left|\|g_\alpha\|-1\right| + \left|\|h_\alpha\|-1\right| + \right.$$
$$\left. + \left|(f_\alpha, g_\alpha)-1\right| + \left|(g_\alpha, h_\alpha)-1\right|\right\}$$

for some constant D, a finite number of exceptions α being permissible. (Use Lemma 2.3.1.)

Put $\|f_\alpha\| = 1+\eta$, $\|g_\alpha\| = 1+\theta$, $\|h_\alpha\| = 1+\zeta$,

$$(f_\alpha, g_\alpha) = 1+\varkappa, \quad (g_\alpha, h_\alpha) = 1+\lambda.$$

So $|\eta|$, $|\theta|$, $|\zeta|$, $|\varkappa|$, $|\lambda| \leqq C$, and except for a finite number of α's, $|\theta| \leqq \frac{1}{2}$ $\left(\text{that is } \left|\|g_\alpha\|-1\right| \leqq \frac{1}{2}\right)$. (Use Lemma 2.3.2.)

„Orthogonalise" g_α, f_α, h_α (in this order):

$$g_\alpha = a_{11}\varphi_\alpha,$$
$$f_\alpha = a_{21}\varphi_\alpha + a_{22}\varphi'_\alpha,$$
$$h_\alpha = a_{31}\varphi_\alpha + a_{32}\varphi'_\alpha + a_{33}\varphi''_\alpha,$$
$$\|\varphi_\alpha\| = \|\varphi'_\alpha\| = \|\varphi''_\alpha\| = 1, \quad (\varphi_\alpha, \varphi'_\alpha) = (\varphi_\alpha, \varphi''_\alpha) = (\varphi'_\alpha, \varphi''_\alpha) = 0.$$

Then

$$|a_{11}|^2 = (\|g_\alpha\|)^2 = (1+\theta)^2,$$
$$|a_{21}|^2 + |a_{22}|^2 = (\|f_\alpha\|)^2 = (1+\eta)^2,$$
$$|a_{31}|^2 + |a_{32}|^2 + |a_{33}|^2 = (\|h_\alpha\|)^2 = (1+\zeta)^2,$$
$$a_{21}\overline{a_{11}} = (f_\alpha, g_\alpha) = 1+\varkappa,$$
$$a_{11}\overline{a_{31}} = (g_\alpha, h_\alpha) = 1+\lambda.$$

Now

$$\left|(f_\alpha, h_\alpha)-1\right| = \left|a_{21}\overline{a_{31}} + a_{22}\overline{a_{32}} - 1\right| =$$
$$= \left|\left\{a_{21}\overline{a_{11}}\cdot a_{11}\overline{a_{31}}\cdot|a_{11}|^{-2} - 1\right\} + a_{22}\overline{a_{32}}\right| \leqq$$
$$\leqq \left|a_{21}\overline{a_{11}}\cdot a_{11}\overline{a_{31}}\cdot|a_{11}|^{-2} - 1\right| + \left|a_{22}\overline{a_{32}}\right|.$$

We will now evaluate this expression, using $|\theta| \leqq \frac{1}{2}$, and $|\eta|, \ldots, |\lambda| \leqq C$.

D_1, D_2 will be constants which depend on C. The first term is clearly

$$\leqq (1+|\varkappa|)(1+|\lambda|)(1-|\theta|)^{-2} - 1 \leqq D_1(|\theta| + |\varkappa| + |\lambda|).$$

As to the second term, observe that

$$|a_{22}|^2 = \left\{|a_{21}|^2 + |a_{22}|^2\right\} - \frac{|a_{21}\overline{a_{11}}|^2}{|a_{11}|^2} \leqq (1+|\eta|)^2 - \frac{(1-|\varkappa|)^2}{(1+|\theta|^2)},$$

$$|a_{32}|^2 \leqq \left\{|a_{31}|^2 + |a_{32}|^2 + |a_{33}|^2\right\} - \frac{|a_{11}\overline{a_{31}}|^2}{|a_{11}|^2} \leqq (1+|\zeta|)^2 - \frac{(1-|\lambda|)^2}{(1+|\theta|)^2},$$

and both expressions are

$$\leqq D_2(|\eta| + |\theta| + |\zeta| + |\varkappa| + |\lambda|).$$

Thus

$$|a_{22}\, a_{32}| \leqq D_2(|\eta| + |\theta| + |\zeta| + |\varkappa| + |\lambda|).$$

Combining, we obtain that

$$|(f_\alpha, h_\alpha) - 1| \leqq D(|\eta| + |\theta| + |\zeta| + |\varkappa| + |\lambda|)$$

with a finite number of exceptions α. But this is the desired inequality, and so the proof is completed.

Definition 3.3.3. The equivalence $\approx$ decomposes the set of all C_0-sequences into mutually disjoint equivalence-classes. (Cf. Lemma 3.3.3.) Denote the set formed by these equivalence classes by Γ, and the equivalence-class of a given C_0-sequence f_α, $\alpha \in I$, by $\mathfrak{C}(f_\alpha;\, \alpha \in I)$.

Theorem I. If two C_0-sequences f_α, $\alpha \in I$, and g_α, $\alpha \in I$, belong to two different equivalence-classes, then $(\Pi\otimes_{\alpha \in I} f_\alpha,\ \Pi\otimes_{\alpha\in I}\, g_\alpha) = 0$. If they belong to the same equivalence-class, then $(\Pi\otimes_{\alpha \in I} f_\alpha,\ \Pi\otimes_{\alpha\in I}\, g_\alpha) = 0$ if and only if some $(f_\alpha, g_\alpha) = 0$.

Proof: Clearly $(\Pi\otimes_{\alpha \in I} f_\alpha,\ \Pi\otimes_{\alpha\in I}\, g_\alpha) = \Pi_{\alpha \in I}\,(f_\alpha, g_\alpha)$ (in the sence of quasi-convergence), and so our statement coincides with that of Lemma 2.5.1.

Some additional information about $\approx$:

Lemma 3.3.4. $(f_\alpha;\, \alpha \in I) \approx (g_\alpha;\, \alpha \in I)$ if and only if both $\Sigma_{\alpha \in I}\, (\|f_\alpha - g_\alpha\|)^2$, $\Sigma_{\alpha\in I}\, |\mathfrak{J}(f_\alpha, g_\alpha)|$ converge.

Proof: In other words: These convergences are equivalent to that one of $\Sigma_{\alpha\in I}\, |(f_\alpha, g_\alpha) - 1|$. As $\Sigma_{\alpha\in I}\, |(\|f_\alpha\|)^2 - 1|$, $\Sigma_{\alpha\in I} |(\|g_\alpha\|)^2 - 1|$ are convergent by Lemma 3.3.2, we may as well compare with $\Sigma_{\alpha\in I}\, |(f_\alpha, g_\alpha) - \frac{1}{2}(\|f_\alpha\|)^2 - \frac{1}{2}(\|g_\alpha\|)^2|$.

Now

$$\mathfrak{R}\{(f_\alpha, g_\alpha) - \tfrac{1}{2}(\|f_\alpha\|)^2 - \tfrac{1}{2}(\|g_\alpha\|)^2\} =$$
$$= -\tfrac{1}{2}\{(\|f_\alpha\|)^2 + (\|g_\alpha\|)^2 - 2\mathfrak{R}(f_\alpha, g_\alpha)\} = -\tfrac{1}{2}(\|f_\alpha - g_\alpha\|)^2,$$
$$\mathfrak{J}\{(f_\alpha, g_\alpha) - \tfrac{1}{2}(\|f_\alpha\|)^2 - \tfrac{1}{2}(\|g_\alpha\|)^2\} = \mathfrak{J}(f_\alpha, g_\alpha)$$

and the convergence of $\Sigma_{\alpha\in I}\, z_\alpha$ is always equivalent to the combined ones of $\Sigma_{\alpha\in I}\, |\mathfrak{R} z_\alpha|$, $\Sigma_{\alpha\in I}\, |\mathfrak{J} z_\alpha|$ (cf. the beginning of the proof of Lemma 2.3.3). This completes the proof.

Lemma 3.3.5. $(f_\alpha;\, \alpha \in I) \approx (g_\alpha;\, \alpha\in I)$ if $f_\alpha \neq g_\alpha$ occurs for a finite number of α's only.

Proof: Clear by Lemma 3.3.4, as $f_\alpha = g_\alpha$ implies $(\|f_\alpha - g_\alpha\|)^2 = 0 = \mathfrak{J}(f_\alpha, g_\alpha)$.

The Lemma which follows deserves some attention, as it is a very characteristic consequence of our conventions attributing values to quasi-convergent (but not convergent) expressions $\Pi_{\alpha \in I} z_\alpha$.

LEMMA 3.3.6. Let the z_α be arbitrary complex numbers, and $\Pi_{\alpha \in I} z_\alpha$ quasi-convergent.

(I) If $f_\alpha, \alpha \in I$ is a C-sequence, so is $z_\alpha f_\alpha, \alpha \in I$.

(II) If $\Sigma_{\alpha \in I} \big| |z_\alpha| - 1 \big|$ converges [20]), then if $f_\alpha, \alpha \in I$, is a C_0-sequence, so is $z_\alpha f_\alpha, \alpha \in I$.

(III) We have always

$$(\dagger) \quad \Pi\otimes_{\alpha \in I} z_\alpha f_\alpha = \Pi_{\alpha \in I} z_\alpha \cdot \Pi\otimes_{\alpha \in I} f_\alpha$$

except when $\Pi_{\alpha \in I} z_\alpha$ is (quasi-convergent but) not convergent, and $\Pi\otimes_{\alpha \in I} f_\alpha \neq 0$. In this case z_α, f_α fulfill the assumptions of (II), and all $z_\alpha \neq 0$.

(IV) If z_α, f_α fulfill the assumptions of (II), then the C_0-sequences $f_\alpha, \alpha \in I$, and $z_\alpha f_\alpha, \alpha \in I$, are equivalent if and only if $\Sigma_{\alpha \in I} |z_\alpha - 1|$ converges. If all $z_\alpha \neq 0$, then this is equivalent to the convergence of $\Pi_{\alpha \in I} z_\alpha$ (beyond mere quasi-convergence).

Remark: Combining (III), (IV) with Theorem I shows, that whenever $(\dagger)$ fails to hold, we have

$$(\Pi\otimes_{\alpha \in I} f_\alpha, \ \Pi\otimes_{\alpha \in I} z_\alpha f_\alpha) = 0 .$$

Proof: Ad (I): As $\Pi_{\alpha \in I} |z_\alpha|$, $\Pi_{\alpha \in I} \|f_\alpha\|$ converges, so does

$$\Pi_{\alpha \in I} \|z_\alpha f_\alpha\| = \Pi_{\alpha \in I} |z_\alpha| \cdot \Pi_{\alpha \in I} \|f_\alpha\| .$$

Ad (II): $\Sigma_{\alpha \in I} \big| |z_\alpha| - 1 \big|$, $\Sigma_{\alpha \in I} \big| \|f_\alpha\| - 1 \big|$ converge, therefore $\big| |z_\alpha| - 1 \big|$ is bounded, and with it $|z_\alpha|$, say $|z_\alpha| \leqq C$. Now

$$\big| \|z_\alpha f_\alpha\| - 1 \big| = \big| |z_\alpha| \cdot \|f_\alpha\| - 1 \big| = \big| (|z_\alpha| - 1) + |z_\alpha| (\|f_\alpha\| - 1) \big| \leqq$$
$$\leqq \big| |z_\alpha| - 1 \big| + C \big| \|f_\alpha\| - 1 \big|$$

and thus $\Sigma_{\alpha \in I} \big| \|z_\alpha f_\alpha\| - 1 \big|$ converges. (Use Lemma 2.3.1.)

Ad (III): $(\dagger)$ means, that $(\Pi\otimes_{\alpha \in I} z_\alpha f_\alpha)(g_\alpha; \alpha \in I) = \Pi_{\alpha \in I} z_\alpha \cdot (\Pi\otimes_{\alpha \in I} f_\alpha)(g_\alpha; \alpha \in I)$ for all C-sequences $g_\alpha, \alpha \in I$, that is $\Pi_{\alpha \in I} (z_\alpha f_\alpha, g_\alpha) = \Pi_{\alpha \in I} z_\alpha \cdot \Pi_{\alpha \in I} (f_\alpha, g_\alpha)$. Put $z'_\alpha = (f_\alpha, g_\alpha)$, then this formula becomes

$$\Pi_{\alpha \in I} z_\alpha z'_\alpha = \Pi_{\alpha \in I} z_\alpha \cdot \Pi_{\alpha \in I} z'_\alpha .$$

[20]) By Lemma 2.4.1, (II), this is certainly so, if $\Pi_{\alpha \in I} |z_\alpha| \neq 0$. A fortiori, if $\Pi_{\alpha \in I} z_\alpha$ does not converge or is $\neq 0$.

Now it is easy to verify, that this formula holds, if one of the two factors on the right side is convergent, while the other need only be quasi-convergent [21]). (Use Lemma 2.4.2.) Thus in case (†) fails, both $\Pi_{\alpha\in I} z_\alpha$ and $\Pi_{\alpha\in I} z'_\alpha$ are not convergent. This excludes (by Lemma 2.4.2, (I)) $\Pi_{\alpha\in I}|z'_\alpha| = 0$. As $|z'_\alpha| = |(f_\alpha, g_\alpha)| \leqq \|f_\alpha\|\,\|g_\alpha\|$ and as $\Pi_{\alpha\in I}\|f_\alpha\|$, $\Pi_{\alpha\in I}\|g_\alpha\|$ converge, it excludes $\Pi_{\alpha\in I}\|f_\alpha\| = 0$ too, and $\Pi_{\alpha\in I}(\|f_\alpha\|)^2 = 0$ too. Now $(\Pi\otimes_{\alpha\in I} f_\alpha)(f_\alpha;\alpha\in I) = \Pi_{\alpha\in I}(f_\alpha, f_\alpha) = \Pi_{\alpha\in I}(\|f_\alpha\|)^2 \neq 0$, $\Pi\otimes_{\alpha\in I} f_\alpha \neq 0$.

So $\Pi_{\alpha\in I} z_\alpha$ is not convergent, and $\Pi\otimes_{\alpha\in I} f_\alpha \neq 0$ if (†) fails. This implies $\Sigma_{\alpha\in I}|z_\alpha| \neq 0$ (cf. Lemma 2.4.2, (I)), and so the convergence of $\Sigma_{\alpha\in I}\big||z_\alpha| - 1\big|$ (cf. [20])), together with $z_\alpha \neq 0$; and further the C_0-sequence character of f_α, $\alpha\in I$ (cf. Lemma 3.3.1). Thus z_α, f_α fulfill the assumptions of (II).

Ad (IV): $\Sigma_{\alpha\in I}\big||z_\alpha| - 1\big|$, $\Sigma_{\alpha\in I}\big|\|f_\alpha\| - 1\big|$ converge by assumption. Equivalence of f_α, $\alpha\in I$, and $z_\alpha f_\alpha$, $\alpha\in I$, means that $\Sigma_{\alpha\in I}|(z_\alpha f_\alpha, f_\alpha) - 1| = \Sigma_{\alpha\in I}|z_\alpha(\|f_\alpha\|)^2 - 1|$ converges. Now $\big||z_\alpha| - 1\big|$ is bounded, so $|z_\alpha|$ is too, say $\leqq C$. Thus

$$\big||z_\alpha - 1| - |z_\alpha(\|f_\alpha\|)^2 - 1|\big| \leqq |(z_\alpha - 1) - (z_\alpha(\|f_\alpha\|)^2 - 1)| =$$
$$= |z_\alpha((\|f_\alpha\|)^2 - 1)| \leqq C|(\|f_\alpha\|)^2 - 1|,$$

therefore $\Sigma_{\alpha\in I}\big||z_\alpha - 1| - |z_\alpha(\|f_\alpha\|)^2 - 1|\big|$ converges (use Lemma 2.3.1), and with it $\Sigma_{\alpha\in I}(|z_\alpha - 1| - |z_\alpha(\|f_\alpha\|)^2 - 1|)$. (Use Lemma 2.3.3.) Thus the convergence of $\Sigma_{\alpha\in I}|z_\alpha(\|f_\alpha\|)^2 - 1|$ is equivalent to the one of $\Sigma_{\alpha\in I}|z_\alpha - 1|$. This proves the first part of (IV).

Make now the additional assumption that all $z_\alpha \neq 0$. The convergence of $\Pi_{\alpha\in I} z_\alpha$ is equivalent to the one of $\Sigma_{\alpha\in I}|\text{arcus}\, z_\alpha|$ (use Lemma 2.4.2, (II), we have $\Pi_{\alpha\in I}|z_\alpha| \neq 0$ by Lemma 2.4.1, (II)), and this is equivalent to the convergence of $\Sigma_{\alpha\in I}|z_\alpha - 1|$, because of

$$\frac{1}{\pi}|\text{arcus}\, z_\alpha| \leqq |z_\alpha - 1| \leqq \big||z_\alpha| - 1\big| + |\text{arcus}\, z_\alpha|.$$

We have $\Pi_{\alpha\in I}|z_\alpha| \neq 0$ (by Lemma 2.4.1, (II)), and so if $\Pi_{\alpha\in I} z_\alpha$ converges, it is necessarily $\neq 0$ too (by Lemma 2.4.2) [22]). Thus the convergence of $\Pi_{\alpha\in I} z_\alpha$ can be characterised by Lemma 2.5.1: It is equivalent (as all $z_\alpha \neq 0$) to the convergence of $\Sigma_{\alpha\in I}|z_\alpha - 1|$. This proves the second part of (IV).

We will see the effect of this Lemma later in § 6.2. This is an inference, which could have been obtained directly, too:

---

[21]) If both are quasi-convergent, it may not hold: Put $z_\alpha = e^{i\theta_\alpha}$, $z'_\alpha = e^{-i\theta_\alpha}$, where $-\pi \leqq \theta_\alpha < \pi$, and $\Sigma_{\alpha\in I}|\theta_\alpha|$ does not converge.

[22]) Mere quasi-convergence would not imply this!

LEMMA 3.3.7. Each equivalence-class $\mathfrak{C}$ contains a (C_0-)sequence f_α, $\alpha \in I$, with $\|f_\alpha\| = 1$ for all $\alpha \in I$.

Proof: Choose $(f'_\alpha;\ \alpha \in I) \in \mathfrak{C}$. As $\sum_{\alpha \in I} \big| \|f'_\alpha\| - 1 \big|$ converges, therefore we have, except for a finite number of α's, $\big| \|f'_\alpha\| - 1 \big| \leqq \frac{1}{2}$, $\|f'_\alpha\| \geqq \frac{1}{2}$. For these exceptional α's we may, however, replace f'_α by any f''_α with $\|f''_\alpha\| \geqq \frac{1}{2}$. (Use Lemma 3.3.5.) So we may assume, that all $\|f'_\alpha\| \geqq \frac{1}{2}$.

Now $\left| \frac{1}{\|f'_\alpha\|} - 1 \right| \leqq 2 \big| \|f'_\alpha\| - 1 \big|$, so $\sum_{\alpha \varepsilon I} \left| \frac{1}{\|f'_\alpha\|} - 1 \right|$ converges, (Use Lemma 2.3.1.) Therefore Lemma 3.3.6, (II) and (IV) apply to $z_\alpha = |z_\alpha| = \frac{1}{\|f'_\alpha\|}$ and our f'_α: The $f_\alpha = \frac{1}{\|f'_\alpha\|} f'_\alpha$ form a C_0-sequence too, and one which is equivalent to f'_α, $\alpha \in I$, thus belonging to $\mathfrak{C}$. $\|f_\alpha\| = 1$ is obvious, therefore the proof is completed.

3.4. LEMMA 3.4.1. $(\Phi, \Phi) \geqq 0$.

Proof: We have $\Phi \in \Pi' \otimes_{\alpha \in I} \mathfrak{H}_\alpha$, so

$$\Phi = \sum_{\nu=1}^{p} \Pi \otimes_{\alpha \in I} f^0_{\alpha,\nu}.$$

Every sequence $f^0_{\alpha,\nu}$, $\alpha \in I$ is a C-sequence. For each one which is not a C_0-sequence, Lemma 3.3.1 gives $\Pi \otimes_{\alpha \in I} f^0_{\alpha,\nu} = 0$, so we can omit all such terms. We may therefore assume, that all $f_{\alpha,\nu}$, $\alpha \in I$, are C_0-sequences. Denote the equivalence-class of $f^0_{\alpha,\nu}$, $\alpha \in I$, by $\mathfrak{C}_\nu$. Denote the different ones among the $\mathfrak{C}_1, \ldots, \mathfrak{C}_p$ by $\mathfrak{D}_1, \ldots, \mathfrak{D}_q$ (clearly $q = 1, \ldots, p$). Let N_i be the set of those $\nu = 1, \ldots, p$, for which $\mathfrak{C}_\nu = \mathfrak{D}_i$ $(i = 1, \ldots, q)$.

Now put

$$\Phi_i = \sum_{\nu \in N_i} \Pi \otimes_{\alpha \in I} f^0_{\alpha,\nu},$$

then $\Phi = \sum_{i=1}^{q} \Phi_i$. If $i \neq j$, then $\mathfrak{D}_i \neq \mathfrak{D}_j$, so $\nu \in N_i$, $\mu \in N_j$ imply $\mathfrak{C}_\nu \neq \mathfrak{C}_\mu$, $(f^0_{\alpha,\nu};\ \alpha \in I) \not\approx (f^0_{\alpha,\mu};\ \alpha \in I)$. Theorem I gives therefore $(\Pi \otimes_{\alpha \in I} f^0_{\alpha,\nu}, \Pi \otimes_{\alpha \in I} f^0_{\alpha,\mu}) = 0$, and thus $(\Phi_i, \Phi_j) = 0$. So we have $(\Phi, \Phi) = (\sum_{i=1}^{q} \Phi_i, \sum_{j=1}^{q} \Phi_j) = \sum_{i,j=1}^{q} (\Phi_i, \Phi_j) = \sum_{i=1}^{q} (\Phi_i, \Phi_i)$. Therefore $(\Phi, \Phi) \geqq 0$ would follow, if we proved $(\Phi_i, \Phi_i) \geqq 0$ for all $i = 1, \ldots, q$. Writing again Φ for Φ_i, p for q, $1, \ldots, p$ for the elements of N_i, and $\mathfrak{C}$ for $\mathfrak{C}_i$, we see: We need to prove $(\Phi, \Phi) \geqq 0$ only for the case where all sequences $f^0_{\alpha,\nu}$, $\alpha \in I$ belong to the same equivalence-class $\mathfrak{C}$. That is: For all $\mu, \nu = 1, \ldots, p$ $(f^0_{\alpha,\nu};\ \alpha \in I) \approx (f^0_{\alpha,\mu};\ \alpha \in I)$, $\sum_{\alpha \in I} \big| (f^0_{\alpha,\nu}, f^0_{\alpha,\mu}) - 1 \big|$ convergent. Thus $\Pi_{\alpha \in I} (f^0_{\alpha,\nu}, f^0_{\alpha,\mu})$ is convergent (by Lemma 2.4.1, and not merely quasi-convergent).

But

$$(\Phi, \Phi) = \Sigma_{\nu,\mu=1}^{p} \Pi_{\alpha \in I} (f_{\alpha,\nu}^{0}, f_{\alpha,\mu}^{0}) \, .$$

Each term is convergent, therefore this will certainly be $\geqq 0$ if we can show

$$\Sigma_{\nu,\mu=1}^{p} (f_{\alpha_1,\nu}^{0}, f_{\alpha_1,\mu}^{0}) \cdot \ldots \cdot (f_{\alpha_s,\nu}^{0}, f_{\alpha_s,\mu}^{0}) \geqq 0$$

for every set of mutually different $\alpha_1, \ldots, \alpha_s$.

Put $(f_{\alpha,\nu}^{0}, f_{\alpha,\mu}^{0}) = a_{\nu\mu}^{\alpha}$. Then for any (complex) $x_1, \ldots, x_p$

$$\Sigma_{\nu,\mu=1}^{p} a_{\nu\mu}^{\alpha} x_\nu \bar{x}_\mu = \Sigma_{\nu,\mu=1}^{p} (f_{\alpha,\nu}^{0}, f_{\alpha,\mu}^{0}) x_\nu \bar{x}_\mu =$$
$$= (\Sigma_{\nu=1}^{p} x_\nu f_{\alpha,\nu}^{0}, \Sigma_{\mu=1}^{p} x_\mu f_{\alpha,\mu}^{0}) = (\|\Sigma_{\nu=1}^{p} x_\nu f_{\alpha,\nu}^{0}\|)^2 \geqq 0 \, .$$

So the matrix $(a_{\nu\mu}^{\alpha})_{\nu,\mu=1,\ldots,p}$ is semi-definite for each $\alpha \in I$. Therefore it is the sum of p semi-definite matrices of rank 1 [23]), that is of p terms of the form $u_\nu^{\alpha} \overline{u_\mu^{\alpha}}$. Thus

$$\Sigma_{\nu,\mu=1}^{p} (f_{\alpha_1,\nu}^{0}, f_{\alpha_1,\mu}^{0}) \cdot \ldots \cdot (f_{\alpha_s,\nu}^{0}, f_{\alpha_s,\mu}^{0}) = \Sigma_{\nu,\mu=1}^{p} a_{\nu\mu}^{\alpha_1} \cdot \ldots \cdot a_{\nu\mu}^{\alpha_s}$$

is a sum of p^s terms of the form

$$\Sigma_{\nu,\mu=1}^{p} u_\nu^{\alpha_1} \cdot \overline{u_\mu^{\alpha_1}} \cdot \ldots \cdot u_\nu^{\alpha_s} \cdot \overline{u_\mu^{\alpha_s}} = \Sigma_{\nu,\mu=1}^{p} u_\nu^{\alpha_1} \cdot \ldots \cdot u_\nu^{\alpha_s} \cdot \overline{u_\mu^{\alpha_1} \cdot \ldots \cdot u_\mu^{\alpha_s}} =$$
$$= \Sigma_{\nu=1}^{p} u_\nu^{\alpha_1} \cdot \ldots \cdot u_\nu^{\alpha_s} \cdot \overline{\Sigma_{\mu=1}^{p} u_\mu^{\alpha_1} \cdot \ldots \cdot u_\mu^{\alpha_s}} = \left|\Sigma_{\nu=1}^{p} u_\nu^{\alpha_1} \cdot \ldots \cdot u_\nu^{\alpha_s}\right|^2 \geqq 0 \, .$$

Therefore it is $\geqq 0$ itself, and the proof is completed.

DEFINITION 3.4.1. Define $\|\Phi\| = \sqrt{(\Phi, \Phi)} \geqq 0$ $((\Phi, \Phi) \geqq 0$ by Lemma 3.4.1$)$.

LEMMA 3.4.2. $|(\Phi, \Psi)| \leqq \|\Phi\| \cdot \|\Psi\|$. (Schwarz's inequality.)

Proof: Use Lemmata 3.2.2, 3.4.1: For any two real x, y

$$0 \leqq (x\Phi + y\Psi, x\Phi + y\Psi) =$$
$$= x^2(\Phi, \Phi) + y^2(\Psi, \Psi) + xy((\Phi, \Psi) + (\Psi, \Phi)) =$$
$$= x^2\|\Phi\|^2 + y^2\|\Psi\|^2 + 2xy\Re(\Phi, \Psi) ,$$

therefore this polynomial in x, y has a non-negative discriminant: $|\Re(\Phi, \Psi)| \leqq \|\Phi\| \cdot \|\Psi\|$. Replace Φ by $e^{-i\theta}\Phi$, θ real. Then $\|\Phi\| = \sqrt{(\Phi, \Phi)}$ is unchanged, while $|\Re(\Phi, \Psi)|$ becomes $|\Re\{e^{-i\theta}(\Phi, \Psi)\}|$. (Use Lemma 3.2.2.) Put $\theta = \text{arcus}\,(\Phi, \Psi)$, then $|(\Phi, \Psi)| \leqq \|\Phi\| \cdot \|\Psi\|$ results.

---

[23]) This statement is orthogonal-invariant; therefore it suffices to verify it when $a_{\nu\mu}^{\alpha}$ has the diagonal form: $a_{\nu\mu}^{\alpha} = a_\nu^{\alpha} \delta_{\nu-\mu}$. ($\delta_\varrho = 1$ or 0 if $\varrho = 0$ resp. $\neq 0$). The semi-definiteness implies $a_\nu^{\alpha} \geqq 0$. Now $a_{\nu\mu}^{\alpha} = \Sigma_{\sigma=1}^{p} a_{\sigma;\nu\mu}^{\alpha}$, the $a_{\sigma;\nu\mu}^{\alpha} = a_\sigma^{\alpha} \delta_{\nu-\sigma} \delta_{\mu-\sigma}$ being the desired matrices.

LEMMA 3.4.3. $\Phi \neq 0$ implies $\|\Phi\|^2 = (\Phi, \Phi) > 0$.

Proof: By Lemma 3.4.1 always $\|\Phi\|^2 = (\Phi, \Phi) \geqq 0$, so we must only infer $\Phi = 0$ from $\|\Phi\| = 0$.

Now $\|\Phi\| = 0$ implies $|(\Phi, \Psi)| \leqq \|\Phi\| \cdot \|\Psi\| = 0$, $(\Phi, \Psi) = 0$ for all Ψ by Lemma 3.4.2. Put $\Psi = \Pi\otimes_{\alpha\in I} f_\alpha$, then this becomes $\Phi(f_\alpha;\ \alpha\in I) = 0$ by Lemma 3.2.3. As f_α, $\alpha \in I$, was an arbitrary C-sequence, necessarily $\Phi = 0$.

Theorem II. With the (Φ, Ψ) of Lemma 3.2.1, $\Pi'\otimes_{\alpha\in I} \mathfrak{H}_\alpha$ is a (complex) linear space with a (Hermitean and definite) linear inner product, that is, it satisfies the conditions **A**, **B** of (8), p. 64.

Thus it can be metrised by defining:

Distance $(\Phi, \Psi) = \|\Phi - \Psi\|$, where $\|\Phi\| = \sqrt{(\Phi, \Phi)} \geqq 0$ (cf. Definition 3.4.1).

Proof: This follows from Lemmata 3.2.2, 3.4.3, remembering (8), pp. 64—65.

$\Pi\otimes_{\alpha\in I} \mathfrak{H}_\alpha$ is not necessarily complete (condition **E** of (8), p. 66), and this prescribes the course of our further constructions.

3.5. LEMMA 3.5.1. We have $\|\Pi\otimes_{\alpha\in I} f^0_\alpha\| = \Pi_{\alpha\in I} \|f^0_\alpha\|$ and for every Φ $(\in \Pi\otimes_{\alpha\in I} \mathfrak{H}_\alpha)$

$$|\Phi(f_\alpha;\ \alpha\in I)| \leqq \|\Phi\|\ \Pi_{\alpha\in I}\|f^0_\alpha\|.$$

Proof: By definition

$$\|\Pi\otimes_{\alpha\in I} f^0_\alpha\|^2 = (\Pi\otimes_{\alpha\in I} f^0_\alpha, \Pi\otimes_{\alpha\in I} f^0_\alpha) = \Pi_{\alpha\in I}(f^0_\alpha, f^0_\alpha) = \Pi_{\alpha\in I}\left(\|f^0_\alpha\|\right)^2 = \left(\Pi_{\alpha\in I}\|f^0_\alpha\|\right)^2,$$

$$\|\Pi\otimes_{\alpha\in I} f^0_\alpha\| = \Pi_{\alpha\in I}\|f^0_\alpha\|.$$

Now Lemmata 3.2.3 and 3.4.2 (Schwarz's inequality) give

$$|\Phi(f^0_\alpha;\ \alpha\in I)| = |(\Phi, \Pi\otimes_{\alpha\in I} f^0_\alpha)| \leqq \|\Phi\| \cdot \|\Pi\otimes_{\alpha\in I} f^0_\alpha\| = \|\Phi\| \cdot \Pi_{\alpha\in I}\|f^0_\alpha\|.$$

DEFINITION 3.5.1. Consider those functions $\Phi \in \Pi\otimes_{\alpha\in I} \mathfrak{H}_\alpha$ for which a sequence $\Phi_1, \Phi_2, \ldots \in \Pi'\otimes_{\alpha\in I} \mathfrak{H}_\alpha$ exists, such that

(I) $\Phi(f_\alpha;\ \alpha\in I) = \lim\limits_{r\to\infty} \Phi_r(f_\alpha;\ \alpha\in I)$ for all C-sequences f_α, $\alpha\in I$,

(II) $\lim\limits_{r,s\to\infty} \|\Phi_q - \Phi_s\| = 0$.

The set they form is the *complete direct product of the* $\mathfrak{H}_\alpha$, $\alpha\in I$, to be denoted by $\Pi\otimes_{\alpha\in I} \mathfrak{H}_\alpha$. The relations (I), (II) will be denoted by $\Phi = \underset{r\to\infty}{\mathrm{L}}\ \Phi_r$.

LEMMA 3.5.2. If a sequence $\Phi_1, \Phi_2, \ldots \in \Pi'\otimes_{\alpha\in I} \mathfrak{H}_\alpha$ satisfies condition (II) in Definition 3.5.1, then there exists exactly one $\Phi \in \Pi \otimes_{\alpha \in I} \mathfrak{H}_\alpha$ with $\Phi = \underset{r \to \infty}{\mathrm{L}} \Phi_r$.

If $\Phi = \underset{r \to \infty}{\mathrm{L}} \Phi_r$, then all sequences $\Psi_1, \Psi_2, \ldots \in \Pi \otimes_{\alpha \in I} \mathfrak{H}_\alpha$ with $\Phi = \underset{r \to \infty}{\mathrm{L}} \Psi_r$ are characterised by $\lim_{r \to \infty} \|\Phi_r - \Psi_r\| = 0$.

Proof: We have (by Lemma 3.5.1)

$$\left|\Phi_r(f_\alpha; \alpha \in I) - \Phi_s(f_\alpha; \alpha \in I)\right| \leqq \|\Phi_r - \Phi_s\| \cdot \Pi_{\alpha \in I} \|f_\alpha\|$$

for every C-sequence f_α. So (II) gives

$$\lim_{r,s \to \infty} \left|\Phi_r(f_\alpha; \alpha \in I) - \Phi_s(f_\alpha; \alpha \in I)\right| = 0,$$

and thus the (numerical) $\lim_{r \to \infty} \Phi_r(f_\alpha; \alpha \in I)$ exists. Denote it by $\Phi(f_\alpha; \alpha \in I)$. Clearly Φ is a functional $\in \Pi \otimes_{\alpha \in I} \mathfrak{H}_\alpha$. By construction (I), (II) hold, so $\Phi = \underset{r \to \infty}{\mathrm{L}} \Phi_r$. Thus $\Phi \in \Pi \otimes_{\alpha \in I} \mathfrak{H}_\alpha$, and Φ is unique by (I).

As to the second statement, we may replace Φ, Φ_r, Ψ_r by 0, $\Phi_r - \Psi_r$, 0. So we may assume $\Phi = \Psi_r = 0$. Thus we must prove, that

(I)′ $\lim_{r \to \infty} \Phi_r(f_\alpha; \alpha \in I) = 0$ for all C-sequences f_α, $\alpha \in I$,

(II)′ $\lim_{r,s \to \infty} \|\Phi_r - \Phi_s\| = 0$

are equivalent to $\lim_{r \to \infty} \|\Phi_r\| = 0$.

Sufficiency: If $\lim_{r \to \infty} \|\Phi_r\| = 0$, then $\left|\Phi_r(f_\alpha; \alpha \in I)\right| \leqq \|\Phi_r\| \cdot \Pi_{\alpha \in I} \|f_\alpha\|$ (use Lemma 3.5.1) gives (I)′, and $\|\Phi_r - \Phi_s\| \leqq \|\Phi_r\| + \|\Phi_s\|$ gives (II)′.

Necessity: Assume (I)′, (II)′, and the invalidity of $\lim_{r \to \infty} \|\Phi_r\| = 0$. Then there would be $\|\Phi_r\| \geqq a$ for a fixed $a > 0$ and infinitely many r's. Now (I)′ means $\lim_{r \to \infty} (\Phi_r, \Pi \otimes_{\alpha \in I} f_\alpha) = 0$ (use Lemma 3.2.3), and so $\lim_{r \to \infty} (\Phi_r, \Omega) = 0$ for all $\Omega \in \Pi' \otimes_{\alpha \in I} \mathfrak{H}_\alpha$. Now choose an r_0 with $\|\Phi_{r_0}\| \geqq a$ so great, that $r, s \geqq r_0$ imply $\|\Phi_r - \Phi_s\| \leqq \frac{a}{2}$ (use (II)′), and put $\Omega = \Phi_{r_0}$. Then we have for $r \geqq r_0$

$$\begin{aligned}\left|(\Phi_r, \Phi_{r_0})\right| &\geqq \left|(\Phi_{r_0}, \Phi_{r_0})\right| - \left|(\Phi_r - \Phi_{r_0}, \Phi_{r_0})\right| \geqq \\ &\geqq \|\Phi_{r_0}\|^2 - \|\Phi_r - \Phi_{r_0}\| \, \|\Phi_{r_0}\| = \\ &= \|\Phi_{r_0}\| \left(\|\Phi_{r_0}\| - \|\Phi_r - \Phi_{r_0}\|\right) \geqq a\left(a - \frac{a}{2}\right) = \frac{a^2}{2} > 0\end{aligned}$$

contradicting $\lim_{r \to \infty} (\Phi_r, \Phi_{r_0}) = 0$.

LEMMA 3.5.3. If $\Phi, \Psi \in \Pi\otimes_{\alpha \in I} \mathfrak{H}_\alpha$, that is

$$\Phi = \underset{r \to \infty}{\mathrm{L}} \Phi_r, \quad \Psi = \underset{r \to \infty}{\mathrm{L}} \Psi_r, \quad \Phi_r, \Psi_r \in \Pi'\otimes_{\alpha \in I} \mathfrak{H}_\alpha,$$

then $\lim_{r \to \infty} (\Phi_r, \Psi_r)$ exists. Denote it by (Φ, Ψ). This quantity depends on Φ, Ψ only, and not on the particular representation used. If $\Phi, \Psi \in \Pi'\otimes_{\alpha \in I} \mathfrak{H}_\alpha$, then it agrees with the previous definition.

Proof: We have $\lim_{r, s \to \infty} \|\Phi_r - \Phi_s\| = 0$, $\lim_{r, s \to \infty} \|\Psi_r - \Psi_s\| = 0$, these imply by Schwarz's inequality $\lim_{r, s \to \infty} |(\Phi_r, \Psi_r) - (\Phi_s, \Psi_s)| = 0$, hence $\lim_{r \to \infty} (\Phi_r, \Psi_r)$ exists. If further $\Phi = \underset{r \to \infty}{\mathrm{L}} \Phi'_r$, $\Psi = \underset{r \to \infty}{\mathrm{L}} \Psi'_r$ then we have $\lim_{r \to \infty} \|\Phi_r - \Phi'_r\| = 0$, $\lim_{r \to \infty} \|\Psi_r - \Psi'_r\| = 0$ (use Lemma 3.5.2), and Schwarz's inequality gives $\lim_{r \to \infty} |(\Phi_r, \Psi_r) - (\Phi'_r, \Psi'_r)| = 0$, hence $\lim_{r \to \infty} (\Phi_r, \Psi_r) = \lim_{r \to \infty} (\Phi'_r, \Psi'_r)$. Thus $\lim_{r \to \infty} (\Phi_r, \Psi_r)$ depends on Φ, Ψ only.

If $\Phi, \Psi \in \Pi'\otimes_{\alpha \in I} \mathfrak{H}_\alpha$, then we may put all $\Phi_r = \Phi$, $\Psi_r = \Psi$ which makes it clear, that the new (Φ, Ψ) agrees with the old one.

LEMMA 3.5.4. In $\Pi\otimes_{\alpha \in I} \mathfrak{H}_\alpha$, (Φ, Ψ) is linear in Φ, conjugate-linear in Ψ, of Hermitean symmetry in Φ, Ψ, and definite. Together with $\|\Phi\| = \sqrt{(\Phi, \Phi)} \geqq 0$ it fulfills Schwarz's inequality. (Cf. Lemmata 3.2.2, 3.4.3, 3.4.2, where the corresponding statements are made for $\Pi'\otimes_{\alpha \in I} \mathfrak{H}_\alpha$.)

Proof: All these properties, except definiteness, follow by continuity from Lemmata 3.2.2, 3.4.2.

$(\Phi, \Phi) \geqq 0$ follows by continuity from Lemma 3.4.1. If $(\Phi, \Phi) = 0$, then choose $\Phi_1, \Phi_2, \ldots \in \Pi'\otimes_{\alpha \in I} \mathfrak{H}_\alpha$ with $\Phi = \underset{r \to \infty}{\mathrm{L}} \Phi_r$. Then $\lim_{r \to \infty} \|\Phi_r\|^2 = \lim_{r \to \infty} (\Phi_r, \Phi_r) = (\Phi, \Phi) = 0$, $\lim_{r \to \infty} \|\Phi_r\| = 0$. Thus Lemma 3.5.2 gives $\Phi = \underset{r \to \infty}{\mathrm{L}} 0 = 0$. So $\Phi \neq 0$ implies $(\Phi, \Phi) > 0$, proving the definiteness.

LEMMA 3.5.5. With the (Φ, Ψ) of Lemma 3.5.3 $\Pi\otimes_{\alpha \in I} \mathfrak{H}_\alpha$ is a (complex) linear space with a (Hermitean and definite) linear inner product, that is, it satisfies the conditions **A**, **B** of (8), p. 64.

Thus it can be metrised by defining:

Distance $(\Phi, \Psi) = \|\Phi - \Psi\|$, where $\|\Phi\| = \sqrt{(\Phi, \Phi)} \geqq 0$ (cf. Lemma 3.5.4).

$\Pi'\otimes_{\alpha \in I} \mathfrak{H}_\alpha$, as described in Theorem II, is a linear subset of

$\Pi \otimes_{\alpha \in I} \mathfrak{H}_\alpha$, with the same definitions of $u\Phi$, $\Phi \pm \Psi$, (Φ, Ψ), $\|\Phi\|$.

Proof: The first and the second part follow from Lemma 3.5.4, remembering (8), pp. 64—65; the last part follows from Lemma 3.5.3.

Lemma 3.5.6. Lemmata 3.2.3 and 3.5.1 hold for all $\Phi \in \Pi \otimes_{\alpha \in I} \mathfrak{H}_\alpha$.

Proof: They hold for all elements of $\Pi' \otimes_{\alpha \in I} \mathfrak{H}_\alpha$. So if $\Phi = \underset{r \to \infty}{\mathrm{L}} \Phi_r$, $\Phi_1, \Phi_2, \ldots \in \Pi' \otimes_{\alpha \in I} \mathfrak{H}_\alpha$, then they hold for all Φ_r. Now Definition 3.5.1, (I), Lemma 3.5.3, and $\|\Phi\| = \sqrt{(\Phi, \Phi)} = \sqrt{\lim_{r \to \infty} (\Phi_r, \Phi_r)} = \lim_{r \to \infty} \sqrt{(\Phi_r, \Phi_r)} = \lim_{r \to \infty} \|\Phi_r\|$ extend them by continuity to Φ.

Lemma 3.5.7. If $\Phi \in \Pi \otimes_{\alpha \in I} \mathfrak{H}_\alpha$ and $\Phi_1, \Phi_2, \ldots \in \Pi' \otimes_{\alpha \in I} \mathfrak{H}_\alpha$, then $\Phi = \underset{r \to \infty}{\mathrm{L}} \Phi_r$ is equivalent to $\lim_{r \to \infty} \|\Phi - \Phi_r\| = 0$.

Proof: Necessity: $\Phi = \underset{r \to \infty}{\mathrm{L}} \Phi_r$ implies $\|\Phi - \Phi_s\| = \sqrt{(\Phi - \Phi_s, \Phi - \Phi_s)} = \sqrt{\lim_{r \to \infty} (\Phi_r - \Phi_s, \Phi_r - \Phi_s)} = \lim_{r \to \infty} \sqrt{(\Phi_r - \Phi_s, \Phi_r - \Phi_s)} = \lim_{r \to \infty} \|\Phi_r - \Phi_s\|$, hence $\lim_{r \to \infty} \|\Phi - \Phi_s\| = \lim_{r, s \to \infty} \|\Phi_r - \Phi_s\| = 0$ (by Definition 3.5.1, (II)).

Sufficiency: $\lim_{r \to \infty} \|\Phi - \Phi_r\| = 0$ implies Definition 3.5.1, (I), by Lemma 3.5.6 (3.5.1), and (II) eod. by

$$\|\Phi_r - \Phi_s\| = \|-(\Phi - \Phi_r) + (\Phi - \Phi_s)\| \leqq \|\Phi - \Phi_r\| + \|\Phi - \Phi_s\|.$$

Lemma 3.5.8. $\Pi' \otimes_{\alpha \in I} \mathfrak{H}_\alpha$ is everywhere dense in $\Pi \otimes_{\alpha \in I} \mathfrak{H}_\alpha$.

Proof: If $\Phi \in \Pi \otimes_{\alpha \in I} \mathfrak{H}_\alpha$, then $\Phi = \underset{r \to \infty}{\mathrm{L}} \Phi_r$, $\Phi_r \in \Pi' \otimes_{\alpha \in I} \mathfrak{H}_\alpha$. By Lemma 3.5.7 $\lim_{r \to \infty} \|\Phi - \Phi_r\| = 0$, so Φ is a limit-point of $\Pi' \otimes_{\alpha \in I} \mathfrak{H}_\alpha$.

Lemma 3.5.9. $\Pi \otimes_{\alpha \in I} \mathfrak{H}_\alpha$ is topologically complete, that is, it satisfies condition E of (8), p. 65.

Proof: Assume that $\Phi^{(1)}, \Phi^{(2)}, \ldots \in \Pi \otimes_{\alpha \in I} \mathfrak{H}_\alpha$, $\lim_{r, s \to \infty} \|\Phi^{(r)} - \Phi^{(s)}\| = 0$. For each $\Phi^{(r)}$ choose a $\Phi_r \in \Pi' \otimes_{\alpha \in I} \mathfrak{H}_\alpha$ with $\|\Phi^{(r)} - \Phi_r\| \leqq \frac{1}{r}$. (Use Lemma 3.5.8.) Then $\|\Phi_r - \Phi_s\| \leqq \|\Phi^{(r)} - \Phi^{(s)}\| + \frac{1}{r} + \frac{1}{s}$, so $\lim_{r, s \to \infty} \|\Phi_r - \Phi_s\| = 0$. Thus Lemma 3.5.2 secures the existence of a $\Phi \in \Pi \otimes_{\alpha \in I} \mathfrak{H}_\alpha$ with $\Phi = \underset{r \to \infty}{\mathrm{L}} \Phi_r$, that is $\lim_{r \to \infty} \|\Phi - \Phi_r\| = 0$. (Use Lemma 3.5.7.) Since $\|\Phi - \Phi^{(r)}\| \leqq \|\Phi - \Phi_r\| + \frac{1}{r}$, $\lim_{r \to \infty} \|\Phi - \Phi^{(r)}\| = 0$, this completes the proof.

Theorem III. With the (Φ, Ψ) of Lemma 3.5.3 $\Pi\otimes_{\alpha\in I}\mathfrak{H}_\alpha$ is a unitary space, that is, it satisfies conditions **A**, **B**, **E** of (8), pp. 64—66. It is metrised as described in Lemma 3.5.5.

$\Pi'\otimes_{\alpha\in I}\mathfrak{H}_\alpha$, as described in Theorem II, is a linear and everywhere dense subset of $\Pi\otimes_{\alpha\in I}\mathfrak{H}_\alpha$, with the same definition of $u\Phi$, $\Phi \pm \Psi$, (Φ, Ψ), $\|\Phi\|$.

Further essential properties are given in Lemma 3.5.6.

Proof: This follows immediately from Lemmata 3.5.5, 3.5.9, 3.5.8.

3.6. The importance of this unitary space $\Pi\otimes_{\alpha\in I}\mathfrak{H}_\alpha$ becomes clear in the light of the following Theorem:

Theorem IV. Consider a unitary space $\mathfrak{H}$ with the following properties:

(I) For every C-sequence (or alternatively: for every C_0-sequence) f_α, $\alpha\in I$, an element $\Pi\otimes^*_{\alpha\in I} f_\alpha$ of $\mathfrak{H}$ is defined.

(II) $(\Pi\otimes^*_{\alpha\in I} f_\alpha, \Pi\otimes^*_{\alpha\in I} g_\alpha) = \Pi_{\alpha\in I}(f_\alpha, g_\alpha)$.

(III) The finite linear aggregates of the $\Pi\otimes^*_{\alpha\in I} f_\alpha$ form a set $\mathfrak{H}'$ which is everywhere dense in $\mathfrak{H}$.

This is equivalent to the existence of an isomorphism of $\mathfrak{H}$ and $\Pi\otimes_{\alpha\in I}\mathfrak{H}_\alpha$, under which each $\Pi\otimes^*_{\alpha\in I} f_\alpha$ corresponds to $\Pi\otimes_{\alpha\in I} f_\alpha$. This isomorphism is unique.

Proof: Sufficiency: $\mathfrak{H} = \Pi\otimes_{\alpha\in I}\mathfrak{H}_\alpha$, $\Pi\otimes^*_{\alpha\in I} f_\alpha = \Pi\otimes_{\alpha\in I} f_\alpha$ possess the properties (I)—(III) by Theorem III. (As to (III), in the case of C_0-sequence, observe that the non-C_0-sequences f_α, $\alpha\in I$, do not matter: Their $\Pi\otimes_{\alpha\in I} f_\alpha = 0$, by Lemma 3.3.1.) And this is unaffected by isomorphisms.

Necessity: Assume, that $\mathfrak{H}$ and $\Pi\otimes^*_{\alpha\in I} f_\alpha$ possess the properties (I)—(III) (either for all C-sequences, or for all C_0-sequences). Let $\Pi\otimes^*_{\alpha\in I} f_\alpha$ in $\mathfrak{H}$ correspond to $\Pi\otimes_{\alpha\in I} f_\alpha$ in $\Pi'\otimes_{\alpha\in I}\mathfrak{H}_\alpha$. This correspondence leaves (Φ, Ψ) invariant, as (II) holds both in $\mathfrak{H}$ and in $\Pi'\otimes_{\alpha\in I}\mathfrak{H}_\alpha$. Therefore we can extend it (in a unique way) to a linear correspondence between $\mathfrak{H}'$ and $\Pi'\otimes_{\alpha\in I}\mathfrak{H}_\alpha$. (In the case of C_0-sequences remember the remark in the sufficiency-proof.) This still leaves (Φ, Ψ) invariant, and with it $\|\Phi\| = \sqrt{(\Phi, \Phi)}$ and Distance $(\Phi, \Psi) = \|\Phi - \Psi\|$. Thus it is one-to-one and isometric. Therefore this correspondence extends by continuity (in a unique way) to a one-to-one and isometric correspondence between the closures of $\mathfrak{H}'$ and of $\Pi'\otimes_{\alpha\in I}\mathfrak{H}_\alpha$, that is between $\mathfrak{H}$ and $\Pi\otimes_{\alpha\in I}\mathfrak{H}_\alpha$. By continuity this correspondence is again lineair, and leaves (Φ, Ψ) invariant. Thus it is an isomorphism. And we have already observed, that it maps $\Pi\otimes^*_{\alpha\in I} f_\alpha$ on $\Pi\otimes_{\alpha\in I} f_\alpha$.

Uniqueness: Obvious by the above construction of the isomorphism.

We see in particular, that if I has only one element, say α_0, then $\mathfrak{H} = \mathfrak{H}_{\alpha_0}$ with $\Pi\otimes^*_{\alpha\in I} f_\alpha = f_{\alpha_0}$ fulfills the requirements of Theorem IV. So $\Pi\otimes_{\alpha\in I}\mathfrak{H}_\alpha$ is isomorphic to $\mathfrak{H}_{\alpha_0}$. But $\mathfrak{H}_{\alpha_0}$ is simply the set of all $\Pi\otimes^*_{\alpha\in I} f_\alpha$, so $\Pi\otimes_{\alpha\in I}\mathfrak{H}_\alpha$ is the set of all $\Pi\otimes_{\alpha\in I} f_\alpha$, and so a fortiori equal to $\Pi'\otimes_{\alpha\in I}\mathfrak{H}_\alpha$. It is easy to verify, that this is not generally true, if I has two or more elements.

We make use of this isomorphism to identify the elements which correspond under it. So we have $\Pi\otimes_{\alpha\in I}\mathfrak{H}_\alpha = \mathfrak{H}_{\alpha_0}$, $\Pi\otimes_{\alpha\in I} f_\alpha = f_{\alpha_0}$ if α_0 is the only element of I.

For every finite set I our construction of $\Pi\otimes_{\alpha\in I}\mathfrak{H}_\alpha$ coincides with that one of (7), pp. 127—131. (There I is the set $\mathfrak{S}(1, \ldots, n)$ and $\Pi\otimes_{\alpha\in I}\mathfrak{H}_\alpha$ is written $\prod_{i=1}^{n}\otimes\,\mathfrak{H}_i$.) The discussion of these I can be found there, too.

If I is finite, then our Definitions 3.3.1, 3.3.2 show, that every sequence f_α, $\alpha\in I$ is a C- and even a C_0-sequence, and that any two such sequences are equivalent. So Γ (cf. Definition 3.3.3) consists of one equivalence class only. As we will see in Lemma 6.4.1, Γ consists of infinitely many equivalence classes, whenever I is infinite.

If a closed, linear subset $\mathfrak{M}_\alpha \neq (0)$ of each $\mathfrak{H}_\alpha$ is given, we can form $\Pi\otimes_{\alpha\in I}\mathfrak{M}_\alpha$. The $\Pi\otimes_{\alpha\in I} f_\alpha$ ($f_\alpha\in\mathfrak{M}_\alpha$, and $f_\alpha, \alpha\in I$, a C-sequence) which we need for this construction, may be denoted by $\bar{\Pi}\otimes_{\alpha\in I} f_\alpha$ in order to distinguish them from the $\Pi\otimes_{\alpha\in I} f_\alpha$ of $\Pi\otimes_{\alpha\in I}\mathfrak{H}_\alpha$. Consider finally the $\Pi\otimes_{\alpha\in I} f_\alpha$ (in $\Pi\otimes_{\alpha\in I}\mathfrak{H}_\alpha$) with $f_\alpha\in\mathfrak{M}_\alpha$, and denote the closed, linear set, which they determine, by $\bar{\bar{\Pi}}\otimes_{\alpha\in I}\mathfrak{M}_\alpha$ ($\subset \Pi\otimes_{\alpha\in I}\mathfrak{H}_\alpha$). Now apply Theorem IV (to $\Pi\otimes_{\alpha\in I}\mathfrak{M}_\alpha$, $\bar{\Pi}\otimes_{\alpha\in I} f_\alpha$ in place of $\Pi\otimes_{\alpha\in I}\mathfrak{H}_\alpha$, $\Pi\otimes_{\alpha\in I} f_\alpha$), putting $\mathfrak{H} = \bar{\bar{\Pi}}\otimes_{\alpha\in I}\mathfrak{M}_\alpha$, $\Pi\otimes^*_{\alpha\in I} f_\alpha = \Pi\otimes_{\alpha\in I} f_\alpha$ (here $f_\alpha\in\mathfrak{M}_\alpha$). Then an isomorphism of $\Pi\otimes_{\alpha\in I}\mathfrak{M}_\alpha$ with $\bar{\bar{\Pi}}\otimes_{\alpha\in I}\mathfrak{M}_\alpha$ results, which carries $\bar{\Pi}\otimes_{\alpha\in I} f_\alpha$ into $\Pi\otimes_{\alpha\in I} f_\alpha$.

We make use of this isomorphism, to identify the elements which correspond under it. Thus $\Pi\otimes_{\alpha\in I}\mathfrak{M}_\alpha = \bar{\bar{\Pi}}\otimes_{\alpha\in I}\mathfrak{M}_\alpha$, $\bar{\Pi}\otimes_{\alpha\in I} f_\alpha = \Pi\otimes_{\alpha\in I} f_\alpha$. So we have now $\Pi\otimes_{\alpha\in I}\mathfrak{M}_\alpha \subset \Pi\otimes_{\alpha\in I}\mathfrak{H}_\alpha$ if all $\mathfrak{M}_\alpha \subset \mathfrak{H}_\alpha$, and their $\Pi\otimes_{\alpha\in I} f_\alpha$ agree if all $f_\alpha\in\mathfrak{M}_\alpha$.

Chapter 4: Decomposition of the complete direct product into incomplete ones.

4.1. Definition 4.1.1. If $\mathfrak{C} \in \Gamma$ is an equivalence-class, then let $\Pi\otimes^{\mathfrak{C}}_{\alpha \in I} \mathfrak{H}_\alpha$ be the closed linear set determined by all $\Pi\otimes_{\alpha \in I} f_\alpha$ where f_α, $\alpha \in I$, is any C_0-sequence from $\mathfrak{C}$. Clearly $\Pi\otimes^{\mathfrak{C}}_{\alpha \in I} \mathfrak{H}_\alpha \subset \Pi\otimes_{\alpha \in I} \mathfrak{H}_\alpha$.

This $\Pi\otimes^{\mathfrak{C}}_{\alpha \in I} \mathfrak{H}_\alpha$ is an *incomplete direct product of the* $\mathfrak{H}_\alpha$, $\alpha \in I$, more specifically: It is the *$\mathfrak{C}$-adic incomplete direct product.*

Lemma 4.1.1. The $\Pi\otimes^{\mathfrak{C}}_{\alpha \in I} \mathfrak{H}_\alpha$, for all $\mathfrak{C} \in \Gamma$, are mutually orthogonal, and the closed linear set they determine is $\Pi\otimes_{\alpha \in I} \mathfrak{H}_\alpha$.

Proof: Assume $\mathfrak{C}$, $\mathfrak{D} \in \Gamma$, $\mathfrak{C} \neq \mathfrak{D}$. Consider two C_0-sequences $(f_\alpha;\ \alpha \in I) \in \mathfrak{C}$, $(g_\alpha;\ \alpha \in I) \in \mathfrak{D}$. Then $(f_\alpha;\ \alpha \in I) \not\approx (g_\alpha;\ \alpha \in I)$, so by Theorem I $\Pi\otimes_{\alpha \in I} f_\alpha$ is orthogonal to $\Pi\otimes_{\alpha \in I} g_\alpha$. Therefore $\Pi\otimes_{\alpha \in I} f_\alpha$ is orthogonal to all $\Pi\otimes^{\mathfrak{D}}_{\alpha \in I} \mathfrak{H}_\alpha$, and again all $\Pi\otimes^{\mathfrak{C}}_{\alpha \in I} \mathfrak{H}_\alpha$ are orthogonal to all $\Pi\otimes^{\mathfrak{D}}_{\alpha \in I} \mathfrak{H}_\alpha$.

$\Pi\otimes_{\alpha \in I} \mathfrak{H}_\alpha$ is the closed, linear set determined by all $\Pi\otimes_{\alpha \in I} f_\alpha$ where f_α, $\alpha \in I$, runs over all C-sequences. By Lemma 3.3.1. it suffices to let it run over all C_0-sequences. But each C_0-sequence belongs to some $\mathfrak{C} \in \Gamma$, so $\Pi\otimes_{\alpha \in I} \mathfrak{H}_\alpha$ is a fortiori determined by all $\Pi\otimes^{\mathfrak{C}}_{\alpha \in I} \mathfrak{H}_\alpha$, $\mathfrak{C} \in \Gamma$.

Lemma 4.1.2. Consider an $\mathfrak{C} \in \Gamma$, and a C_0-sequence $(f^0_\alpha;\ \alpha \in I) \in \mathfrak{C}$ with $\|f^0_\alpha\| = 1$. (Cf. Lemma 3.3.7.) Then $\Pi\otimes^{\mathfrak{C}}_{\alpha \in I} \mathfrak{H}_\alpha$ is the closed, linear set determined by all C_0-sequences f_α, $\alpha \in I$, for which $f_\alpha \neq f^0_\alpha$ occurs for a finite number of α's only.

Proof: Let $\mathfrak{H}^*$ be the closed, linear set which these $\Pi\otimes_{\alpha \in I} f_\alpha$ determine. Lemma 3.3.5 secures $\mathfrak{H}^* \subset \Pi\otimes^{\mathfrak{C}}_{\alpha \in I} \mathfrak{H}_\alpha$. If we can prove, that $\Pi\otimes_{\alpha \in I} f_\alpha \in \mathfrak{H}^*_\alpha$ whenever $(f_\alpha;\ \alpha \in I) \in \mathfrak{C}$, then necessarily $\Pi\otimes^{\mathfrak{C}}_{\alpha \in I} \mathfrak{H}_\alpha \subset \mathfrak{H}^*$, and so $\mathfrak{H}^* = \Pi\otimes^{\mathfrak{C}}_{\alpha \in I} \mathfrak{H}_\alpha$, completing the proof.

Assume therefore $(f_\alpha;\ \alpha \in I) \in \mathfrak{C}$, that is $(f_\alpha;\ \alpha \in I) \approx (f^0_\alpha;\ \alpha \in I)$. That is: $\Sigma_{\alpha \in I} \left|(f_\alpha, f^0_\alpha) - 1\right|$ converges.

If $\|\Pi\otimes_{\alpha \in I} f_\alpha\| = \Pi_{\alpha \in I} \|f_\alpha\| = 0$, then $\Pi\otimes_{\alpha \in I} f_\alpha = 0 \in \mathfrak{H}^*$. We may assume therefore, that $\Pi_{\alpha \in I} \|f_\alpha\| \neq 0$. So $\|f_\alpha\| \neq 0$, and $\Pi_{\alpha \in I} \frac{1}{\|f_\alpha\|}$ converges and is $\neq 0$. So for $z_\alpha = \frac{1}{\|f_\alpha\|} > 0$, $\Sigma_{\alpha \in I} |z_\alpha - 1|$ converges by Lemma 2.4.1, (II). Now Lemma 3.3.6, (III), (IV), apply: $(z_\alpha f_\alpha;\ \alpha \in I) \in \mathfrak{C}$ too, and $\Pi\otimes_{\alpha \in I} z_\alpha f_\alpha = \Pi_{\alpha \in I} z_\alpha \cdot \Pi\otimes_{\alpha \in I} f_\alpha = \frac{1}{\Pi_{\alpha \in I} \|f_\alpha\|} \cdot \Pi\otimes_{\alpha \in I} f_\alpha$, so $\Pi\otimes_{\alpha \in I} f_\alpha = \Pi_{\alpha \in I} \|f_\alpha\| \cdot \Pi\otimes_{\alpha \in I} z_\alpha f_\alpha$. So we may consider the C_0-sequence $z_\alpha f_\alpha$, $\alpha \in I$, instead of f_α, $\alpha \in I$. But $\|z_\alpha f_\alpha\| = z_\alpha \|f_\alpha\| \doteqdot 1$. So we see: We may assume $\|f_\alpha\| = 1$ for all $\alpha \in I$.

Choose a δ with $0 < \delta < 1$, and then a finite set $J = \mathfrak{S}(\alpha_1, \ldots, \alpha_n)$, the $\alpha_1, \ldots, \alpha_n$ mutually different, so that $|(f_{\alpha_1}, f^0_{\alpha_1}) - 1| + \ldots + |(f_{\alpha_n}, f^0_{\alpha_n}) - 1| \geqq \Sigma_{\alpha \in I} |(f_\alpha, f^0_\alpha) - 1| - \delta$, that is $\Sigma_{\alpha \in I - J} |(f_\alpha, f^0_\alpha) - 1| \leqq \delta$. Clearly $\Pi_{\alpha \in I} (f_\alpha, f^0_\alpha)$ is convergent (not only quasi-convergent). For any mutually different $\beta_1, \ldots, \beta_m \in I - J$ we have [24])

$$|(f_{\beta_1}, f^0_{\beta_1}) \cdot \ldots \cdot (f_{\beta_m}, f^0_{\beta_m}) - 1| \leqq e^{|(f_{\beta_1}, f^0_{\beta_1}) - 1| + \ldots + |(f_{\beta_m}, f^0_{\beta_m}) - 1|} - 1 \leqq$$
$$\leqq e^{\Sigma_{\alpha \in I - J} |(f_\alpha, f^0_\alpha) - 1|} - 1 \leqq e^\delta - 1 \leqq e\,\delta.$$

Therefore $|\Pi_{\alpha \in I - J} (f_\alpha, f^0_\alpha) - 1| \leqq e\,\delta$.

Define now $g_\alpha \begin{cases} = f_\alpha \text{ for } \alpha \in J \\ = f^0_\alpha \text{ for } \alpha \in I - J. \end{cases}$

Then clearly $\Pi \otimes_{\alpha \in I} g_\alpha \in \mathfrak{H}^*_\alpha$. And

$$\|\Pi \otimes_{\alpha \in I} f_\alpha - \Pi \otimes_{\alpha \in I} g_\alpha\|^2 = \|\Pi \otimes_{\alpha \in I} f_\alpha\|^2 +$$
$$+ \|\Pi \otimes_{\alpha \in I} g_\alpha\|^2 - 2\Re(\Pi \otimes_{\alpha \in I} f_\alpha, \Pi \otimes_{\alpha \in I} g_\alpha) =$$
$$= (\Pi_{\alpha \in I} \|f_\alpha\|)^2 + (\Pi_{\alpha \in I} \|g_\alpha\|)^2 - 2\Re(\Pi_{\alpha \in I} (f_\alpha, g_\alpha)) =$$
$$= 1 + 1 - 2\Re(\Pi_{\alpha \in I} (f_\alpha, g_\alpha) \cdot \Pi_{\alpha \in I - J} (f_\alpha, g_\alpha)) =$$
$$= 2 - 2\Re(\Pi_{\alpha \in I} (f_\alpha, f_\alpha) \cdot \Pi_{\alpha \in I - J} (f_\alpha, f^0_\alpha)) =$$
$$= 2 - 2\Re(\Pi_{\alpha \in I - J} (f_\alpha, f^0_\alpha)) =$$
$$= 2\Re(1 - \Pi_{\alpha \in I - J} (f_\alpha, f^0_\alpha)) \leqq 2|\Pi_{\alpha \in I - J} (f_\alpha, f^0_\alpha) - 1| \leqq 2e\delta,$$
$$\|\Pi \otimes_{\alpha \in I} f_\alpha - \Pi \otimes_{\alpha \in I} g_\alpha\| \leqq \sqrt{2e\delta}.$$

As our δ, $0 < \delta < 1$, was otherwise arbitrary, this means that $\Pi \otimes_{\alpha \in I} f_\alpha$ is a limit-point of $\mathfrak{H}^*$, and therefore belongs to $\mathfrak{H}^*$. Thus the proof is completed.

LEMMA 4.1.3. If f_{α_0} is a fixed element of $\mathfrak{H}_{\alpha_0}$, and if the f_α, $\alpha \neq \alpha_0$, are held fixed, so as to form a C- (or even a C_0-) sequence [25]) then $\Pi \otimes_{\alpha \in I} f_\alpha$ is a linear and continuous function of f_{α_0}.

Proof: We have clearly [26])

---

[24]) Due to the easily verifiable inequality

$$|z_1 \cdot \ldots \cdot z_m - 1| \leqq e^{|z_1 - 1| + \ldots + |z_m - 1|} - 1.$$

[25]) Clearly neither fact depends on the choice of f_{α_0}.

[26]) In order to be able to handle the factor $\alpha = \alpha_0$ separately, we shall write $\mathfrak{H}_{\alpha_0} \otimes \Pi \otimes_{\alpha \in I, \alpha \neq \alpha_0} \mathfrak{H}_\alpha$ and $f_{\alpha_0} \otimes \Pi \otimes_{\alpha \in I, \alpha \neq \alpha_0} f_\alpha$ for $\Pi \otimes_{\alpha \in I} \mathfrak{H}_\alpha$ and $\Pi \otimes_{\alpha \in I} f_\alpha$ respectively.

$$(zf_{\alpha_0}) \otimes \Pi\otimes_{\alpha\in I,\alpha\neq\alpha_0} f_\alpha = z\cdot(f_{\alpha_0}\otimes \Pi_{\alpha\in I,\alpha\neq\alpha_0} f_\alpha),$$

$$(f_{\alpha_0}+g_{\alpha_0})\otimes \Pi\otimes_{\alpha\in I,\alpha\neq\alpha_0} f_\alpha =$$
$$= f_{\alpha_0}\otimes \Pi\otimes_{\alpha\in I,\alpha\neq\alpha_0} f_\alpha + g_\alpha \otimes \Pi\otimes_{\alpha\in I,\alpha\neq\alpha_0} f_\alpha,$$

$$\|f_{\alpha_0}\otimes \Pi\otimes_{\alpha\in I,\alpha\neq\alpha_0} f_\alpha - g_{\alpha_0}\otimes \Pi\otimes_{\alpha\in I,\alpha\neq\alpha_0} f_\alpha\| =$$
$$= \|(f_{\alpha_0}-g_{\alpha_0})\otimes \Pi\otimes_{\alpha\in I,\alpha\neq\alpha_0} f_\alpha\| =$$
$$= \|f_{\alpha_0}-g_{\alpha_0}\|\cdot \Pi_{\alpha\in I,\alpha\neq\alpha_0}\|f_\alpha\|.$$

These formulae prove all statements of our Lemma.

LEMMA 4.1.4. Consider an $\mathfrak{C}\in\Gamma$. Select from $\mathfrak{C}$ a sequence f_α^0, $\alpha\in I$, with $\|f_\alpha^0\|=1$. (Cf. Lemma 3.3.7.)

Let $\aleph_\alpha$ be the dimensionality of $\mathfrak{H}_\alpha$. Use a set of indices K_α of power $\aleph_\alpha$, and form a complete normalised orthogonal set $\varphi_{\alpha,\beta}$, $\beta\in K_\alpha$, in $\mathfrak{H}_\alpha$. Make these choices in such a manner, that $0\in K_\alpha$, and $\varphi_{\alpha,0}=f_\alpha^0$.

Let $\mathbf{F}$ be the set of all functions $\beta(\alpha)$, which are defined for all $\alpha\in I$ and for those only, such that $\alpha\in I$ implies $\beta(\alpha)\in K_\alpha$, and such that $\beta(\alpha)\neq 0$ occurs for a finite number of α's only. Then, if $\beta(\alpha)$ runs over all $\mathbf{F}$, all $\Pi\otimes_{\alpha\in I}\varphi_{\alpha,\beta(\alpha)}$ exist, and they form a complete, normalised orthogonal set in $\Pi\otimes_{\alpha\in I}^{\mathfrak{C}}\mathfrak{H}_\alpha$.

Proof: As $\Sigma_{\alpha\in I}\big|\|\varphi_{\alpha,\beta(\alpha)}\|-1\big| = \Sigma_{\alpha\in I}|1-1| = \Sigma_{\alpha\in I}0$ converges, all sequences $\varphi_{\alpha,\beta(\alpha)}$, $\alpha\in I$ are C_0-sequences, and so all $\Pi\otimes_{\alpha\in I}\varphi_{\alpha,\beta(\alpha)}$ exist. As $\varphi_{\alpha,\beta(\alpha)}\neq\varphi_{\alpha,0}=f_\alpha^0$ occurs only if $\beta(\alpha)\neq 0$, that is for a finite number of α's, we have $(\varphi_{\alpha,\beta(\alpha)};\alpha\in I)\approx(f_\alpha^0;\alpha\in I)$ (use Lemma 3.3.5) and so $(\varphi_{\alpha,\beta(\alpha)};\alpha\in I)\in\mathfrak{C}$, $\Pi\otimes_{\alpha\in I}\varphi_{\alpha,\beta(\alpha)}\in\Pi\otimes_{\alpha\in I}^{\mathfrak{C}}\mathfrak{H}_\alpha$.

$(\Pi\otimes_{\alpha\in I}\varphi_{\alpha,\beta(\alpha)}, \Pi\otimes_{\alpha\in I}\varphi_{\alpha,\gamma(\alpha)}) = \Pi_{\alpha\in I}(\varphi_{\alpha,\beta(\alpha)},\varphi_{\alpha,\gamma(\alpha)})$. If $\beta(\alpha)=\gamma(\alpha)$ for all $\alpha\in I$, then all factors on the right side are $=1$, and so this expression is $=1$. If $\beta(\alpha)\neq\gamma(\alpha)$ ever occurs, then the corresponding factor on the right side is 0, and so the expression is 0.

Thus the $\Pi\otimes_{\alpha\in I}\varphi_{\alpha,\beta(\alpha)}$, $\beta(\alpha)\in\mathbf{F}$, form a normalised, orthogonal set in $\Pi\otimes_{\alpha\in I}^{\mathfrak{C}}\mathfrak{H}_\alpha$. It remains to prove, that it is complete. Let $\mathfrak{H}^*$ be the closed, linear set determined by the $\Pi\otimes_{\alpha\in I}\varphi_{\alpha,\beta(\alpha)}$, $\beta(\alpha)\in\mathbf{F}$. Then we must prove, that $\mathfrak{H}^*$ contains $\Pi\otimes_{\alpha\in I}^{\mathfrak{C}}\mathfrak{H}_\alpha$. Owing to Lemma 4.1.2 it suffices to prove, that it contains all those $\Pi\otimes_{\alpha\in I}f_\alpha\in\mathfrak{H}^*$ for which $f_\alpha\neq f_\alpha^0$ occurs for a finite number of α's only.

Consider this statement S_n $(n=0,1,2,\ldots)$: If

$(\mathrm{I})_n$ the number of those α's for which $f_\alpha\neq\varphi_{\alpha,\beta}$ (for all $\beta\in K_\alpha$) is $=n$, and if

(II) the number of those α's for which $f_\alpha \neq \varphi_{\alpha,0} = f_\alpha^0$ is finite, then $\Pi\otimes_{\alpha\in I} f_\alpha \in \mathfrak{H}^*$.

S_0 is true: If f_α, $\alpha \in I$, satisfies $(\mathrm{I})_0$, (II), then $f_\alpha = \varphi_{\alpha,\beta(\alpha)}, \beta(\alpha) \in \mathbf{F}$, so $\Pi\otimes_{\alpha\in I} f_\alpha = \Pi\otimes_{\alpha\in I} \varphi_{\alpha,\beta(\alpha)} \in \mathfrak{H}^*$. Assume now, that S_{n-1} holds for some $n-1$, $n = 1, 2, \ldots$, and consider S_n. Let f_α, $\alpha \in I$, fulfill $(\mathrm{I})_n$, (II). Denote the α for which $f_\alpha \neq \varphi_{\alpha,\beta}$ by $\alpha_1, \ldots, \alpha_n$. We ask: For which $g_{\alpha_n} \in \mathfrak{H}_{\alpha_n}$ is $g_{\alpha_n} \otimes \Pi\otimes_{\alpha\in I, \alpha\neq\alpha_n} f_\alpha \in \mathfrak{H}^*$? These g_{α_n} form a closed, linear set $\mathfrak{N}$ in $\mathfrak{H}_{\alpha_n}$, considering that $\mathfrak{H}^*$ is one, by Lemma 4.1.3. Every $\varphi_{\alpha_n,\beta} \in \mathfrak{N}$, as we assumed the validity of S_{n-1}. So $\mathfrak{N} = \mathfrak{H}_{\alpha_n}$, and in particular $f_{\alpha_n} \in \mathfrak{N}$. That is, $\Pi\otimes_{\alpha\in I} f_\alpha = f_{\alpha_n} \otimes \Pi\otimes_{\alpha\in I, \alpha\neq\alpha_n} f_\alpha \in \mathfrak{H}^*$, proving S_n. Thus all statements S_n are true.

Now let f_α, $\alpha \in I$, fulfill (II) only. The number of the α's with $f_\alpha \neq S_{\alpha,\beta}$ (being a subset of those with $f_\alpha \neq \varphi_{\alpha,0} = f_\alpha^0$) is necessarily finite, $= 0, 1, 2, \ldots$ Let n be this number. Then $(\mathrm{I})_n$ holds, too, and so we have $\Pi\otimes_{\alpha\in I} f_\alpha \in \mathfrak{H}^*$ by S_n. But this is exactly what we needed in order to complete our proof.

Theorem V. Using the notations of Lemma 4.1.4 for $\mathfrak{C}$, f_α^0, $\aleph_\alpha$, K_α, $\varphi_{\alpha,\beta}$, and the set $\mathbf{F}$ of functions $\beta(\alpha)$, there exists a one-to-one correspondence between the $\Phi \in \Pi\otimes_{\alpha\in I}^{\mathfrak{C}} \mathfrak{H}_\alpha$ and the coefficient-systems $a[\beta(\alpha); \alpha \in I]$ such that

(I) $a[\beta(\alpha); \alpha \in I]$ is defined for the functions $\beta(\alpha) \in \mathbf{F}$ and for those only, its values [27]) being complex numbers,

(II) $\Sigma_{\beta(\alpha)\in\mathbf{F}} |a[\beta(\alpha); \alpha \in I]|^2$ converges.

This correspondence is established by the following equations:

$$(1) \qquad a[\beta(\alpha); \alpha \in I] = (\Phi, \Pi\otimes_{\alpha\in I} \varphi_{\alpha,\beta(\alpha)}) = \Phi(\varphi_{\alpha,\beta(\alpha)}; \alpha \in I).$$

If Φ, Ψ correspond to $a[\beta(\alpha); \alpha \in I]$, $b[\beta(\alpha); \alpha \in I]$ respectively, then

$$(2) \qquad (\Phi, \Psi) = \Sigma_{\beta(\alpha)\in\mathbf{F}} a[\beta(\alpha); \alpha \in I] \cdot \overline{b[\beta(\alpha); \alpha \in I]}.$$

(This $\Sigma_{\beta(\alpha)\in\mathbf{F}}$ is convergent.)

Proof: The $\Pi\otimes_{\alpha\in I} \varphi_{\alpha,\beta(\alpha)}$, $\beta(\alpha) \in \mathbf{F}$, form a complete, normalised, orthogonal set in $\Pi\otimes_{\alpha\in I}^{\mathfrak{C}} \mathfrak{H}_\alpha$, by Lemma 4.1.4. Therefore the first equation of (I) creates a one-to-one correspondence of the $\Phi \in \Pi\otimes_{\alpha\in I}^{\mathfrak{C}} \mathfrak{H}_\alpha$ with the $a[\beta(\alpha); \alpha \in I]$ as described in (I), (II). And equation 2) holds. (Cf. the respective theorems of (4), (12), (13) or (15)). The second equation of (1) coincides with Lemma 3.2.3.

---

[27]) $a[\beta(\alpha); \alpha \in I]$ does not depend on α, the argument is the *function* $\beta(\alpha)$!

Theorem V (or Lemma 4.1.4, to which it is practically equivalent) clarifies the structure of the $\Pi\otimes^{\mathfrak{C}}_{\alpha\in I}\mathfrak{H}_\alpha$, while Lemma 4.1.1, describes how $\Pi\otimes_{\alpha\in I}\mathfrak{H}_\alpha$ is built up from the $\Pi\otimes^{\mathfrak{C}}_{\alpha\in I}\mathfrak{H}_\alpha$, $\mathfrak{C}\in\Gamma$. We see in particular, that each $\Pi\otimes^{\mathfrak{C}}_{\alpha\in I}\mathfrak{H}_\alpha$ has the same dimension: The power of $\mathbf{F}$. It is easy to determine this power with the help of the $\aleph_\alpha$, $\alpha\in I$; we shall do this in some special cases (cf. § 7.2). The dimension of $\Pi\otimes_{\alpha\in I}\mathfrak{H}_\alpha$ is clearly $=$ power of $\Gamma\cdot$ power of $\mathbf{F}$. We shall reconsider this in Lemma 6.4.1 and immediately after it.

4.2. The *associative law* holds for our products in a restricted form only: It is particularly noteworthy, that it applies to the incomplete direct products $\Pi\otimes^{\mathfrak{C}}_{\alpha\in I}\mathfrak{H}_\alpha$ only, and not to the complete direct product $\Pi\otimes_{\alpha\in I}\mathfrak{H}_\alpha$. There are no restrictions, however, as to the number (power) of factors [28]). The Theorem which follows, describes the situation exhaustively.

Theorem VI. Let I be a set of indices α, L a set of indices γ, and let for each $\gamma\in\mathrm{L}$ a set I_γ of indices α be given. Assume, that the I_γ, $\gamma\in\mathrm{L}$, are mutually disjoint, and that their sum is I. Let for each $\alpha\in I$ a unitary space $\mathfrak{H}_\alpha$ be given.

(I) If f_α, $\alpha\in I$, is a C- or a C_0-sequence, then every sequence f_α, $\alpha\in I_\gamma$, as well as the sequence $\Pi\otimes_{\alpha\in I_\gamma}f_\alpha$, $\gamma\in\mathrm{L}$, is a C- or a C_0-sequence respectively. In the case of C_0-sequences form the equivalence-classes $\mathfrak{C}=\mathfrak{C}(f_\alpha;\alpha\in I)$ (in $\Pi\otimes_{\alpha\in I}\mathfrak{H}_\alpha$), $\mathfrak{C}_\gamma=\mathfrak{C}(f_\alpha;\alpha\in I_\gamma)$ (in $\Pi\otimes_{\alpha\in I_\gamma}\mathfrak{H}_\alpha$), $\mathfrak{C}^0=\mathfrak{C}(\Pi\otimes_{\alpha\in I_\gamma}f_\alpha;\gamma\in\mathrm{L})$ (in $\Pi\otimes_{\gamma\in\mathrm{L}}(\Pi\otimes_{\alpha\in I_\gamma}f_\alpha)$).

(II) The classes $\mathfrak{C}_\gamma$, $\mathfrak{C}^0$ depend on $\mathfrak{C}$ only (and not on the particular choice of the f_α, $\alpha\in I$).

(III) There exists a unique isomorphism of $\Pi\otimes^{\mathfrak{C}}_{\alpha\in I}\mathfrak{H}_\alpha$ $(\subset\Pi\otimes_{\alpha\in I}\mathfrak{H}_\alpha)$ and $\Pi\otimes^{\mathfrak{C}^0}_{\gamma\in\mathrm{L}}(\Pi\otimes^{\mathfrak{C}_\gamma}_{\alpha\in I_\gamma}\mathfrak{H}_\alpha)$ $(\subset\Pi\otimes_{\gamma\in\mathrm{L}}(\Pi\otimes_{\alpha\in I_\gamma}\mathfrak{H}_\alpha))$ where $\Pi\otimes_{\alpha\in I}f_\alpha$ corresponds to $\Pi\otimes_{\gamma\in\mathrm{L}}(\Pi\otimes_{\alpha\in I_\gamma}f_\alpha)$ for all C_0-sequences f_α, $\alpha\in I$, of $\mathfrak{C}$.

Proof: Ad (I): If $\Pi_{\alpha\in I}\|f_\alpha\|$ converges, then all $\Pi_{\alpha\in I_\gamma}\|f_\alpha\|$ do so by Lemma 2.4.1, (I), and if $\Sigma_{\alpha\in I}\big|\|f_\alpha\|-1\big|$ converges, then all $\Sigma_{\alpha\in I_\gamma}\big|\|f_\alpha\|-1\big|$ do so by Lemma 2.3.1 (owing to $I_\gamma\subset I$). So the f_α, $\alpha\in I_\gamma$ are C- resp. C_0-sequences along with f_α, $\alpha\in I$. Thus only the C- resp. C_0-sequence-character of $\Pi\otimes_{\alpha\in I_\gamma}f_\alpha$, $\gamma\in\mathrm{L}$, must be established.

---

[28]) It is evident from our constructions, in which no ordering of I occurs, that the *commutative law* holds unrestrictedly.

For any mutually different $\beta_1, \ldots, \beta_m \in I_\gamma$ we have [29])

$$\text{Max}\,(\|f_{\beta_1}\| \cdot \ldots \cdot \|f_{\beta_m}\| - 1, 0) \leqq e^{\text{Max}\,(\|f_{\beta_1}\|-1,0) + \ldots + \text{Max}\,(\|f_{\beta_m}\|-1,0)} - 1$$

$$\leqq e^{\Sigma_{\alpha \in I_\gamma} \text{Max}\,(\|f_\alpha\|-1,0)} - 1 \leqq e \Sigma_{\alpha \in I_\gamma} \text{Max}\,(\|f_\alpha\| - 1, 0)$$

if $\Sigma_{\alpha \in I_\gamma} \text{Max}\,(\|f_\alpha\| - 1, 0) \leqq 1$. Thus $\text{Max}\,(\|\Pi\otimes_{\alpha \in I_\gamma} f_\alpha\| - 1, 0) = \text{Max}\,(\Pi_{\alpha \in I_\gamma} \|f_\alpha\| - 1, 0) \leqq e \cdot \Sigma_{\alpha \in I_\gamma} \text{Max}\,(\|f_\alpha\| - 1, 0)$. Now it is clear, for every C-sequence f_α, $\alpha \in I$, that

$$\Sigma_{\gamma \in \mathrm{L}} (\Sigma_{\alpha \in I_\gamma} \text{Max}\,(\|f_\alpha\| - 1, 0)) = \Sigma_{\alpha \in I} \text{Max}\,(\|f_\alpha\| - 1, 0)$$

converges (use Lemma 2.3.1), so $\Sigma_{\alpha \in I_\gamma} \text{Max}\,(\|f_\alpha\| - 1, 0) \leqq 1$, except for a finite number of γ's. Our above inequality establishes therefore the convergence of $\Sigma_{\gamma \in \mathrm{L}} \text{Max}\,(\|\Pi\otimes_{\alpha \in I_\gamma} f_\alpha\| - 1, 0)$. (Use Lemma 2.3.1.) So $\Pi\otimes_{\alpha \in I_\gamma} f_\alpha$, $\gamma \in \mathrm{L}$ is a C-sequence, too.

Next

$$\big|\|f_{\beta_1}\| \cdot \ldots \cdot \|f_{\beta_m}\| - 1\big| \leqq e^{|\|f_{\beta_1}\|-1| + \ldots + |\|f_{\beta_m}\|-1|} - 1 \leqq$$

(use the inequality in [24]) on page [36] 36)

$$\leqq e^{\Sigma_{\alpha \in I_\gamma} |\|f_\alpha\|-1|} - 1 \leqq e\, \Sigma_{\alpha \in I_\gamma} \big|\|f_\alpha\| - 1\big|$$

if $\Sigma_{\alpha \in I_\gamma} \big|\|f_\alpha\| - 1\big| \leqq 1$. Thus

$$\big|\|\Pi\otimes_{\alpha \in I_\gamma} f_\alpha\| - 1\big| = \big|\Pi_{\alpha \in I_\gamma} \|f_\alpha\| - 1\big| \leqq e \Sigma_{\alpha \in I_\gamma} \big|\|f_\alpha\| - 1\big|.$$

Again for every C_0-sequence f_α, $\alpha \in I$,

$$\Sigma_{\gamma \in \mathrm{L}} (\Sigma_{\alpha \in I_\gamma} \big|\|f_\alpha\| - 1\big|) = \Sigma_{\alpha \in I} \big|\|f_\alpha\| - 1\big|$$

converges. (Use Lemma 2.3.1.) So $\Sigma_{\alpha \in I_\gamma} \big|\|f_\alpha\| - 1\big| \leqq 1$ except for a finite number of γ's. Our inequality now establishes the convergence of $\Sigma_{\gamma \in \mathrm{L}} \big|\|\Pi\otimes_{\alpha \in I_\gamma} f_\alpha\| - 1\big|$. (Use Lemma 2.3.1.) So $\Pi\otimes_{\alpha \in I_\gamma} f_\alpha$, $\gamma \in \mathrm{L}$, is a C_0-sequence too.

Ad (II): We must prove: If $(f_\alpha; \alpha \in I) \approx (g_\alpha; \alpha \in I)$, then $(f_\alpha; \alpha \in I_\gamma) \approx (g_\alpha; \alpha \in I_\gamma)$ and $(\Pi\otimes_{\alpha \in I_\gamma} f_\alpha; \gamma \in \mathrm{L}) \approx (\Pi\otimes_{\alpha \in I_\gamma} g_\alpha; \gamma \in \mathrm{L})$. That is: If $\Sigma_{\alpha \in I} |(f_\alpha, g_\alpha) - 1|$ converges, then $\Sigma_{\alpha \in I_\gamma} |(f_\alpha, g_\alpha) - 1|$ and $\Sigma_{\gamma \in \mathrm{L}} |(\Pi\otimes_{\alpha \in I_\gamma} f_\alpha, \Pi\otimes_{\alpha \in I_\gamma} g_\alpha) - 1|$ converge. The first statement is obvious. As to the second one observe that we have (just as

---

[29]) Due to the easily verifiable inequality ($z_1, \ldots, z_m$ real and $\geqq 0$)

$$\text{Max}\,(z_1 \cdot \ldots \cdot z_m - 1, 0) \leqq e^{\text{Max}\,(z_1 - 1, 0) + \ldots + \text{Max}\,(z_m - 1, 0)} - 1.$$

in the last part of the proof of (I)) for any mutually different $\beta_1, \ldots, \beta_m$

$$\left|(f_{\beta_1}, g_{\beta_1}) \cdot \ldots \cdot (f_{\beta_m}, g_{\beta_m}) - 1\right| \leqq e^{|(f_{\beta_1}, g_{\beta_1})-1|+\ldots+|(f_{\beta_m}, g_{\beta_m})-1|} - 1 \leqq$$

$$\leqq e^{\Sigma_{\alpha\in I_\gamma}|(f_\alpha, g_\alpha)-1|} - 1 \leqq e\,\Sigma_{\alpha\in I_\gamma}\left|(f_\alpha, g_\alpha) - 1\right|$$

if $\Sigma_{\alpha\in I_\gamma}\left|(f_\alpha, g_\alpha) - 1\right| \leqq 1$. Thus $\left|(\Pi\otimes_{\alpha\in I_\gamma} f_\alpha, \Pi\otimes_{\alpha\in I_\gamma} g_\alpha) - 1\right| = \left|\Pi_{\alpha\in I_\gamma}(f_\alpha, g_\alpha) - 1\right| \leqq e\,\Sigma_{\alpha\in I_\gamma}\left|(f_\alpha, g_\alpha) - 1\right|$. As

$$\Sigma_{\gamma\in \mathrm{L}}\left(\Sigma_{\alpha\in I_\gamma}\left|(f_\alpha, g_\alpha) - 1\right|\right) = \Sigma_{\alpha\in I}\left|(f_\alpha, g_\alpha) - 1\right|$$

converges, so $\Sigma_{\alpha\in I_\gamma}\left|(f_\alpha, g_\alpha) - 1\right| \leqq 1$ except for a finite number of γ's. Our inequality now establishes the convergence of

$$\Sigma_{\gamma\in \mathrm{L}}\left|(\Pi\otimes_{\alpha\in I_\gamma} f_\alpha, \Pi\otimes_{\alpha\in I_\gamma} g_\alpha) - 1\right|.$$

Ad (III): Consider the equation

$$(*) \qquad \Pi_{\alpha\in I}(f_\alpha, g_\alpha) = \Pi_{\gamma\in \mathrm{L}}\left(\Pi_{\alpha\in I_\gamma}(f_\alpha, g_\alpha)\right)$$

for two C_0-sequences $(f_\alpha;\ \alpha\in I)$ and $(g_\alpha;\ \alpha\in I) \in \mathfrak{C}$. By (I), (II) all these Π are convergent (and not merely quasi-convergent; use Lemma 2.4.1). Therefore (*) holds, as one verifies easily [30]).

Let now correspond to each $\Pi\otimes_{\alpha\in I} f_\alpha$ in $\Pi\otimes^{\mathfrak{C}}_{\alpha\in I}\mathfrak{H}_\alpha((f_\alpha;\ \alpha\in I)\in\mathfrak{C})$, the element $\Pi\otimes_{\gamma\in \mathrm{L}}(\Pi\otimes_{\alpha\in I_\gamma} f_\alpha)$ in $\Pi\otimes^{\mathfrak{C}^0}_{\alpha\in \mathrm{L}}(\Pi\otimes^{\mathfrak{C}_\gamma}_{\alpha\in I_\gamma}\mathfrak{H}_\alpha)$ (we have $(f_\alpha;\ \alpha\in I_\gamma)\in\mathfrak{C}_\gamma$, $(\Pi\otimes_{\alpha\in I_\gamma} f_\alpha;\ \gamma\in \mathrm{L})\in\mathfrak{C}^0$). This correspondence leaves (Φ, Ψ) invariant by (*). Therefore it extends in a unique way, literally as in the proof of necessity in the proof of Theorem IV, to an isomorphism of the closed, linear sets determined by the $\Pi\otimes_{\alpha\in I} f_\alpha$ resp. the $\Pi\otimes_{\gamma\in \mathrm{L}}(\Pi\otimes_{\alpha\in I_\gamma} f_\alpha)$. The former set is $\Pi\otimes^{\mathfrak{C}}_{\alpha\in I}\mathfrak{H}_\alpha$ by definition, denote the latter by $\overline{\mathfrak{H}}$. So we must only prove $\overline{\mathfrak{H}} = \Pi\otimes^{\mathfrak{C}^0}_{\gamma\in \mathrm{L}}(\Pi\otimes^{\mathfrak{C}_\gamma}_{\alpha\in I_\gamma}\mathfrak{H}_\alpha)$. But $\subset$ is obvious, so we need only show $\supset$.

By (II) we can use any sequence $(f_\alpha;\ \alpha\in I)\in\mathfrak{C}$, so we may assume that all $\|f_\alpha\| = 1$ (use Lemma 3.3.7). Choose a complete, normalised, orthogonal set $\varphi_{\alpha,\beta}$, $\beta\in K_\alpha$, in each $\mathfrak{H}_\alpha$, with $0\in K_\alpha$, $\varphi_{\alpha,0} = f_\alpha$. Let $\mathbf{F}_\gamma$ be the set of all functions $\beta(\gamma, \alpha)$ which are defined for the $\alpha\in I_\gamma$, $\beta(\gamma, \alpha)\in K_\alpha$ and which are $\neq 0$ for a finite number of α's $(\alpha\in I_\gamma)$ only — all this for a fixed $\gamma\in \mathrm{L}$. $(\varphi_{\alpha,0};\ \alpha\in I_\gamma) = (f_\alpha;\ \alpha\in I_\gamma)\in\mathfrak{C}_\gamma$. So Lemma 4.1.4 applies to

[30]) It would not be so in case of quasi-convergence. Thus $\Pi_{\alpha\in\gamma(1,2,\ldots)}(-1) = 0$ (quasi-convergence), while $\Pi_{\gamma\in\gamma(1,2,\ldots)}\left(\Pi_{\alpha\in\gamma(2\gamma-1,2\gamma)}(-1)\right) = \Pi_{\gamma\in\gamma(1,2,\ldots)} 1 = 1$.

$\Pi\otimes^{\mathfrak{C}_\gamma}_{\alpha\in I_\gamma}\mathfrak{H}_\alpha$: The $\Phi_{\gamma,\beta(\gamma,\alpha)} = \Pi\otimes_{\alpha\in I_\gamma}\varphi_{\alpha,\beta(\gamma,\alpha)}$, $\beta(\gamma,\alpha)\in\mathbf{F}_\gamma$, form a complete, normalised orthogonal set there. Denote the function $\beta(\gamma,\alpha)=0$ (for all $\alpha\in I_\gamma$) by $\mathbf{0}$, then $\Phi_{\gamma,\mathbf{0}}=\Pi\otimes_{\alpha\in I_\gamma}\varphi_{\alpha,0}=\Pi\otimes_{\alpha\in I_\gamma}f_\alpha$. Let $\mathbf{F}^0$ be the set of all functions $\beta^0(\gamma)$, which are defined for all $\gamma\in\mathrm{L}$, $\beta^0(\gamma)\in\mathbf{F}_\gamma$, and which are $\neq\mathbf{0}$ for a finite number of γ's ($\gamma\in\mathrm{L}$) only. As $\beta^0(\gamma)$ is really a function of α, $\alpha\in I_\gamma$, we prefer to write $\beta^0(\gamma,\alpha)$ for it. $(\Phi_{\gamma,\mathbf{0}};\ \gamma\in\mathrm{L}) = (\Pi\otimes_{\alpha\in I_\gamma}\varphi_{\alpha,0};\ \gamma\in\mathrm{L}) = (\Pi\otimes_{\alpha\in I_\gamma}f_\alpha;\ \gamma\in\mathrm{L})\in\mathfrak{D}^0$. So Lemma 4.1.4 may be applied again, now to $\Pi\otimes^{\mathfrak{C}^0}_{\gamma\in\mathrm{L}}(\Pi\otimes^{\mathfrak{C}_\gamma}_{\alpha\in I_\gamma}\mathfrak{H}_\alpha)$: The $\Pi\otimes_{\gamma\in\mathrm{L}}\Phi_{\gamma,\beta^0(\gamma,\alpha)} = \Pi\otimes_{\gamma\in\mathrm{L}}(\Pi\otimes_{\alpha\in I_\gamma}\varphi_{\alpha,\beta^0(\gamma,\alpha)})$ form a complete, normalised, orthogonal set in $\Pi\otimes^{\mathfrak{C}^0}_{\gamma\in I}(\Pi\otimes^{\mathfrak{C}_\gamma}_{\alpha\in I_\gamma}\mathfrak{H}_\alpha)$.

Now we have, except for a finite number of γ's, $\beta^0(\gamma)=\mathbf{0}$ that is $\beta^0(\gamma,\alpha)=0$ for all $\alpha\in I_\gamma$. And for the remaining γ's $\beta^0(\gamma)\in\mathbf{F}_\gamma$ so $\beta^0(\gamma,\alpha)=0$ again, except for a finite number of α's (γ being fixed). Thus $\beta^0(\gamma,\alpha)=0$ holds always, except for a finite number of α's (γ is free). As the domain of α in $\beta^0(\gamma,\alpha)$ is I_γ, and as the I_γ, $\gamma\in\mathrm{L}$, are mutually exclusive, we may write $\beta^0(\gamma,\alpha)$ as a function of α only: $\beta^0(\alpha)$, $\alpha\in I$. So the $\Pi\otimes_{\gamma\in\mathrm{L}}(\Pi\otimes_{\alpha\in I_\gamma}\varphi_{\alpha,\beta^0(\alpha)})$ form a complete, normalised, orthogonal set in $\Pi\otimes^{\mathfrak{C}^0}_{\gamma\in\mathrm{L}}(\Pi\otimes^{\mathfrak{C}_\gamma}_{\alpha\in I_\gamma}\mathfrak{H}_\alpha)$, where $\beta^0(\alpha)=0$ except for a finite number of α's. Thus $\varphi_{\alpha,\beta^0(\alpha)}=\varphi_{\alpha,0}=f_\alpha$ with these exceptions, and therefore $(\varphi_{\alpha,\beta^0(\alpha)};\ \alpha\in I)\approx(f_\alpha;\ \alpha\in I)\in\mathfrak{C}$ (by Lemma 3.3.5). Therefore $\Pi\otimes_{\gamma\in\mathrm{L}}(\Pi\otimes_{\alpha\in I_\gamma}\varphi_{\alpha,\beta(\alpha)})\in\overline{\mathfrak{H}}$ and as $\overline{\mathfrak{H}}$ is a closed, linear set, this implies $\overline{\mathfrak{H}}\supset\Pi\otimes^{\mathfrak{C}^0}_{\gamma\in\mathrm{L}}(\Pi\otimes^{\mathfrak{C}_\gamma}_{\alpha\in I_\gamma}\mathfrak{H}_\alpha)$. This completes the proof of $\overline{\mathfrak{H}}=\Pi\otimes^{\mathfrak{C}^0}_{\gamma\in\mathrm{L}}(\Pi\otimes^{\mathfrak{C}_\gamma}_{\alpha\in I_\gamma}\mathfrak{H}_\alpha)$.

We know already that our isomorphism maps $\Pi\otimes_{\alpha\in I}f_\alpha$ on $\Pi\otimes_{\gamma\in\mathrm{L}}(\Pi\otimes_{\alpha\in I_\gamma}f_\alpha)$. Its uniqueness (with this restriction) is obvious by its construction.

This associative law cannot be extended to the complete direct products $\Pi\otimes_{\alpha\in I}\mathfrak{H}_\alpha$ and $\Pi\otimes_{\gamma\in\mathrm{L}}(\Pi\otimes_{\alpha\in I_\gamma}\mathfrak{H}_\alpha)$, when L is infinite, for two reasons.

First, different $\mathfrak{C}$'s may give rise to the same $\mathfrak{C}_\gamma$ and $\mathfrak{C}^0$: Put $\mathrm{L}=\mathfrak{S}(1,2,\ldots)$, $I=\mathfrak{S}(1,2,\ldots)$, $I_\gamma=\mathfrak{S}(2\gamma-1,\ 2\gamma)$ (cf. [30])). Choose in each $\mathfrak{H}_\alpha$, $\alpha\in I$, an $f_\alpha\in\mathfrak{H}_\alpha$ with $\|f_\alpha\|=1$, and put $g_\alpha=-f_\alpha$. Then f_α, $\alpha\in I$, and g_α, $\alpha\in I$, are clearly two inequivalent C_0-sequences for $\Pi\otimes_{\alpha\in I}\mathfrak{H}_\alpha$, but they are equivalent for each $\Pi\otimes_{\alpha\in I_\gamma}\mathfrak{H}_\alpha$ (I_γ is finite), and as $\Pi\otimes_{\alpha\in I_\gamma}f_\alpha=f_{2\gamma-1}\otimes f_{2\gamma}$, $\Pi\otimes_{\alpha\in I_\gamma}g_\alpha=g_{2\gamma-1}\otimes g_{2\gamma}=(-f_{2\gamma-1})\otimes(-f_{2\gamma})=f_{2\gamma-1}\otimes f_{2\gamma}$ so

$\Pi\otimes_{\alpha\in I_\gamma} f_\alpha$, $\gamma\in L$, and $\Pi\otimes_{\alpha\in I_\gamma} g_\alpha$, $\gamma\in L$, are equivalent, too (for $\Pi\otimes_{\gamma\in L}(\Pi\otimes_{\alpha\in I_\gamma}\mathfrak{H}_\alpha)$). Thus $\mathfrak{C}\neq\mathfrak{D}$, but $\mathfrak{C}_\gamma=\mathfrak{D}_\gamma$ and $\mathfrak{C}^0=\mathfrak{D}^0$. (It must be admitted, however, that this phenomenon is due to our way of disposing of quasi-convergence in Definition 2.5.1 (II).)

Second, $\Pi\otimes_{\gamma\in L}(\Pi\otimes_{\alpha\in I_\gamma}\mathfrak{H}_\alpha)$ may possess such equivalence-classes $\mathfrak{C}^0=\mathfrak{C}^0(\Phi_\gamma;\,\gamma\in L)$ too, for which $\Phi_\gamma\in\Pi\otimes_{\alpha\in I_\gamma}\mathfrak{H}_\alpha$ cannot be chosen of the form $\Pi\otimes_{\alpha\in I_\gamma} f_\alpha$. This phenomenon will be decisive in § 7.3.

These complications cannot arise, however, if L is finite (the I_γ may be arbitrary).

Theorem VII. If L is finite, then the isomorphisms mentioned in Theorem VI, (III) can be extended simultaneously to a unique isomorphism of $\Pi\otimes_{\alpha\in I}\mathfrak{H}_\alpha$ and $\Pi\otimes_{\gamma\in L}(\Pi\otimes_{\alpha\in I_\gamma}\mathfrak{H}_\alpha)$.

Proof: As L is finite, only one equivalence-class $\mathfrak{C}^0$ exists for $\Pi\otimes_{\gamma\in L}(\Pi\otimes_{\alpha\in I_\gamma}\mathfrak{H}_\alpha)$ (cf. the second remark at the end of 3.6). Thus Theorem VI, (III) establishes an isomorphism between $\Pi\otimes^{\mathfrak{C}}_{\alpha\in I}\mathfrak{H}_\alpha$ and $\Pi\otimes^{\mathfrak{C}^0}_{\gamma\in L}(\Pi\otimes^{\mathfrak{C}_\gamma}_{\alpha\in I_\gamma}\mathfrak{H}_\alpha)=\Pi\otimes_{\gamma\in L}(\Pi\otimes^{\mathfrak{C}_\gamma}_{\alpha\in I_\gamma}\mathfrak{H}_\alpha)$.

As L is finite, $\Sigma_{\alpha\in I}|(f_\alpha, g_\alpha)-1|$ converges if and only if all $\Sigma_{\alpha\in I_\gamma}|(f_\alpha, g_\alpha)-1|$, $\gamma\in L$, converge (use Lemma 2.3.1). Thus a change of $\mathfrak{C}$ changes at least one $\mathfrak{C}_\gamma$, $\gamma\in L$. On the other hand any prescribed combination $\mathfrak{C}_\gamma$, $\gamma\in L$, belongs to an $\mathfrak{C}$: Choose all representatives f_α with $\|f_\alpha\|=1$ (use Lemma 3.3.7). Thus $\mathfrak{C}$ and all combinations $\mathfrak{C}_\gamma$, $\gamma\in L$, are in a one-to-one correspondence.

The $\Pi\otimes^{\mathfrak{C}}_{\alpha\in I}\mathfrak{H}_\alpha$ are mutually orthogonal, and so are the $\Pi\otimes^{\mathfrak{C}_\gamma}_{\alpha\in I_\gamma}\mathfrak{H}_\alpha$ (use Lemma 4.1.1). Thus the $\Pi\otimes_{\gamma\in L}(\Pi\otimes^{\mathfrak{C}_\gamma}_{\alpha\in I_\gamma}\mathfrak{H}_\alpha)$ are mutually orthogonal too. Therefore the isomorphisms of the various $\Pi\otimes^{\mathfrak{C}}_{\alpha\in I}\mathfrak{H}_\alpha$ with their $\Pi\otimes_{\gamma\in L}(\Pi\otimes^{\mathfrak{C}_\gamma}_{\alpha\in I_\gamma}\mathfrak{H}_\alpha)$ extend in a unique way to an isomorphism of the closed, linear sets determined by the $\Pi\otimes^{\mathfrak{C}}_{\alpha\in I}\mathfrak{H}_\alpha$ resp. the $\Pi\otimes_{\gamma\in L}(\Pi\otimes^{\mathfrak{C}_\gamma}_{\alpha\in I_\gamma}\mathfrak{H}_\alpha)$. The former set is $\Pi\otimes_{\alpha\in I}\mathfrak{H}_\alpha$ (by Lemma 4.1.1), denote the latter by $\overline{\mathfrak{H}}$. So we must only prove $\overline{\mathfrak{H}}=\Pi\otimes_{\gamma\in L}(\Pi\otimes_{\alpha\in I_\gamma}\mathfrak{H}_\alpha)$. But $\subset$ is obvious, so we need only show $\supset$. This is established if we prove $\Pi\otimes_{\gamma\in L}\Phi_\gamma\in\overline{\mathfrak{H}}$ for any $\Phi_\gamma\in\Pi\otimes_{\alpha\in I_\gamma}\mathfrak{H}_\alpha$.

L is a finite set, put $L=\mathfrak{S}(\gamma_1,\ldots,\gamma_m)$, the $\gamma_1,\ldots,\gamma_m$ mutually different. Consider now this statement, R_n ($n=0,1,2,\ldots,m$): If $\Phi_\gamma\in\Pi\otimes_{\alpha\in I_\gamma}\mathfrak{H}_\alpha$ for $\gamma\in L$, but for $\gamma=\gamma_{n+1},\ldots,\gamma_m$ even $\Phi_\gamma\in\Pi\otimes^{\mathfrak{C}_\gamma}_{\alpha\in I_\gamma}\mathfrak{H}_\alpha$ ($\mathfrak{C}_\gamma$ arbitrary), then $\Pi\otimes_{\gamma\in L}\Phi_\gamma\in\overline{\mathfrak{H}}$.

R_0 is true: $\Pi\otimes_{\gamma\in L}\Phi_\gamma \in \Pi\otimes_{\gamma\in L}(\Pi\otimes^{\mathfrak{C}_\gamma}_{\alpha\in I_\gamma}\mathfrak{H}_\alpha) \subset \bar{\mathfrak{H}}$. Assume now, that R_{n-1} holds for some $n-1$, $n=1,2,\ldots,m$, and consider R_n. Choose some $\Phi_\gamma \in \Pi\otimes_{\alpha\in I_\gamma}\mathfrak{H}_\alpha$ for $\gamma\in L$, with $\Phi_\gamma \in \Pi\otimes^{\mathfrak{C}_\gamma}_{\alpha\in I_\gamma}\mathfrak{H}_\alpha$ for $\gamma=\gamma_{n+1},\ldots,\gamma_m$. We ask: For which $\Psi_{\gamma_n}\in\Pi\otimes_{\alpha\in I_{\gamma_n}}\mathfrak{H}_\alpha$ is $\Psi_{\gamma_n}\otimes\Pi\otimes_{\gamma\in L,\gamma\neq\gamma_n}\Phi_\gamma\in\bar{\mathfrak{H}}$? These Ψ_{γ_n} form a closed, linear set $\mathfrak{N}$ in $\Pi\otimes_{\alpha\in I_{\gamma_n}}\mathfrak{H}_\alpha$, considering $\bar{\mathfrak{H}}$ is one, by Lemma 4.1.3. Every $\Pi\otimes^{\mathfrak{C}_{\gamma_n}}_{\alpha\in I_{\gamma_n}}\mathfrak{H}_\alpha\subset\mathfrak{N}$, as we assumed the validity of R_{n-1}. So $\mathfrak{N}=\Pi\otimes_{\alpha\in I_{\gamma_n}}\mathfrak{H}_\alpha$ (by Lemma 4.1.1), and in particular $\Phi_{\gamma_n}\in\mathfrak{N}$. That is $\Pi\otimes_{\gamma\in L}\Phi_\gamma=\Phi_{\gamma_n}\otimes\Pi\otimes_{\gamma\in L,\gamma\neq\gamma_n}\Phi_\gamma\in\bar{\mathfrak{H}}$, proving R_n. Thus all statements R_n are true.

But R_m states this: If $\Phi_\gamma\in\Pi\otimes_{\alpha\in I_\gamma}\mathfrak{H}_\alpha$ for $\gamma\in L$, then $\Pi\otimes_{\gamma\in L}\Phi_\gamma\in\bar{\mathfrak{H}}$, and this completes the proof.

Part III: Operator-rings and the direct product.

Chapter 5: Extension of operators and the direct product.

5.1. We now wish to study the relationship of operators in the various $\mathfrak{H}_\alpha$, $\alpha\in I$, to those in $\Pi\otimes_{\alpha\in I}\mathfrak{H}_\alpha$. We shall denote the ring of all bounded (everywhere defined, linear, and closed) operators in $\mathfrak{H}_\alpha$ by $\mathcal{B}_\alpha$, and the ring of those in $\Pi\otimes_{\alpha\in I}\mathfrak{H}_\alpha$ by $\mathcal{B}_\otimes$ (cf. (7), p. 135).

Lemma 5.1.1. If an operator $A_{\alpha_0}\in\mathcal{B}_{\alpha_0}$ is given, then there exists a unique operator $\bar{A}_{\alpha_0}\in\mathcal{B}_\otimes$, such that for all C-sequences f_α, $\alpha\in I$,

$$\bar{A}_{\alpha_0}(\Pi\otimes_{\alpha\in I}f_\alpha)=\bar{A}_{\alpha_0}(f_{\alpha_0}\otimes\Pi\otimes_{\alpha\in I,\alpha\neq\alpha_0}f_\alpha)=$$
$$=(A_{\alpha_0}f_{\alpha_0})\otimes\Pi\otimes_{\alpha\in I,\alpha\neq\alpha_0}f_\alpha.$$

Proof: $\bar{A}_{\alpha_0}$'s values are prescribed for every $\Pi\otimes_{\alpha\in I}f_\alpha$, so if $\bar{A}_{\alpha_0}\in\mathcal{B}_\otimes$ exists at all, then its values are uniquely determined in the entire closed, linear set determined by the $\Pi\otimes_{\alpha\in I}f_\alpha$, which is $\Pi\otimes_{\alpha\in I}\mathfrak{H}_\alpha$. Thus $\bar{A}_{\alpha_0}\in\mathcal{B}_\otimes$ is unique, if it exists.

If f_α, $\alpha\in I$, is a C-sequence, but not a C_0-sequence, then $A_{\alpha_0}f_{\alpha_0}$ and f_α, $\alpha\in I$, $\alpha\neq\alpha_0$, is such, too. So $\Pi\otimes_{\alpha\in I}f_\alpha=0$ and $(A_{\alpha_0}f_{\alpha_0})\otimes\Pi\otimes_{\alpha\in I,\alpha\neq\alpha_0}f_\alpha=0$ and our requirement for $\bar{A}_{\alpha_0}$ becomes vacuous in this case.

Thus we must only prove the existence of $\bar{A}_{\alpha_0}\in\mathcal{B}_\otimes$, and we need to consider C_0-sequences f_α, $\alpha\in I$, only.

Apply Theorem VII, with $\mathrm{L} = \mathfrak{S}(1, 2,), I_1 = \mathfrak{S}(\alpha_0), I_2 = I - \mathfrak{S}(\alpha_0)$. We have an isomorphism between $\Pi\otimes_{\alpha \in I} \mathfrak{H}_\alpha = \mathfrak{H}_{\alpha_0} \otimes \Pi\otimes_{\alpha \in I, \alpha \neq \alpha_0} \mathfrak{H}_\alpha$ and $\Pi\otimes_{\gamma \in \mathfrak{S}(1,2)} (\Pi\otimes_{\alpha \in I_\gamma} \mathfrak{H}_\alpha) = \mathfrak{H}_{\alpha_0} \otimes \Pi\otimes_{\alpha \in I - \mathfrak{S}(\alpha_0)} \mathfrak{H}_\alpha$ [31]), which makes correspond $\Pi\otimes_{\alpha \in I} f_\alpha = f_{\alpha_0} \otimes \Pi\otimes_{\alpha \in I, \alpha \neq \alpha_0} f_\alpha$ to

$$\Pi\otimes_{\gamma \in \mathfrak{S}(1,2)} (\Pi\otimes_{\alpha \in I_\gamma} f_\alpha) = f_{\alpha_0} \otimes \Pi\otimes_{\alpha \in I - \mathfrak{S}(\alpha_0)} f_\alpha \text{ (cf. }^{31}))$$

for all C_0-sequences f_α, $\alpha \in I$. Thus if our Lemma holds for $\Pi\otimes_{\gamma \in \mathfrak{S}(1,2)} \mathfrak{H}_\gamma$, where $\mathfrak{G}_1 = \mathfrak{H}_{\alpha_0}$, $\mathfrak{G}_2 = \Pi\otimes_{\alpha \in I - \mathfrak{S}(\alpha_0)} \mathfrak{H}_\alpha$ (1 replacing the α_0), then it holds for $\Pi\otimes_{\alpha \in I} \mathfrak{H}_\alpha$, too. In other words: We may assume $I = \mathfrak{S}(1, 2)$, $\alpha_0 = 1$.

Let φ_ϱ, $\varrho \in K$, be a complete, normalised orthogonal set in $\mathfrak{H}_2$. We want an $\bar{A}_1 \in \mathcal{B}_\otimes$ with

$$(*) \qquad \bar{A}_1(f_1 \otimes f_2) = (A_1 f_1) \otimes f_2$$

for all $f_1 \in \mathfrak{H}_1, f_2 \in \mathfrak{H}_2$. Now it suffices to secure (*) for $f_2 = \varphi_\varrho$, $\varrho \in K$, only: For any fixed f_1, the f_2 for which (*) holds, form a closed, linear set in $\mathfrak{H}_2$ (by Lemma 4.1.3), and as it contains all φ_ϱ, $\varrho \in K$, it must be $= \mathfrak{H}_2$.

Consider now the set $\mathfrak{H}'$ of all finite linear aggregates $f_1^{(1)} \otimes \varphi_{\varrho_1} + \ldots + f_1^{(n)} \otimes \varphi_{\varrho_n}$ ($f_1^{(1)}, \ldots, f_1^{(n)} \in \mathfrak{H}_1$, the $\varrho_1, \ldots, \varrho_n \in K$ are mutually different). The closed, linear set determined by $\mathfrak{H}'$ contains all $f_1 \otimes \varphi_\varrho$, and so (by Lemma 4.1.3) all $f_1 \otimes f_2$, therefore it is $\mathfrak{H}_1 \otimes \mathfrak{H}_2$. But $\mathfrak{H}'$ is a linear set, therefore it must be dense in $\mathfrak{H}_1 \otimes \mathfrak{H}_2$.

Define now an operator $\bar{A}_1^-$ in $\mathfrak{H}'$ by

$$(**) \quad \bar{A}_1^-(f_1^{(1)} \otimes \varphi_{\varrho_1} + \ldots + f_1^{(n)} \otimes \varphi_{\varrho_n}) = \\ = (A_1 f_1^{(1)}) \otimes \varphi_{\varrho_1} + \ldots + (A_1 f_1^{(n)}) \otimes \varphi_{\varrho_n}.$$

$\bar{A}_1^-$ is clearly linear. As $A_1 \in \mathcal{B}_1$, there is a C, such that always $\|A_1 f_1\|_1 \leqq C \|f_1\|_1$. Now (**) gives, as the addends on both sides are mutually orthogonal (the $\varrho_1, \ldots, \varrho_n$ being mutually different),

$$\|\bar{A}_1^-(f_1^{(1)} \otimes \varphi_{\varrho_1} + \ldots + f_1^{(n)} \otimes \varphi_{\varrho_n})\|^2 = \\ = \|(A_1 f_1^{(1)}) \otimes \varphi_{\varrho_1} + \ldots + (A_1 f_1^{(n)}) \otimes \varphi_{\varrho_n}\|^2 = \\ = \|(A_1 f_1^{(1)}) \otimes \varphi_{\varrho_1}\|^2 + \ldots + \|(A_1 f_1^{(n)}) \otimes \varphi_{\varrho_n}\|^2 = \\ = \|A_1 f_1^{(1)}\|^2 \cdot \|\varphi_{\varrho_1}\|^2 + \ldots + \|A_1 f_1^{(n)}\|^2 \cdot \|\varphi_{\varrho_n}\|^2 = \\ = \|A_1 f_1^{(1)}\|^2 + \ldots + \|A_1 f_1^{(n)}\|^2 \leqq C^2 (\|f_1^{(1)}\|^2 + \ldots + \|f_1^{(n)}\|^2),$$

[31]) Observe, that $\mathfrak{H}_{\alpha_0} \otimes \Pi\otimes_{\alpha \in I, \alpha \neq \alpha_0} \mathfrak{H}_\alpha$ is just another way to write $\Pi\otimes_{\alpha \in I} \mathfrak{H}_\alpha$ (cf. [26])), while $\mathfrak{H}_{\alpha_0} \otimes \Pi\otimes_{\alpha \in I - \mathfrak{S}(\alpha_0)} \mathfrak{H}_\alpha$ denotes a different object. But Theorem VII establishes their isomorphic character.

$$\|f_1^{(1)} \otimes \varphi_{\varrho_1} + \dots + f_1^{(n)} \otimes \varphi_{\varrho_n}\|^2 = \|f_1^{(1)} \otimes \varphi_{\varrho_1}\|^2 + \dots + f_1^{(n)} \otimes \varphi_{\varrho_n}\|^2 =$$
$$= \|f_1^{(1)}\|^2 \cdot \|\varphi_{\varrho_1}\|^2 + \dots + \|f_1^{(n)}\|^2 \cdot \|\varphi_{\varrho_n}\|^2 = \|f_1^{(1)}\|^2 + \dots + \|f_1^{(n)}\|^2,$$

that is

$$\|\bar{A}_1^- \Phi\|^2 \leqq C^2 \|\Phi\|^2, \ \|\bar{A}_1^- \Phi\| \leqq C \|\Phi\|$$

for all $\Phi \in \mathfrak{H}'$.

This relation, together with the linearity of $\bar{A}_1^-$ and the fact, that $\mathfrak{H}'$ is dense in $\mathfrak{H}_1 \otimes \mathfrak{H}_2$, secure the existence of a closure $\overline{A_1} \in \mathcal{B}_\otimes$ of $\bar{A}_1^-$. (Cf. (10), top of p. 296.) Now

$$\overline{A_1}(f_1 \otimes \varphi_\varrho) = \bar{A}_1^-(f_1 \otimes \varphi_\varrho) = (A_1 f_1) \otimes \varphi_\varrho$$

which is the desired special case of (*). Thus the proof is completed.

Definition 5.1.1. Denote the set of all $\overline{A}_{\alpha_0}$, $A_{\alpha_0} \in \mathcal{B}_{\alpha_0}$, by $\overline{\mathcal{B}}_{\alpha_0}$ ($\subset \mathcal{B}_\otimes$). We will call $\overline{A}_{\alpha_0}$ the extension of A_{α_0}.

Lemma 5.1.2. The correspondence $A_{\alpha_0} \rightleftarrows \overline{A}_{\alpha_0}$ is a one-to-one mapping of $\mathcal{B}_{\alpha_0}$ on $\overline{\mathcal{B}}_{\alpha_0}$, isomorphic for the operations uA (u any complex number), A^* (adjoint), $A + B$, AB. It carries the operators 0_{α_0}, 1_{α_0} (null and unit in $\mathfrak{H}_{\alpha_0}$) into $0_\otimes$, $1_\otimes$ (null and unit in $\Pi\otimes_{\alpha \in I} \mathfrak{H}_\alpha$).

Proof: One-to-one character: $A_{\alpha_0} = B_{\alpha_0}$ implies clearly $\overline{A}_{\alpha_0} = \overline{B}_{\alpha_0}$. Assume conversely $\overline{A}_{\alpha_0} = \overline{B}_{\alpha_0}$. Choose an $f_\alpha \in \mathfrak{H}_\alpha$, $\|f_\alpha\| = 1$ for each $\alpha \in I$, $\alpha \neq \alpha_0$. Then for any $f_{\alpha_0} \in \mathfrak{H}_{\alpha_0}$, f_α, $\alpha \in I$, is a C_0-sequence, and

$$0 = \|\overline{A}_{\alpha_0}(\Pi\otimes_{\alpha \in I} f_\alpha) - \overline{B}_{\alpha_0}(\Pi\otimes_{\alpha \in I} f_\alpha)\| =$$
$$= \|(A_{\alpha_0} f_{\alpha_0}) \otimes \Pi\otimes_{\alpha \in I, \alpha \neq \alpha_0} f_\alpha - (B_{\alpha_0} f_{\alpha_0}) \otimes \Pi\otimes_{\alpha \in I, \alpha \neq \alpha_0} f_\alpha\| =$$
$$= \|(A_{\alpha_0} f_{\alpha_0} - B_{\alpha_0} f_{\alpha_0}) \otimes \Pi\otimes_{\alpha \in I, \alpha \neq \alpha_0} f_\alpha\| =$$
$$= \|A_{\alpha_0} f_{\alpha_0} - B_{\alpha_0} f_{\alpha_0}\| \cdot \Pi_{\alpha \in I, \alpha \neq \alpha_0} \|f_\alpha\| =$$
$$= \|A_{\alpha_0} f_{\alpha_0} - B_{\alpha_0} f_{\alpha_0}\|.$$

So $A_{\alpha_0} f_{\alpha_0} = B_{\alpha_0} f_{\alpha_0}$ and as $f_{\alpha_0} \in \mathfrak{H}_{\alpha_0}$ was arbitrary, therefore $A_{\alpha_0} = B_{\alpha_0}$.

Isomorphism for uA, $A + B$, AB, 0,1: Obvious.

Isomorphism for A^*: For any two C-sequences f_α, $\alpha \in I$, and g_α, $\alpha \in I$, we have

$$(\overline{(A_{\alpha_0})^*} \Pi\otimes_{\alpha \in I} f_\alpha, \Pi\otimes_{\alpha \in I} g_\alpha) =$$
$$= ((A_{\alpha_0})^* f_{\alpha_0} \otimes \Pi\otimes_{\alpha \in I, \alpha \neq \alpha_0} f_\alpha, \ g_{\alpha_0} \otimes \Pi\otimes_{\alpha \in I, \alpha \neq \alpha_0} g_\alpha) =$$
$$= ((A_{\alpha_0})^* f_{\alpha_0}, g_{\alpha_0}) \cdot \Pi_{\alpha \in I, \alpha \neq \alpha_0} (f_\alpha, g_\alpha) =$$
$$= (f_{\alpha_0}, A_{\alpha_0} g_{\alpha_0}) \cdot \Pi_{\alpha \in I, \alpha \neq \alpha_0} (f_\alpha, g_\alpha),$$

$$((\overline{A_{\alpha_0}})^* \Pi\otimes_{\alpha\in I} f_\alpha, \Pi\otimes_{\alpha\in I} g_\alpha) = (\Pi\otimes_{\alpha\in I} f_\alpha, \overline{A_{\alpha_0}}\, \Pi\otimes_{\alpha\in I} g_\alpha) =$$
$$= (f_{\alpha_0} \otimes \Pi\otimes_{\alpha\in I,\, \alpha\neq\alpha_0} f_\alpha,\ A_{\alpha_0} g_{\alpha_0} \otimes \Pi\otimes_{\alpha\in I,\, \alpha\neq\alpha_0} g_\alpha) =$$
$$= (f_{\alpha_0}, A_{\alpha_0} g_{\alpha_0}) \cdot \Pi\otimes_{\alpha\in I,\, \alpha\neq\alpha_0} (f_\alpha, g_\alpha).$$

Thus $(\overline{(A_{\alpha_0})^*}\Phi, \Psi) = ((\overline{A_{\alpha_0}})^* \Phi, \Psi)$ whenever Φ, Ψ have both the form $\Pi\otimes_{\alpha\in I} f_\alpha$. Now this equation extends by continuity to all Φ, $\Psi \in \Pi\otimes_{\alpha\in I} \mathfrak{H}_\alpha$. Hence always $\overline{(A_{\alpha_0})^*}\Phi = (\overline{A_{\alpha_0}})^* \Phi$ and thus $\overline{(A_{\alpha_0})^*} = (\overline{A_{\alpha_0}})^*$.

5.2. Our next objective is the study of the isomorphism $A_{\alpha_0} \rightleftarrows \overline{A_{\alpha_0}}$, the extension, on operator-rings. We will have to use therefore the notions which were introduced and the properties which were established in (9) and (11). These papers dealt with separable spaces, but in most cases no use was made of the separability. It is necessary to discuss therefore, how they apply to arbitrary unitary spaces.

We will use the various operator-topologies: The „weak", the „strong" (cf. (9), pp. 378—388) and the „strongest" (cf. (11), pp. 111—112) topology, cf. our discussion in 1.1, (e). The notion of a *ring* will be used in the same sense as in (9), p. 388, Definition 1: A ring is a subset $\mathcal{M}$ of $\mathcal{B}$ (that is a set of bounded operators in the unitary space $\mathfrak{H}$), which contains uA, A^*, $A+B$, AB along with A, B and which is „weakly" closed. The last condition can be replaced equivalently by „strongly" closed or even by „strongest" closed (assuming the preceding algebraic condition): The proofs given in (9), pp. 393—396, and (11), pp. 112—113, hold verbatim for every unitary $\mathfrak{H}$.

For any subset $\mathcal{S}$ of $\mathcal{B}$ we again define $\mathcal{S}'$ as the set of those $A \in \mathcal{B}$, which commute with B, B^* for all $B \in \mathcal{S}$. The considerations of (9), pp. 388—398, apply verbatim, while those on pp. 398—404 (on Abelian rings) make use of the separability of $\mathfrak{H}$, and are therefore invalid [32]).

Thus $\mathcal{S}'$ is always a ring and contains 1, and $\mathcal{S}'' = \mathcal{S}$ holds if and only if $\mathcal{S}$ is a ring containing 1.

I is again an arbitrary set of indices, each $\mathfrak{H}_\alpha$, $\alpha \in I$, a unitary space, α_0 a fixed element of I.

LEMMA 5.2.1. $\overline{\mathcal{B}_{\alpha_0}}$ is a ring containing 1.

---

[32]) This seems to be the only part of the general theory, where separability of $\mathfrak{H}$ is essential.

Proof: [33]) Proceed as in the proof of Lemma 5.1.1: Apply Theorem VII, with $\mathrm{L} = \mathfrak{S}(1, 2)$, $I_1 = \mathfrak{S}(\alpha_0)$, $I_2 = I - \mathfrak{S}(\alpha_0)$. We have an isomorphism between $\Pi\otimes_{\alpha\epsilon I}\mathfrak{H}_\alpha = \mathfrak{H}_{\alpha_0}\otimes\Pi\otimes_{\alpha\epsilon I,\,\alpha\neq\alpha_0}\mathfrak{H}_\alpha$ and $\Pi\otimes_{\gamma\epsilon\mathfrak{S}(1,2)}(\Pi\otimes_{\alpha\epsilon I_\gamma}\mathfrak{H}_\alpha) = \mathfrak{H}_{\alpha_0}\otimes\Pi\otimes_{\alpha\epsilon I-\mathfrak{S}(\alpha_0)}\mathfrak{H}_\alpha$ which makes correspond

$\Pi\otimes_{\alpha\epsilon I} f_\alpha = f_{\alpha_0}\otimes\Pi\otimes_{\alpha\epsilon I,\,\alpha\neq\alpha_0} f_\alpha$ to

$$\Pi\otimes_{\gamma\epsilon\mathfrak{S}(1,2)}(\Pi\otimes_{\alpha\epsilon I_\gamma} f_\alpha) = f_{\alpha_0}\otimes\Pi\otimes_{\alpha\epsilon I-\mathfrak{S}(\alpha_0)} f_\alpha$$

for all C_0-sequences f_α, $\alpha\epsilon I$ (cf. [31])). Thus if our Lemma holds for $\Pi\otimes_{\gamma\epsilon\mathfrak{S}(1,2)}\mathfrak{G}_\gamma$ where $\mathfrak{G}_1 = \mathfrak{H}_{\alpha_0}$, $\mathfrak{G}_2 = \Pi\otimes_{\alpha\epsilon I,\,\alpha\neq\alpha_0}\mathfrak{H}_\alpha$ (1 replacing the α_0), it holds for $\Pi\otimes_{\alpha\epsilon I}\mathfrak{H}_\alpha$ too. In other words: We may assume $I = \mathfrak{S}(1, 2)$, $\alpha_0 = 1$.

If $A_1\epsilon B_1$, $A_2\epsilon B_2$ then $\overline{A_1}\,\overline{A_2}\Phi = \overline{A_2}\,\overline{A_1}\Phi$ for $\Phi = f_1\otimes f_2$ (both are $= A_1 f_1\otimes A_2 f_2$), so for every $\Phi\epsilon\mathfrak{H}_1\otimes\mathfrak{H}_2$. Thus A_1, A_2 commute, that is $\overline{B_1}\subset(\overline{B_2})'$.

Assume now conversely $A\epsilon(\overline{B_2})'$. Consider any $f_2\epsilon\mathfrak{H}_2$, $\|f_2\| = 1$, and the operator $P_{[f_2]}\epsilon\mathfrak{B}_2$ [34]). Then $E = \overline{P_{[f_2]}}\epsilon\overline{\mathfrak{B}_2}$, and for every $\Phi = g_1\otimes g_2$, $E\Phi = \overline{P_{[f_2]}}g_1\otimes g_2 = g_1\otimes P_{[f_2]}g_2 = g_1\otimes(g_2, f_2)f_2 =$ $= (g_2, f_2)\,g_1\otimes f_2$, so $E\Phi$ has the form $h_1\otimes f_2$. Thus this holds for every $\Phi\epsilon\mathfrak{H}_1\otimes\mathfrak{H}_2$. Conversely, if $\Phi = h_1\otimes f_2$, then the above formula gives $E\Phi = h_1\otimes f_2 = \Phi$. So E is the projection of the closed, linear set of the $h_1\otimes f_2$, $h_1\epsilon\mathfrak{H}_1$. Now A commutes with this $E\epsilon\overline{\mathfrak{B}_2}$ by assumption, so $A(h_1\otimes f_2)$ has again this form, say $h_1'\otimes f_2$; h_1' could depend on both h_1, f_2.

Consider next any other $\widehat{f_2}\epsilon\mathfrak{H}_2$, $\|\widehat{f_2}\| = 1$, then $A(h_1\otimes\widehat{f_2}) = \widehat{h_1'}\otimes\widehat{f_2}$. Choose a $U_2\epsilon\mathfrak{B}_2$ with $U_2 f_2 = \widehat{f_2}$. $\overline{U_2}\epsilon\overline{\mathfrak{B}_2}$, so A commutes with $\overline{U_2}$ by assumption. But $\overline{U_2}(h_1\otimes f_2) = h_1\otimes U_2 f_2 = h_1\otimes\widehat{f_2}$, $\overline{U_2}(h_1'\otimes f_2) = h_1'\otimes U_2 f_2 = h_1'\otimes\widehat{f_2}$ so $h_1'\otimes\widehat{f_2} = \widehat{h_1'}\otimes\widehat{f_2}$, $(h_1' - \widehat{h_1'})\otimes\widehat{f_2} = 0$ and thus (form the $\|\ldots\|$) $h_1' - \widehat{h_1'} = 0$. So h_1' does not depend on f_2. Therefore an operator A_1 can be defined by $A_1 h_1 = h_1'$. Thus

$$(*)\qquad A(h_1\otimes f_2) = (A_1 h_1)\otimes f_2$$

for $\|f_2\| = 1$. But (*) extends immediately to all $f_2\epsilon\mathfrak{H}_2$.

A_1 is clearly linear. As $A\epsilon\mathfrak{B}_\otimes$, so a C with $\|A\Phi\|\leqq C\|\Phi\|$ exists. Choose again $\|f_2\| = 1$, then

---

[33]) $\overline{\mathfrak{B}_{\alpha_0}}$ contains uA, A^*, $A+B$, AB along with A, B as well as 1, by Lemma 5.1.2. The essential point is to prove, that it is weakly closed.

[34]) The projection of the closed, linear set $[f_2]$:

$$P_{[f_2]}g_2 = (g_2, f_2)\cdot f_2.$$

$$\|A_1 h_1\| = \|A_1 h_1\| \cdot \|f_2\| = \|(A_1 h_1) \otimes f_2\| = \|A(h_1 \otimes f_2)\| \leqq$$
$$\leqq C\|h_1 \otimes f_2\| = C\|h_1\| \cdot \|f_2\| = C\|h_1\|.$$

Thus $A_1 \in \mathfrak{B}_1$. Now (*) makes clear, that $A = \overline{A_1} \in \overline{\mathfrak{B}_1}$. Therefore $(\overline{\mathfrak{B}_2})' \subset \overline{\mathfrak{B}_1}$.

So $\overline{\mathfrak{B}_1} = (\overline{\mathfrak{B}_2})'$, and $(\overline{\mathfrak{B}_2})'$ is clearly a ring containing 1.

LEMMA 5.2.2. Every $\Phi \in \mathfrak{H}_1 \otimes \mathfrak{H}_2$ can be written as a finite or enumerably infinite (strongly convergent) sum

$$\Phi = f_1 \otimes \omega_1 + f_2 \otimes \omega_2 + \dots$$

where $f_1, f_2, \dots \in \mathfrak{H}_1$, $\omega_1, \omega_2, \dots \in \mathfrak{H}_2$ and the latter form a normalised, orthogonal set.

Proof: Let φ_ϱ, $\varrho \in K_1$, be a complete, normalised, orthogonal set in $\mathfrak{H}_1$, and ψ_σ, $\sigma \in K_2$, one in $\mathfrak{H}_2$. Then $\varphi_\varrho \otimes \psi_\sigma$, $\varrho \in K_1$, $\sigma \in K_2$, is one in $\mathfrak{H}_1 \otimes \mathfrak{H}_2$ (by Lemma 4.1.6, remembering the second remark at the end of 3.6). Thus

$$\Phi = u_1 \varphi_{\varrho_1} \otimes \psi_{\sigma_1} + u_2 \varphi_{\varrho_2} \otimes \psi_{\sigma_2} + \dots$$

where $|u_1|^2 + |u_2|^2 + \dots$ is finite, and the pairs (ϱ_1, σ_1), $(\varrho_2, \sigma_2), \dots$ are mutually different. Let $\tau_1, \tau_2, \dots$ be the different ones among the $\varrho_1, \varrho_2, \dots$, and $v_1, v_2, \dots$ the different ones among the $\sigma_1, \sigma_2, \dots$, then

$$\Phi = \Sigma_{i=1,2,\dots} \Sigma_{j=1,2,\dots} v_{ij} \varphi_{\tau_i} \otimes \psi_{v_i}$$

$\Sigma_{i=1,2,\dots} \Sigma_{j=1,2,\dots} |v_{ij}|^2$ being finite. $\left(\text{Put } v_{ij} \begin{cases} = u_k \text{ if } (\tau_i, v_j) = (\varrho_k, \sigma_k) \\ = 0 \text{ if } (\tau_i, v_j) \neq \text{all } (\varrho_k, \sigma_k) \end{cases}.\right)$
So $\Sigma_{i=1,2,\dots} |v_{ij}|^2$ is finite for every j, and we can form

$$f_j = \Sigma_{i=1,2,\dots} v_{ij} \varphi_{\tau_i} \in \mathfrak{H}_1 .$$

Then $\Sigma_{i=1,2,\dots} v_{ij} \varphi_{\tau_i} \otimes \psi_{v_j} = (\Sigma_{i=1,2,\dots} v_{ij} \varphi_{\tau_i}) \otimes \psi_{v_j} = f_j \otimes \psi_{v_j}$ (use Lemma 4.1.3), and so

$$\Phi = f_1 \otimes \psi_{v_1} + f_2 \otimes \psi_{v_2} + \dots .$$

Thus $\omega_i = \psi_{v_i}$ have the desired properties.

LEMMA 5.2.3. Let $\mathcal{S}_{\alpha_0}$ be a set $\subset \mathfrak{B}_{\alpha_0}$ and $\overline{\mathcal{S}_{\alpha_0}}$ its image under the isomorphism $A_{\alpha_0} \rightleftarrows \overline{A_{\alpha_0}}$. Then $\mathcal{S}_{\alpha_0}$ is a ring if and only if $\overline{\mathcal{S}_{\alpha_0}}$ is one.

Proof: Just as in the proof of Lemma 5.2.1, we may assume $I = \mathfrak{S}(1, 2)$, $\alpha_0 = 1$.

A ring can be defined as follows: It contains uA, A^*, $A + B$, AB along with A, B and it is closed in the strongest topology. (Cf. the discussion at the beginning of 5.2.) The first four properties hold for $\mathcal{S}_1$ if and only if they hold for $\overline{\mathcal{S}_1}$, owing to Lemma 5.1.2. And for the last one, closure in the strongest topology, this follows from (11), p. 114 (§ 4). In fact: The separability of $\mathfrak{H}_1 \otimes \mathfrak{H}_2$ was not used there, the symbols $\langle f, 0, 0, \dots \rangle$ (for a given $f \in \mathfrak{H}_1$) which occur there, may be replaced by $f \otimes \varphi$ (with any $\varphi \in \mathfrak{H}_2$, $\|\varphi\| = 1$), and the symbols $\Phi = \langle f_1, f_2, \dots \rangle$ (for a given $\Phi \in \mathfrak{H}_1 \otimes \mathfrak{H}_2$) by the $f_1 \otimes \omega_1 + f_2 \otimes \omega_2 + \dots$ of Lemma 5.2.2.

Theorem VIII. The correspondence $A_{\alpha_0} \rightleftarrows \overline{A_{\alpha_0}}$, as defined in Lemma 5.1.1, is a one-to-one mapping of $\mathcal{B}_{\alpha_0}$ on a certain subset $\overline{\mathcal{B}_{\alpha_0}}$ of $\mathcal{B}_{\otimes}$, which is a ring containing 1. It is an isomorphism for the operations uA, A^*, $A + B$, AB and for 0, 1. If it maps a set $\mathcal{S}_{\alpha_0} \subset \mathcal{B}_{\alpha_0}$ on the set $\overline{\mathcal{S}_{\alpha_0}} \subset \overline{\mathcal{B}_{\alpha_0}}$, then $\mathcal{S}_{\alpha_0}$ is a ring if and only if $\overline{\mathcal{S}_{\alpha_0}}$ is one.

Proof: This follows immediately from Lemmata 5.1.2, 5.2.1, 5.2.3.

Chapter 6: The ring of all extended operators.

6.1. We saw in Lemma 3.3.6, that the intuitively plausible equation $\Pi\otimes_{\alpha \in I} z_\alpha f_\alpha = \Pi_{\alpha \in I} z_\alpha \cdot \Pi\otimes_{\alpha \in I} f_\alpha$ holds only, if $\Pi_{\alpha \in I} z_\alpha$ is convergent (or $\Pi\otimes_{\alpha \in I} f_\alpha = 0$). Otherwise the sequence $z_\alpha f_\alpha$, $\alpha \in I$, and f_α, $\alpha \in I$, are not even equivalent, that is the $\Pi\otimes_{\alpha \in I} z_\alpha f_\alpha$ and $\Pi\otimes_{\alpha \in I} f_\alpha$ belong to two different incomplete direct products $\Pi\otimes_{\alpha \in I}^{\mathfrak{C}} \mathfrak{H}_\alpha$. This situation provides the motive for the definition which follows:

DEFINITION 6.1.1. Two C_0-sequences f_α, $\alpha \in I$, and g_α, $\alpha \in I$, are *weakly equivalent*, in symbols $(f_\alpha;\ \alpha \in I) \underset{w}{\approx} (g_\alpha;\ \alpha \in I)$, if complex numbers z_α, $\alpha \in I$, can be found, such that $z_\alpha f_\alpha$, $\alpha \in I$, is a C_0-sequence, and equivalent to g_α, $\alpha \in I$.

LEMMA 6.1.1. Without modifying the meaning of Definition 6.1.1, we may require that all $|z_\alpha| = 1$.

Proof: In other words: If f_α, $\alpha \in I$, and $z_\alpha f_\alpha$, $\alpha \in I$, are C_0-sequences we can find such z'_α with $|z'_\alpha| = 1$, that $(z_\alpha f_\alpha;\ \alpha \in I) \approx (z'_\alpha f_\alpha;\ \alpha \in I)$.

As $\Sigma_{\alpha \in I} \big| \|z_\alpha f_\alpha\| - 1 \big|$ converges, therefore $z_\alpha f_\alpha = 0$ (which implies $\big| \|z_\alpha f_\alpha\| - 1 \big| = 1$) can occur for a finite number of α's only. For these we may replace z_α by 1 and f_α by some $f_\alpha^0 \neq 0$ (use Lemma 3.3.5). So we may assume, that always $z_\alpha f_\alpha \neq 0$.

As $\Sigma_{\alpha \in I} \big| \|f_\alpha\| - 1 \big|$, $\Sigma_{\alpha \in I} \big| \|z_\alpha f_\alpha\| - 1 \big|$ converge, and all $\|f_\alpha\|$, $\|z_\alpha f_\alpha\| \neq 0$ so Lemma 2.4.1, (II), secures the convergence of $\Pi_{\alpha \in I} \|f_\alpha\|$, $\Pi_{\alpha \in I} \|z_\alpha f_\alpha\|$ and that their values are $\neq 0$. Thus

$\prod_{\alpha \in I} \frac{1}{|z_\alpha|}$ converges too, and has a value $\neq 0$, because $\frac{\|f_\alpha\|}{\|z_\alpha f_\alpha\|} = \frac{1}{|z_\alpha|}$. Therefore $\sum_{\alpha \in I} \left| \frac{1}{|z_\alpha|} - 1 \right|$ converges by Lemma 2.4.1, (II). Now Lemma 3.3.6, (II), (IV), give $\left(\text{replace their } z_\alpha \text{ by } \frac{1}{|z_\alpha|}\right)$, that $z_\alpha f_\alpha$, $\alpha \in I$, and $\frac{z_\alpha}{|z_\alpha|} f_\alpha$, $\alpha \in I$, are equivalent. Thus $z'_\alpha = \frac{z_\alpha}{|z_\alpha|}$ has all the desired properties.

LEMMA 6.1.2. The weak equivalence $\underset{w}{\approx}$ for C_0-sequences is reflexive, symmetric, and transitive:

(I) $(f_\alpha;\ \alpha \in I) \underset{w}{\approx} (f_\alpha;\ \alpha \in I)$,

(II) $(f_\alpha;\ \alpha \in I) \underset{w}{\approx} (g_\alpha;\ \alpha \in I)$ implies $(g_\alpha;\ \alpha \in I) \underset{w}{\approx} (f_\alpha;\ \alpha \in I)$,

(III) $(f_\alpha;\ \alpha \in I) \underset{w}{\approx} (g_\alpha;\ \alpha \in I)$, $(g_\alpha;\ \alpha \in I) \underset{w}{\approx} (h_\alpha;\ \alpha \in I)$ imply $(f_\alpha;\ \alpha \in I) \underset{w}{\approx} (h_\alpha;\ \alpha \in I)$.

Proof: Obvious, since we may restrict ourselves to z_α with $|z_\alpha| = 1$ by Lemma 6.1.1.

DEFINITION 6.1.2. The weak equivalence $\underset{w}{\approx}$ decomposes the set of all C_0-sequences into mutually disjoint weak equivalence classes. (Cf. Lemma 6.1.2.) Denote the set formed by these equivalence classes by Γ_w, and the equivalence class of a given C_0-sequence f_α, $\alpha \in I$, by $\mathfrak{C}_w(f_\alpha;\ \alpha \in I)$.

Since equivalence implies weak equivalence, therefore every $\mathfrak{C} \in \Gamma$ is $\mathfrak{C} \subset \mathfrak{C}_w$ for exactly one $\mathfrak{C}_w \subset \Gamma_w$, and every $\mathfrak{C}_w \subset \Gamma_w$ is the sum of all $\mathfrak{C} \subset \Gamma$ with $\mathfrak{C} \subset \mathfrak{C}_w$.

DEFITION 6.1.3. If $\mathfrak{C}_w \in \Gamma_w$ is a weak equivalence class, then let $\prod \underset{w}{\otimes}{}_{\alpha \in I}^{\mathfrak{C}_w} \mathfrak{H}_\alpha$ be the closed, linear set determined by all $\prod \otimes_{\alpha \in I} f_\alpha$, where f_α, $\alpha \in I$, is any C_0-sequence from $\mathfrak{C}_w$. Clearly $\prod \underset{w}{\otimes}{}_{\alpha \in I}^{\mathfrak{C}_w} \mathfrak{H}_\alpha \subset \prod \otimes_{\alpha \in I} \mathfrak{H}_\alpha$.

Our previous remarks show, that $\prod \underset{w}{\otimes}{}_{\alpha \in I}^{\mathfrak{C}_w} \mathfrak{H}_\alpha$ is the closed, linear set determined by all $\prod \otimes_{\alpha \in I}^{\mathfrak{C}} \mathfrak{H}_\alpha$ with $\mathfrak{C} \in \Gamma$, $\mathfrak{C} \subset \mathfrak{C}_w$.

An explicit criterium of $\underset{w}{\approx}$:

LEMMA 6.1.3. $(f_\alpha;\ \alpha \in I) \underset{w}{\approx} (g_\alpha;\ \alpha \in I)$ if and only if $\sum_{\alpha \in I} \big| |(f_\alpha, g_\alpha)| - 1 \big|$ converges [35]).

Proof: $(f_\alpha;\ \alpha \in I) \underset{w}{\approx} (g_\alpha;\ \alpha \in I)$ means by Lemma 6.1.1 (and Definition 3.3.2), that we can find such numbers z_α with $|z_\alpha| = 1$

[35]) If $\|f_\alpha\| = \|g_\alpha\| = 1$, then $|(f_\alpha, g_\alpha)| \leqq 1$, and so we may write $\sum_{\alpha \in I} \left(1 - |(f_\alpha, g_\alpha)|\right)$ instead.

that $\sum_{\alpha\in I}|(z_\alpha f_\alpha, g_\alpha)-1|$ converges. Now for every given $\alpha\in I$ $|(z_\alpha f_\alpha, g_\alpha)-1|$ depends on z_α only, and possesses (and assumes) a minimum: $(z_\alpha f_\alpha, g_\alpha)=z_\alpha(f_\alpha, g_\alpha)$ varies over a circle of center 0 and of radius $|(f_\alpha, g_\alpha)|$ when z_α varies over the circle $|z_\alpha|=1$, the nearest point to 1 on which is $|(f_\alpha, g_\alpha)|$. Thus the minimum of $|(z_\alpha f_\alpha, g_\alpha)-1|$ is $||(f_\alpha, g_\alpha)|-1|$. Thus our condition becomes this: $\sum_{\alpha\in I}||(f_\alpha, g_\alpha)|-1|$ must converge.

6.2. LEMMA 6.2.1. Assume that a z_α with $|z_\alpha|=1$ is given for each $\alpha\in I$. Then there exists one and only one closed, linear operator U, such that

$$U\prod\otimes_{\alpha\in I}f_\alpha=\prod\otimes_{\alpha\in I}z_\alpha f_\alpha$$

for every C_0-sequence f_α, $\alpha\in I$. This U is unitary.

Proof: Existence and unitary character: Apply Theorem IV to $\mathfrak{H}=\prod\otimes_{\alpha\in I}\mathfrak{H}_\alpha$, $\prod\otimes^*_{\alpha\in I}f_\alpha=\prod\otimes_{\alpha\in I}z_\alpha f_\alpha$ (use C_0-sequences only). Its conditions are fulfilled: (I) obviously, (III) because every $\prod\otimes_{\alpha\in I}f_\alpha$ is a $\prod\otimes^*_{\alpha\in I}g_\alpha$ (with $g_\alpha=\bar{z}_\alpha f_\alpha$), and (II) owing to

$$(\prod\otimes^*_{\alpha\in I}f_\alpha, \prod\otimes^*_{\alpha\in I}g_\alpha)=(\prod\otimes_{\alpha\in I}z_\alpha f_\alpha, \prod\otimes_{\alpha\in I}z_\alpha g_\alpha)=$$
$$=\prod_{\alpha\in I}(z_\alpha f_\alpha, z_\alpha g_\alpha)=\prod_{\alpha\in I}(f_\alpha, g_\alpha)=(\prod\otimes_{\alpha\in I}f_\alpha, \prod\otimes_{\alpha\in I}g_\alpha).$$

Thus an isomorphism of $\prod\otimes_{\alpha\in I}\mathfrak{H}_\alpha$ on itself exists, which carries every $\prod\otimes_{\alpha\in I}f_\alpha$ into its $\prod\otimes^*_{\alpha\in I}f_\alpha=\prod\otimes_{\alpha\in I}z_\alpha f_\alpha$. This isomorphism may be looked at as a unitary (and therefore closed, linear) operator U, and it possesses the desired properties.

Uniqueness: If U is the above defined unitary operator, and U' another closed, linear operator which meets our requirements, then U, U' agree for all $\prod\otimes_{\alpha\in I}f_\alpha$, and so for the finite linear aggregates of these, too. Thus they agree on an everywhere dense set, but U is continuous and U' is closed, therefore they agree everywhere.

DEFINITION 6.2.1. Denote the U of Lemma 6.2.1 by $U(z_\alpha;\ \alpha\in I)$.

Denote the projection operator of $\prod\otimes^{\mathfrak{C}}_{\alpha\in I}\mathfrak{H}_\alpha$, $\mathfrak{C}\in\Gamma$, by $P[\mathfrak{C}]$ and that one of $\prod\underset{w}{\otimes}{}^{\mathfrak{C}_w}_{\alpha\in I}\mathfrak{H}_\alpha$, $\mathfrak{C}_w\in\Gamma_w$, by $P_w[\mathfrak{C}_w]$.

LEMMA 6.2.2. $U(z_\alpha;\ \alpha\in I)$ maps $\prod\underset{w}{\otimes}{}^{\mathfrak{C}_w}_{\alpha\in I}\mathfrak{H}_\alpha$ on itself, that is, it commutes with $P_w[\mathfrak{C}_w]$.

Proof: $U(z_\alpha;\ \alpha\in I)$ maps $\prod\otimes_{\alpha\in I}f_\alpha$ on $\prod\otimes_{\alpha\in I}z_\alpha f_\alpha$; if $(f_\alpha;\ \alpha\in I)\in\mathfrak{C}_w$ then $(z_\alpha f_\alpha;\ \alpha\in I)\in\mathfrak{C}_w$ too, so $U(z_\alpha;\ \alpha\in I)$ maps $\prod\underset{w}{\otimes}{}^{\mathfrak{C}_w}_{\alpha\in I}\mathfrak{H}_\alpha$ on part of itself. As the same is true for the inverse of $U(z_\alpha;\ \alpha\in I)$, that is $U(\bar{z}_\alpha;\ \alpha\in I)$, therefore it maps $\prod\underset{w}{\otimes}{}^{\mathfrak{C}_w}_{\alpha\in I}\mathfrak{H}_\alpha$ exactly on itself.

Lemma 6.2.3. (I) $U(z_\alpha;\ \alpha\in I)$ maps $\Pi\otimes^{\mathfrak{C}}_{\alpha\in I}\mathfrak{H}_\alpha$ on itself, that is it commutes with $P[\mathfrak{C}]$, if and only if $\Pi_{\alpha\in I}z_\alpha$ converges [36]. Then $U(z_\alpha;\ \alpha\in I)=(\Pi_{\alpha\in I}z_\alpha)\cdot 1$.

(II) If this is not the case, then $U(z_\alpha;\ \alpha\in I)$ maps $\Pi\otimes^{\mathfrak{C}}_{\alpha\in I}\mathfrak{H}_\alpha$ on a $\Pi\otimes^{\mathfrak{D}}_{\alpha\in I}\mathfrak{H}_\alpha$ with $\mathfrak{C}\neq\mathfrak{D}$.

Proof: We proceed in a somewhat changed order:

Ad (II): Assume that $\Pi_{\alpha\in I}z_\alpha$ does not converge. Then $(f_\alpha;\ \alpha\in I)\in\mathfrak{C}$ implies $(z_\alpha f_\alpha;\ \alpha\in I)\in\mathfrak{D}\neq\mathfrak{C}$ owing to Lemma 3.3.6, (IV). $\mathfrak{D}$ depends on $\mathfrak{C}$ only (and not on the choice of f_α, $\alpha\in I$), because $(f_\alpha;\ \alpha\in I)\approx(g_\alpha;\ \alpha\in I)$ implies $(z_\alpha f_\alpha;\ \alpha\in I)\approx(z_\alpha g_\alpha;\ \alpha\in I)$, as $(z_\alpha f_\alpha, z_\alpha g_\alpha)=(f_\alpha, g_\alpha)$ (use Definition 3.3.2). Thus $U(z_\alpha;\ \alpha\in I)$ maps $\Pi\otimes^{\mathfrak{C}}_{\alpha\in I}\mathfrak{H}_\alpha$ on part of $\Pi\otimes^{\mathfrak{D}}_{\alpha\in I}\mathfrak{H}_\alpha$. Similarly its inverse, $U(\bar z_\alpha;\ \alpha\in I)$, maps $\Pi\otimes^{\mathfrak{D}}_{\alpha\in I}\mathfrak{H}_\alpha$ on part of $\Pi\otimes^{\mathfrak{C}}_{\alpha\in I}\mathfrak{H}_\alpha$. Therefore $\Pi\otimes^{\mathfrak{C}}_{\alpha\in I}\mathfrak{H}_\alpha$ has exactly the image $\Pi\otimes^{\mathfrak{D}}_{\alpha\in I}\mathfrak{H}_\alpha$, and we know that $\mathfrak{C}\neq\mathfrak{D}$.

Ad (I): Sufficiency: If $\Pi_{\alpha\in I}z_\alpha$ converges, then Lemma 3.3.6, (III), secures $U(z_\alpha;\ \alpha\in I)=(\Pi_{\alpha\in I}z_\alpha)\cdot 1$, and the remainder is immediate.

Necessity: Obvious by (II).

Lemma 6.2.4. For any operator $A_{\alpha_0}\in\mathfrak{B}_{\alpha_0}$, $\alpha_0\in I$, the extended operator $\overline{A_{\alpha_0}}$ commutes with every $P[\mathfrak{C}]$, $P_w[\mathfrak{C}_w]$ and $U(z_\alpha;\ \alpha\in I)$.

Proof: $P[\mathfrak{C}]$ and $P_w[\mathfrak{C}_w]$: $\overline{A_{\alpha_0}}\,(\Pi\otimes_{\alpha\in I}f_\alpha)=\Pi\otimes_{\alpha\in I}g_\alpha$ where $g_\alpha\begin{cases}=f_\alpha & \text{for } \alpha\neq\alpha_0\\ =A_{\alpha_0}f_{\alpha_0} & \text{for } \alpha=\alpha_0\end{cases}$. So $(f_\alpha;\ \alpha\in I)\approx(g_\alpha;\ \alpha\in I)$(by Lemma 3.3.5), and a fortiori $(f_\alpha;\ \alpha\in I)\underset{w}{\approx}(g_\alpha;\ \alpha\in I)$. Thus $\overline{A_{\alpha_0}}$ maps $\Pi\otimes^{\mathfrak{C}}_{\alpha\in I}\mathfrak{H}_\alpha$ on part of itself, and $\Pi\underset{w}{\otimes}{}^{\mathfrak{C}_w}_{\alpha\in I}\mathfrak{H}_\alpha$ on part of itself, too. The same is true for $(\overline{A_{\alpha_0}})^*=\overline{(A_{\alpha_0})^*}$. Therefore $\overline{A_{\alpha_0}}$ commutes with both $P[\mathfrak{C}]$ and $P_w[\mathfrak{C}_w]$ [37].

$U(z_\alpha;\ \alpha\in I)$: $U(z_\alpha;\ \alpha\in I)^{-1}\overline{A_{\alpha_0}}\,U(z_\alpha;\ \alpha\in I)$ obviously possesess the definitory properties of $\overline{A_{\alpha_0}}$, as given in Lemma 5.1.1. Therefore $U(z_\alpha;\ \alpha\in I)^{-1}\,\overline{A_{\alpha_0}}\,U(z_\alpha;\ \alpha\in I)=\overline{A_{\alpha_0}}$, $\overline{A_{\alpha_0}}\,U(z_\alpha;\ \alpha\in I)=$ $=U(z_\alpha;\ \alpha\in I)\,\overline{A_{\alpha_0}}$.

---

[36]) This is equivalent to the convergence of $\Sigma_{\alpha\in I}\,|\,\text{arcus}\, z_\alpha\,|$, or to that one of $\Sigma_{\alpha\in I}|z_\alpha-1|$. (Use Lemma 2.4.1, (II), and Lemma 3.3.6, (IV), remembering that $|z_\alpha|=1$.)

[37]) Let E be the projection of $\mathfrak{M}$, and A, A^* both map $\mathfrak{M}$ on parts of itself. This means $EAE=AE$ and $EA^*E=A^*E$ Apply * to the second equation, then $EAE=EA$ ensues, and so $AE=EA$.

6.3. Definition 6.3.1. Denote the ring generated by all $\overline{A_{\alpha_0}}$ with arbitrary $A_{\alpha_0} \in \mathcal{B}_{\alpha_0}$ and all $\alpha_0 \in I$, by $\mathcal{B}^{\#}$. Clearly $\mathcal{B}^{\#} \subset \mathcal{B}_{\otimes}$.

Obviously $B^{\#}$ might have just as well been defined as the ring generated by all $\overline{\mathcal{B}_{\alpha_0}}$, $\alpha_0 \in I$. (Cf. Definition 5.1.1 and Lemma 5.2.1.) That is:

$$\mathcal{B}^{\#} = \mathcal{R}\,(\overline{\mathcal{B}_{\alpha_0}};\ \alpha_0 \in I).$$

If I is finite, then $\mathcal{B}^{\#} = \mathcal{B}_{\otimes}$ (cf. (7), p. 135), and we will prove (Theorem IX or X), that this holds only if I is finite. Now $\mathcal{B}^{\#}$ is in a way more important than $\mathcal{B}_{\otimes}$: The elements of $\mathcal{B}^{\#}$ arise from those of the $\mathcal{B}_{\alpha_0}$, $\alpha_0 \in I$, by extension (cf. Definition 5.1.1) and algebraical and topological processes. In other words: They are the only operators in $\Pi \otimes_{\alpha \in I} \mathfrak{H}_{\alpha}$, which are based directly on operators in the $\mathfrak{H}_{\alpha}$, $\alpha \in I$. Therefore it is of importance to determine the structure of $\mathcal{B}^{\#}$, as $\mathcal{B}^{\#}$ may no longer be identical with $\mathcal{B}_{\otimes}$.

Lemma 6.3.1. Every $A \in \mathcal{B}^{\#}$ commutes with all $P[\mathfrak{C}]$, $(\mathfrak{C} \in \Gamma)$, $P_w[\mathfrak{C}_w]$, $(\mathfrak{C}_w \in \Gamma_w)$, and $U(z_{\alpha};\ \alpha \in I)$ $(|z_{\alpha}| = 1)$.

Proof: Put $X = P[\mathfrak{C}]$ or $P_w[\mathfrak{C}_w]$ or $U(z_{\alpha};\ \alpha \in I)$. If $A_{\alpha_0} \in \mathcal{B}_{\alpha_0}$, $\alpha_0 \in I$, then $\overline{A_{\alpha_0}}$ commutes with X, by Lemma 6.2.4, that is $\overline{A_{\alpha_0}} \in (X)'$. As $(X)'$ is a ring, this implies $\mathcal{B}^{\#} \subset (X)'$. So every $A \in \mathcal{B}^{\#}$ commutes with X.

Definition 6.3.2. Given a $\Phi \in \Pi \otimes_{\alpha \in I} \mathfrak{H}_{\alpha}$, denote by $\mathfrak{M}[\Phi]$ the closed, linear set determined by all $U(z_{\alpha};\ \alpha \in I)\Phi$, $(|z_{\alpha}| = 1)$. Denote by $E[\Phi]$ the projection of $\mathfrak{M}[\Phi]$.

Lemma 6.3.2. (I) If $\Phi \in \Pi \otimes_{\alpha \in I}^{\mathfrak{C}} \mathfrak{H}_{\alpha}$, then $E[\Phi]$ commutes with $P[\mathfrak{C}]$ and their product is $P_{[\Phi]}$ [38]).

(II) If $\Phi \in \Pi \otimes_{w\ \alpha \in I}^{\mathfrak{C}_w} \mathfrak{H}_{\alpha}$, then $E[\Phi] \leqq P_w[\mathfrak{C}_w]$.

(III) $E[\Phi]$ commutes with every $U(z_{\alpha};\ \alpha \in I)$.

Proof: Ad (I): Denote the closed, linear sets determined by the $U(z_{\alpha};\ \in \alpha I)\Phi$ $(|z_{\alpha}| = 1)$ with a convergent resp. divergent $\Pi_{\alpha \in I} z_{\alpha}$ by $\mathfrak{M}_1$ resp. $\mathfrak{M}_2$. Clearly $\mathfrak{M}[\Phi] = \mathfrak{S}[\mathfrak{M}_1, \mathfrak{M}_2]$.

If $\Pi_{\alpha \in I} z_{\alpha}$ converges, then $U(z_{\alpha};\ \alpha \in I)\Phi = (\Pi_{\alpha \in I} z_{\alpha})\Phi$ by Lemma 6.2.3, (I). So $\mathfrak{M}_1 = [\Phi] \subset \Pi \otimes_{\alpha \in I}^{\mathfrak{C}} \mathfrak{H}_{\alpha}$.

---

[38]) The projection of the closed, linear set $[\Phi]$:

$$P_{[\Phi]}\,\Psi \begin{cases} = \dfrac{(\Psi, \Phi)}{\|\Phi\|^2}\,\Phi & \text{for } \Phi \neq 0, \\ = 0 & \text{for } \Phi = 0. \end{cases}$$

If $\Pi_{\alpha \epsilon I} z_\alpha$ diverges, then $\Phi \epsilon \Pi \otimes_{\alpha \epsilon I}^{\mathfrak{C}} \mathfrak{H}_\alpha$ gives $U(z_\alpha; \alpha \epsilon I) \Phi \epsilon \Pi \otimes_{\alpha \epsilon I}^{\mathfrak{D}} \mathfrak{H}_\alpha$ for some $\mathfrak{D} \neq \mathfrak{C}$, so $U(z_\alpha; \alpha \epsilon I) \Phi$ is orthogonal to $\Pi \otimes_{\alpha \epsilon I}^{\mathfrak{C}} \mathfrak{H}_\alpha$.

Thus $\mathfrak{M}_1$, $\mathfrak{M}_2$ are orthogonal, and therefore $E[\Phi] = P_{\mathfrak{M}[\Phi]} = P_{\mathfrak{M}_1} + P_{\mathfrak{M}_2}$. Besides

$$E[\Phi]\, P[\mathfrak{C}] = P_{\mathfrak{M}_1} P[\mathfrak{C}] + P_{\mathfrak{M}_2} P[\mathfrak{C}] =$$
$$= P_{[\Phi]} P_{\Pi \otimes_{\alpha \epsilon I}^{\mathfrak{C}} \mathfrak{H}_\alpha} + P_{\mathfrak{M}_2} P_{\Pi \otimes_{\alpha \epsilon I}^{\mathfrak{C}} \mathfrak{H}_\alpha} = P_{[\Phi]} + 0 = P_{[\Phi]}.$$

This is a projection, so $E[\Phi]$, $P[\mathfrak{C}]$ commute (cf. (8), p. 76) and their product is $P_{[\Phi]}$.

Ad (II): $U(z_\alpha; \alpha \epsilon I)$ maps $\Pi \underset{w}{\otimes}{}_{\alpha \epsilon I}^{\mathfrak{C}_w} \mathfrak{H}_\alpha$ on itself by Lemma 6.2.2. So $\Phi \epsilon \Pi \underset{w}{\otimes}{}_{\alpha \epsilon I}^{\mathfrak{C}_w} \mathfrak{H}_\alpha$ implies $U(z_\alpha; \alpha \epsilon I) \Phi \epsilon \Pi \underset{w}{\otimes}{}_{\alpha \epsilon I}^{\mathfrak{C}_w} \mathfrak{H}_\alpha$, and $\mathfrak{M}[\Phi] \subset \Pi \underset{w}{\otimes}{}_{\alpha \epsilon I}^{\mathfrak{C}_w} \mathfrak{H}_\alpha$. Passing to the projections gives $E[\Phi] \leqq P_w[\mathfrak{C}_w]$.

Ad (III): Owing to $U(z_\alpha; \alpha \epsilon I)\, U(z'_\alpha; \alpha \epsilon I) = U(z_\alpha z'_\alpha; \alpha \epsilon I)$ application of $U(z_\alpha; \alpha \epsilon I)$ merely permutes the $U(z'_\alpha; \alpha \epsilon I) \Phi$, $|z'_\alpha| = 1$. Therefore $U(z_\alpha; \alpha \epsilon I)$ maps $\mathfrak{M}[\Phi]$ on itself, that is it commutes with $E[\Phi]$.

LEMMA 6.3.3. For any C_0-sequence f_α^0, $\alpha \epsilon I$, with $\|f_\alpha^0\| = 1$ we have $E[\Pi \otimes_{\alpha \epsilon I} f_\alpha^0] \epsilon \mathcal{B}^\#$.

Proof: The proof will be carried out in several successive stages.

(I) It suffices to show, that $E[\Pi \otimes_{\alpha \epsilon I} f_\alpha]$ is a strongest (and thus a fortiori a strong, cf. § 1.1, (e), and particularly (11), pp. 111—112) condensation point of $\mathcal{B}^\#$. And this is certainly the case, if we can find for an (enumerably infinite) sequence $\Phi_1, \Phi_2, \ldots$ of elements of $\Pi \otimes_{\alpha \epsilon I} \mathfrak{H}_\alpha$ an $F \epsilon B^\#$, such that $F\Phi_n = E[\Pi \otimes_{\alpha \epsilon I} f_\alpha^0]\, \Phi_n$ for $n = 1, 2, \ldots$

Each Φ_n is the limit of a sequence of (finite) linear aggregates of elements $\Pi \otimes_{\alpha \epsilon I} g_\alpha$ (g_α, $\alpha \epsilon I$ a C_0-sequence, cf. Theorem VI, (III)). All $\Pi \otimes_{\alpha \epsilon I} g_\alpha$ which arise in connection with a given Φ_n form again a sequence: $\Pi \otimes_{\alpha \epsilon I} g_\alpha^{n,i}$, $i = 1, 2, \ldots$. $F(\Pi \otimes_{\alpha \epsilon I} g_\alpha^{n,i}) = E[\Pi \otimes_{\alpha \epsilon I} f_\alpha^0](\Pi \otimes_{\alpha \epsilon I} g_\alpha^{n,i})$ for $i = 1, 2, \ldots$ implies clearly $F\Phi_n = E[\Pi \otimes_{\alpha \epsilon I} f_\alpha^0]\, \Phi_n$. Now we may write the $\Pi \otimes_{\alpha \epsilon I} g_\alpha^{n,i}$, $n, i = 1, 2, \ldots$, as a simple sequence, and replace the Φ_n, $n = 1, 2, \ldots$, by them. In other words: We may as well assume $\Phi_n = \Pi \otimes_{\alpha \epsilon I} h_\alpha^n$ (h_α^n, $\alpha \epsilon I$, a C_0-sequence) for each $n = 1, 2, \ldots$.

If any $h_\alpha^n = 0$, then its $\Phi_n = \Pi \otimes_{\alpha \epsilon I} h_\alpha^n = 0$ may be omitted. So we may assume, that all $h_\alpha^n \neq 0$.

Every $U(z_\alpha; \alpha \epsilon I)$ commutes with $E[\Pi \otimes_{\alpha \epsilon I} f_\alpha^0]$ and F by Lemmata 6.3.2, (III), and 6.3.1. Thus we may replace our $\Phi_n = \Pi \otimes_{\alpha \epsilon I} h_\alpha^n$

by any $U(z_\alpha^n;\ \alpha\in I)\,\Phi_n = \Pi\otimes_{\alpha\in I} z_\alpha^n h_\alpha^n$, $n = 1, 2, \ldots$ We can use this freedom to obtain for every $n = 1, 2, \ldots$ for which $(h_\alpha^n;\ \alpha\in I) \underset{w}{\approx} (f_\alpha^0;\ \alpha\in I)$, even $(h_\alpha^n;\ \alpha\in I) \approx (f_\alpha^0;\ \alpha\in I)$. So we may assume: Let $\mathfrak{S}_1$ be the set of all n with $(h_\alpha^n;\ \alpha\in I) \approx (f_\alpha^0;\ \alpha\in I)$ and $\mathfrak{S}_2$ the set of all n with $(h_\alpha^n;\ \alpha\in I)$ not $\underset{w}{\approx} (f_\alpha^0;\ \alpha\in I)$, then every $n = 1, 2, \ldots$ belongs either to $\mathfrak{S}_1$ or to $\mathfrak{S}_2$.

If $n\in\mathfrak{S}_1$, then $\Sigma_{\alpha\in I}\left|(h_\alpha^n, f_\alpha^0) - 1\right|$ converges (by Definition 3.3.2), and so we have $(h_\alpha^n, f_\alpha^0) = 1$, except for a finite or enumerably infinite set of α's, J_n (by Lemma 2.3.2, (I)). If $n\in\mathfrak{S}_2$, then $\Sigma_{\alpha\in I}\left||(h_\alpha^n, f_\alpha^0)| - 1\right|$ diverges (by Lemma 6.1.3), and therefore an enumerably infinite set of α's, K_n, exists, such that $\Sigma_{\alpha\in K_n}\left||(h_\alpha^n, f_\alpha^0)| - 1\right|$ diverges [39]). Finally $\Sigma_{\alpha\in I}\left|\|h_\alpha^n\| - 1\right|$ converges for each n (C_0-sequences), so we have $\|h_\alpha^n\| = 1$ except for a finite or enumerably infinite set of α's, L_n. Now let I^0 be the sum of all $J_n\ (n\in\mathfrak{S}_1)$, $K_n (n\in\mathfrak{S}_2)$, $L_n\ (n = 1, 2, \ldots)$. I^0 is finite or enumerably infinite, and we have:

For $n\in\mathfrak{S}_1$ $\alpha\notin I^0$ implies $(h_\alpha^n, f_\alpha^0) = \|h_\alpha^n\| = \|f_\alpha^0\| = 1$.

For $n\in\mathfrak{S}_2$ $\Sigma_{\alpha\in I^0}\left||(h_\alpha^n, f_\alpha^0)| - 1\right|$ diverges. But $\Pi_{\alpha\in I}\|h_\alpha^n\|$, $\Pi_{\alpha\in I}\|f_\alpha^0\|$ converge, so $\Pi_{\alpha\in I}\|h_\alpha^n\|\,\|f_\alpha^0\|$ does too, and with it $\Sigma_{\alpha\in I}\operatorname{Max}(\|h_\alpha^n\|\,\|f_\alpha^0\| - 1, 0)$ (by Lemma 2.4.1, (I)). As $|(h_\alpha^n, f_\alpha^0)| \leqq \|h_\alpha^n\|\,\|f_\alpha^0\|$,

$$0 \leqq \operatorname{Max}\left(|(h_\alpha^n, f_\alpha^0)| - 1, 0\right) \leqq \operatorname{Max}\left(\|h_\alpha^n\|\,\|f_\alpha^0\| - 1, 0\right),$$

this implies the convergence of $\Sigma_{\alpha\in I^0}\operatorname{Max}\left(|(h_\alpha^n, f_\alpha^0)| - 1, 0\right)$ (by Lemma 2.3.1).

Combining these facts, Lemma 2.4.1 permits us to conclude: $\Pi_{\alpha\in I^0}|(h_\alpha^n, f_\alpha^0)|$ converges and its value is 0.

(II) Let us compute $\left\|E[\Pi\otimes_{\alpha\in I} f_\alpha^0]\,\Phi_n\right\|$.

If $n\in\mathfrak{S}_1$, then $(h_\alpha^n;\ \alpha\in I) \approx (f_\alpha^0;\ \alpha\in I)\in\mathfrak{C}$, $(h_\alpha^n;\ \alpha\in I)\in\mathfrak{C}$, so $\Phi_n = \Pi\otimes_{\alpha\in I} h_\alpha^n \in \Pi\otimes_{\alpha\in I}^{\mathfrak{C}} \mathfrak{H}_\alpha$, and thus by Lemma 6.3.2, (I),

$$\left\|E[\Pi\otimes_{\alpha\in I} f_\alpha^0]\,\Phi_n\right\| = \left\|E[\Pi\otimes_{\alpha\in I} f_\alpha^0]\,P[\mathfrak{C}]\,\Phi_n\right\| = \left\|P_{[\Pi\otimes_{\alpha\in I} f_\alpha^0]}\,\Phi_n\right\| =$$
$$= \left\|P_{[\Pi\otimes_{\alpha\in I} f_\alpha^0]}\left(\Pi\otimes_{\alpha\in I} h_\alpha^n\right)\right\| = \left|\left(\Pi\otimes_{\alpha\in I} h_\alpha^n,\ \Pi\otimes_{\alpha\in I} f_\alpha^0\right)\right| =$$
$$= \left|\Pi_{\alpha\in I}(h_\alpha^n, f_\alpha^0)\right|.$$

But $\Pi_{\alpha\in I}(h_\alpha^n, f_\alpha^0)$ is convergent (not merely quasi-convergent) as $(h_\alpha^n;\ \alpha\in I) \approx (f_\alpha^0;\ \alpha\in I)$, therefore this expression is $= \Pi_{\alpha\in I}|(h_\alpha^n, f_\alpha^0)| = \Pi_{\alpha\in I_0}|(h_\alpha^n, f_\alpha^0)|$.

---

[39]) If $\Sigma_{\alpha\in I} u_\alpha$ ($u_\alpha \geqq 0$) diverges, then the $u_{\alpha_1} + \ldots + u_{\alpha_l}$ ($\alpha_1, \ldots, \alpha_l$ mutually different) are not bounded (use Lemma 2.3.1). Choose $\alpha_1^N, \ldots, \alpha_{l_N}^N$ with $u_{\alpha_1^N} + \ldots + u_{\alpha_{l_N}^N} \geqq N$ for $N = 1, 2, \ldots$ and let K be the set of all α_k^N, $N = 1, 2, \ldots, k = 1, \ldots, l_N$, then $\Sigma_{\alpha\in K} u_\alpha$ is clearly divergent.

If $n \in \mathfrak{S}_2$, then $(h^n_\alpha;\ \alpha \in I)$ not $\underset{w}{\approx} (f^0_\alpha;\ \alpha \in I)$, so if $(f^0_\alpha;\ \alpha \in I) \in \mathfrak{C}_w$, $(h^n_\alpha;\ \alpha \in I) \in \mathfrak{D}_w$, where $\mathfrak{C}_w$, $\mathfrak{D}_w \in \Gamma_w$, then $\mathfrak{C}_w \neq \mathfrak{D}_w$. So $\underset{w}{\Pi\otimes}{}^{\mathfrak{C}_w}_{\alpha \in I}\, \mathfrak{H}_\alpha$ and $\underset{w}{\Pi\otimes}{}^{\mathfrak{D}_w}_{\alpha \in I}\, \mathfrak{H}_\alpha$ are orthogonal. As $\mathfrak{M}[\Pi\otimes_{\alpha \in I} f^0_\alpha] \subset \underset{w}{\Pi\otimes}{}^{\mathfrak{C}_w}_{\alpha \in I}\, \mathfrak{H}_\alpha$ (by Lemma 6.3.2, (II)) and $\Phi_n = \Pi\otimes_{\alpha \in I} h^n_\alpha \in \underset{w}{\Pi\otimes}{}^{\mathfrak{D}_w}_{\alpha \in I}\, \mathfrak{H}_\alpha$, so Φ_n is orthogonal to $\mathfrak{M}[\Pi\otimes_{\alpha \in I} f^0_\alpha]$. Thus $E[\Pi\otimes_{\alpha \in I} f^0_\alpha]\, \Phi_n = 0$. At the same time $\Pi_{\alpha \in I} |(h^n_\alpha, f^0_\alpha)| = 0$.

So

$$(*) \qquad \| E[\Pi\otimes_{\alpha \in I} f^0_\alpha]\, \Phi_n \| = \Pi_{\alpha \in I^0} |(h^n_\alpha, f^0_\alpha)|$$

holds for all $n = 1, 2, \ldots$.

(III) Write I^0 as a (finite or enumerably infinite) sequence $I^0 = \mathfrak{S}(\alpha_1, \alpha_2, \ldots)$ ($\alpha_1, \alpha_2, \ldots$ mutually different). For every α_l form the projection of $[f^0_{\alpha_l}]$ in $\mathfrak{H}_{\alpha_l}$: $P_{[f^0_{\alpha_l}]} \in \mathcal{B}_{\alpha_l}$, $P_{[f^0_{\alpha_l}]}\, h_{\alpha_l} = (h_{\alpha_l}, f^0_{\alpha_l}) f^0_{\alpha_l}$. So

$$\overline{P_{[f^0_{\alpha_l}]}}\, (\Pi\otimes_{\alpha \in I} h_\alpha) = (h_{\alpha_l}, f^0_{\alpha_l}) f^0_{\alpha_l} \otimes \Pi\otimes_{\alpha \in I,\, \alpha \neq \alpha_l} h_\alpha.$$

This equation exhibits two facts: First, that all $\overline{P_{[f^0_{\alpha_l}]}}$, $l = 1, 2, \ldots$, commute. All $\overline{P_{[f^0_{\alpha_l}]}}$ are projections $\in \mathcal{B}_{\alpha_0}$, so all $\overline{P_{[f^0_{\alpha_l}]}}$ are projections $\in \overline{\mathcal{B}_{\alpha_0}} \subset \mathcal{B}^\#$, and as they commute, all

$$Q_l = \overline{P_{[f^0_{\alpha_1}]}} \cdot \ldots \cdot \overline{P_{[f^0_{\alpha_l}]}}$$

too are projections $\in \mathcal{B}^\#$. And clearly

$$Q_1 \geqq Q_2 \geqq \cdots$$

Thus $\lim_l Q_l$ exists [40]), and is again a projection $\in B^\#$.

Second, it shows, that $\overline{P_{[f^0_{\alpha_l}]}}\, (\Pi\otimes_{\alpha \in I} z_\alpha f^0_\alpha) = \Pi\otimes_{\alpha \in I} z_\alpha f^0_\alpha$, and so $\overline{P_{[f^0_{\alpha_l}]}}\, \Psi = \Psi$ for all $\Psi \in \mathfrak{M}[\Pi\otimes_{\alpha \in I} f^0_\alpha]$. This implies $Q_l \Psi = \Psi$ and $\lim_l Q_l \Psi = \Psi$. So

$$\lim_l Q_l \geqq E[\Pi\otimes_{\alpha \in I} f^0_\alpha].$$

Thus

$$(**) \qquad \| (\lim_l Q_l) \Phi_n \| = \| E[\Pi\otimes_{\alpha \in I} f^0_\alpha]\, \Phi_n \|$$

---

[40]) We mean $\lim_{l=\infty} Q_l$ if I^0 is infinite, and Q_{l_0} if $I^0 = \mathfrak{S}(\alpha_1, \ldots, \alpha_{l_0})$.

implies $(\lim_l Q_l)\Phi_n = E[\Pi\otimes_{\alpha\in I} f_\alpha^0]\,\Phi_n$ [41]) and so it suffices to prove (**) for all $n = 1, 2, \ldots$ Then $F = \lim_l Q_l$ meets all requirements.

(IV) We have:

$$\|Q_l \Phi_n\| = \|(\overline{P_{[f^0_{\alpha_1}]}} \cdot \ldots \cdot \overline{P_{[f^0_{\alpha_l}]}})\,(\Pi\otimes_{\alpha\in I} h_\alpha^n)\| =$$

$$= \|(h_{\alpha_1}, f^0_{\alpha_1}) \cdot \ldots \cdot (h_{\alpha_l}, f^0_{\alpha_l}) \cdot f^0_{\alpha_1} \otimes \cdots \otimes f^0_{\alpha_l} \otimes \Pi\otimes_{\alpha\in I,\, \alpha\neq\alpha_1,\ldots,\alpha_l} h_\alpha^n\| =$$

$$= |(h_{\alpha_1}, f^0_{\alpha_1})| \cdot \ldots \cdot |(h_{\alpha_l}, f^0_{\alpha_l})| \cdot \Pi_{\alpha\in I,\, \alpha\neq\alpha_1,\ldots,\alpha_l} \|h_\alpha^n\| =$$

$$= |(h^0_{\alpha_1}, f^0_{\alpha_1})| \cdot \ldots \cdot |(h_{\alpha_l}, f^0_{\alpha_l})| \cdot \Pi_{\alpha\in I^0,\, \alpha\neq\alpha_1,\ldots,\alpha_l} \|h_\alpha^n\|.$$

(Remember, that $\|h_\alpha^n\| = 1$ for $\alpha \notin I^0$.) If we form $\lim_l$, then the second factor on the right side converges to 1, because $\Pi_{\alpha\in I^0}\|h_\alpha^n\|$ converges. Therefore

$$\|(\lim_l Q_l)\,\Phi_n\| = \lim_l \|Q_l \Phi_n\| = \Pi_{\alpha\in I^0} |(h_\alpha, f_\alpha^0)|.$$

Thus (**) follows from (*).

The proof is now completed.

LEMMA 6.3.4. Assume $(f_\alpha^0;\, \alpha\in I) \in \mathfrak{C}$, $\|f_\alpha^0\| = 1$ and $\Phi \in \Pi\otimes^{\mathfrak{C}}_{\alpha\in I}\,\mathfrak{H}_\alpha$. Then there exists an $A \in \mathcal{B}^\#$ with $A(\Pi\otimes_{\alpha\in I} f_\alpha^0) = \Phi$.

Proof: We proceed again in several successive steps.

(I) Introduce, together with our f_α^0, the corresponding $\aleph_\alpha$, K_α, $\varphi_{\alpha,\beta}$, $\mathbf{F}$ $(f_\alpha^0 = \varphi_{\alpha,0})$ of Lemma 4.1.4 and Theorem V. Apply Theorem V to Φ, and let $\beta_1(\alpha), \beta_2(\alpha), \ldots$ be those $\beta(\alpha) \in \mathbf{F}$ (finite or enumerably infinite in number) for which $a[\beta(\alpha);\, \alpha\in I] \neq 0$. (Cf. Theorem V, (II).) Write $a_i = a[\beta_i(\alpha);\, \alpha\in I]$, then the situation described in Theorem V entails:

$$\Phi = \Sigma_i\, a_i\, \Pi\otimes_{\alpha\in I}\, \varphi_{\alpha,\,\beta_i(\alpha)}, \qquad \Sigma_i\,|a_i|^2 \text{ converges.}$$

(II) For every $\alpha\in I$ and $\beta\in K_\alpha$ define

$$P_\alpha^\beta f_\alpha = (f_\alpha, f_\alpha^0) \cdot \varphi_{\alpha,\,\beta}.$$

Clearly $\qquad P_\alpha^\beta \in \mathcal{B}_\alpha$ and $|||\,P_\alpha^\beta\,||| \leqq 1.$

Each $\beta_i(\alpha)$ $(i = 1, 2, \ldots)$ differs from 0 for a finite number

---

[41]) If E, F are two projections, $E \leqq F$, then $\|E\Phi\| = \|F\Phi\|$ implies $E\Phi = F\Phi$. Indeed, since E, F are projections, so $(F\Phi, \Phi) = \|E\Phi\|^2 = \|F\Phi\|^2 = (F\Phi, \Phi)$ and since $F - E$ is a projection, so $\|(F-E)\Phi\|^2 = ((F-E)\Phi, \Phi) = (F\Phi, \Phi) - (E\Phi, \Phi) = 0$. So $(F-E)\Phi = 0$, $F\Phi - E\Phi = 0$, $F\Phi = E\Phi$.

of α's only, say for $\alpha_1^i, \ldots, \alpha_{h_i}^i$. Define

$$R_i = \overline{P_{\alpha_1^i}^{\beta_i(\alpha_1^i)}} \cdot \ldots \cdot \overline{P_{\alpha_{h_i}^i}^{\beta_i(\alpha_{h_i}^i)}} \cdot E[\Pi\otimes_{\alpha\in I} f_\alpha^0].$$

As $P_\alpha^\beta \in \mathfrak{B}_\alpha$, $\overline{P_\alpha^\beta} \in \overline{\mathfrak{B}_\alpha} \subset \mathfrak{B}^\#$ and $E[\Pi\otimes_{\alpha\in I} f_\alpha^0] \in \mathfrak{B}^\#$ (by Lemma 6.3.3), so $R_i \in \mathfrak{B}^\#$. As all $|||P_\alpha^\beta||| \leqq 1$, $|||\overline{P_\alpha^\beta}||| \leqq 1$ and $|||E[\Pi\otimes_{\alpha\in I} f_\alpha^0]||| \leqq 1$ (it is a projection), so $|||R_i||| \leqq 1$.

(III) Assume $i \neq j$. We wish to prove, that $R_i\Psi'$ and $R_j\Psi''$ are orthogonal for all Ψ', Ψ'' ($\in \Pi\otimes_{\alpha\in I} \mathfrak{H}_\alpha$). It is clearly sufficient to consider the $\Psi' \in \Pi\otimes_{\alpha\in I}^{\mathfrak{C}'} \mathfrak{H}_\alpha$, $\Psi'' \in \Pi\otimes_{\alpha\in I}^{\mathfrak{C}''} \mathfrak{H}_\alpha$ for all $\mathfrak{C}'$, $\mathfrak{C}'' \in \Gamma$.

Form the $\mathfrak{C}_w$ with $\mathfrak{C} \subset \mathfrak{C}_w \in \Gamma_w$. Consider first a $\Psi' \in \Pi\otimes_{\alpha\in I}^{\mathfrak{C}'} \mathfrak{H}_\alpha$ where $\mathfrak{C}' \subset \mathfrak{C}_w' \in \Gamma_w$, but $\mathfrak{C}_w' \neq \mathfrak{C}_w$. Then $\Pi\otimes_{w}{}_{\alpha\in I}^{\mathfrak{C}_w'} \mathfrak{H}_\alpha$ and $\Pi\otimes_{w}{}_{\alpha\in I}^{\mathfrak{C}_w} \mathfrak{H}_\alpha$ are orthogonal, and $\Psi' \in \Pi\otimes_{w}{}_{\alpha\in I}^{\mathfrak{C}_w'} \mathfrak{H}_\alpha$, $\mathfrak{M}[\Pi\otimes_{\alpha\in I} f_\alpha^0] \subset$ $\subset \Pi\otimes_{w}{}_{\alpha\in I}^{\mathfrak{C}_w} \mathfrak{H}_\alpha$ (by Lemma 6.3.2, (II)), so Ψ' is orthogonal to $\mathfrak{M}[\Pi\otimes_{\alpha\in I} f_\alpha^0]$. Thus $E[\Pi\otimes_{\alpha\in I} f_\alpha^0]\Psi' = 0$, $R_i\Psi' = 0$. We may therefore assume $\mathfrak{C}' \subset \mathfrak{C}_w$. Similarly we may assume $\mathfrak{C}'' \subset \mathfrak{C}_w$.

Consider next the case, where $\mathfrak{C}' \neq \mathfrak{C}''$. R_i, $R_j \in \mathfrak{B}^\#$ so they commute with $P[\mathfrak{C}']$, $P[\mathfrak{C}'']$ (by Lemma 6.3.1). Thus

$$(R_i\Psi', R_j\Psi'') = (R_iP[\mathfrak{C}']\Psi', R_jP[\mathfrak{C}'']\Psi'') =$$
$$= (P[\mathfrak{C}']R_i\Psi', P[\mathfrak{C}'']R_j\Psi'') = (R_i\Psi', P[\mathfrak{C}']P[\mathfrak{C}'']R_j\Psi'') = 0$$

disposing of this case.

We may assume therefore, that $\mathfrak{C}' = \mathfrak{C}'' \subset \mathfrak{C}_w$. As R_i, $R_j \in \mathfrak{B}^\#$ they commute with $U(z_\alpha;\, \alpha\in I)$ (by Lemma 6.3.1), so we may replace Ψ', Ψ'' by $U(z_\alpha;\, \alpha\in I)\Psi'$, $U(z_\alpha;\, \alpha\in I)\Psi''$. Now $\mathfrak{C}$, $\mathfrak{C}' \subset \mathfrak{C}_w$, so we can choose $U(z_\alpha;\, \alpha\in I)$ so as to map $\mathfrak{C}' = \mathfrak{C}''$ on $\mathfrak{C}$. In other words: We may even assume $\mathfrak{C}' = \mathfrak{C}'' = \mathfrak{C}$.

Thus we must prove the orthogonality of $R_i\Psi'$ and $R_j\Psi''$ for Ψ', $\Psi'' \in \Pi\otimes_{\alpha\in I}^{\mathfrak{C}} \mathfrak{H}_\alpha$ only. It is clearly sufficient, to consider instead the $\Psi' = \Pi\otimes_{\alpha\in I} f_\alpha'$, $\Psi'' = \Pi\otimes_{\alpha\in I} f_\alpha''$ with $(f_\alpha';\, \alpha\in I)$, $(f_\alpha'';\, \alpha\in I) \in \mathfrak{C}$, only.

Now under these conditions

$$E[\Pi\otimes_{\alpha\in I} f_\alpha^0](\Pi\otimes_{\alpha\in I} f_\alpha') = E[\Pi\otimes_{\alpha\in I} f_\alpha^0]\, P[\mathfrak{C}](\Pi\otimes_{\alpha\in I} f_\alpha') =$$

(use Lemma 6.3.2, (I))

$$= P_{[\Pi\otimes_{\alpha\in I} f_\alpha^0]}(\Pi\otimes_{\alpha\in I} f_\alpha') = (\Pi\otimes_{\alpha\in I} f_\alpha', \Pi\otimes_{\alpha\in I} f_\alpha^0)\, \Pi\otimes_{\alpha\in I} f_\alpha^0 =$$
$$= \Pi_{\alpha\in I}(f_\alpha', f_\alpha^0) \cdot \Pi\otimes_{\alpha\in I} f_\alpha^0 = \Pi_{\alpha\in I}(f_\alpha', f_\alpha^0) \cdot \Pi\otimes_{\alpha\in I} \varphi_{\alpha,0},$$

and so

$$R_i(\Pi\otimes_{\alpha\in I} f'_\alpha) = \Pi_{\alpha\in I}(f'_\alpha, f^0_\alpha)\cdot \overline{P^{\beta_i(\alpha^i_1)}_{\alpha^i_1}} \cdot \ldots \cdot \overline{P^{\beta_i(\alpha^i_{k_i})}_{\alpha^i_{k_i}}} (\Pi\otimes_{\alpha\in I}\varphi_{\alpha,0}) =$$
$$(*)\qquad = \Pi_{\alpha\in I}(f'_\alpha, f^0_\alpha)\cdot \Pi\otimes_{\alpha\in I}\varphi_{\alpha,\beta_i(\alpha)}.$$

Similarly

$$R_j(\Pi\otimes_{\alpha\in I} f''_\alpha) = \Pi_{\alpha\in I}(f''_\alpha, f^0_\alpha)\cdot \Pi\otimes_{\alpha\in I}\varphi_{\alpha,\beta_j(\alpha)}.$$

Combining these equations we obtain

$$(R_i(\Pi\otimes_{\alpha\in I} f'_\alpha),\ R_j(\Pi\otimes_{\alpha\in I} f''_\alpha)) =$$
$$= \Pi_{\alpha\in I}(f'_\alpha, f^0_\alpha)\cdot \Pi_{\alpha\in I}(f''_\alpha, f^0_\alpha)\cdot \Pi_{\alpha\in I}(\varphi_{\alpha,\beta_i(\alpha)}, \varphi_{\alpha,\beta_j(\alpha)}).$$

As $i \neq j$, so an $\alpha\in I$ with $\beta_i(\alpha)\neq\beta_j(\alpha)$, $(\varphi_{\alpha,\beta_i(\alpha)}, \varphi_{\alpha,\beta_j(\alpha)}) = 0$ exists, and so the third factor on the right side is 0. Thus the left side is 0 too, and our statement is proved.

(IV) For any Ψ ($\in \Pi\otimes_{\alpha\in I}\mathfrak{H}_\alpha$) the $R_1\Psi$, $R_2\Psi, \ldots$ are mutually orthogonal, as was shown above. Besides $\Sigma_i\|a_iR_i\Psi\|^2 = \Sigma_i|a_i|^2\|R_i\Psi\|^2 \leqq \Sigma_i|a_i|^2\|\Psi\|^2 = (\Sigma_i|a_i|^2)\|\Psi\|^2$ converges, and so the sum $\Sigma_i a_iR_i\Psi$ is (strongly) convergent (in $\Pi\otimes_{\alpha\in I}\mathfrak{H}_\alpha$). So we may define an operator A (in $\Pi\otimes_{\alpha\in I}\mathfrak{H}_\alpha$) by

$$A\Psi = \Sigma_i a_i R_i\Psi.$$

As $\|A\Psi\|^2 = \|\Sigma_i a_iR_i\Psi\|^2 = \Sigma_i\|a_iR_i\Psi\|^2 \leqq (\Sigma_i|a_i|^2)\|\Psi\|^2$, $\|A\Psi\| \leqq \sqrt{\Sigma_i|a_i|^2}\,\|\Psi\|$, so $A\in\mathfrak{B}_\otimes$.

A is the strong limit of all $a_1R_1 + \ldots + a_jR_j$, $j = 1, 2, \ldots$, and so A belongs to $\mathfrak{B}^\#$ along with $R_1, R_2, \ldots$ Finally (*) gives

$$R_i(\Pi\otimes_{\alpha\in I} f^0_\alpha) = \Pi\otimes_{\alpha\in I}\varphi_{\alpha,\beta_i(\alpha)},$$

and therefore

$$A(\Pi\otimes_{\alpha\in I} f^0_\alpha) = \Sigma_i a_i\Pi\otimes_{\alpha\in I}\varphi_{\alpha,\beta_i(\alpha)} = \Phi.$$

Thus A meets all our requirements.

LEMMA 6.3.5. For any $\Phi\in\Pi\otimes^{\mathfrak{C}}_{\alpha\in I}\mathfrak{H}_\alpha$ we have $E[\Phi]\in\mathfrak{B}^\#$ [42]).

Proof: Choose $(f^0_\alpha;\ \alpha\in I)\in\mathfrak{C}$, $\|f^0_\alpha\| = 1$, then $E[\Pi\otimes_{\alpha\in I} f^0_\alpha]\in\mathfrak{B}^\#$ by Lemma 6.3.3. Choose an $A\in\mathfrak{B}^\#$ with $A(\Pi\otimes_{\alpha\in I} f^0_\alpha) = \Phi$ by

---

[42]) This is an extension of Lemma 6.3.3. Observe, that the condition $\Phi\in\Pi\otimes^{\mathfrak{C}}_{\alpha\in I}\mathfrak{H}_\alpha$ cannot be omitted: If $E[\Phi]\in\mathfrak{B}^\#$ held for all Φ, then we could omit in Lemma 6.3.6. and in Theorem IX the condition, that F resp. A must commute with all $P(\mathfrak{C})$. Then this fact, together with Theorem IX (or Lemma 6.3.1), would give: If A commutes with every $U(z_\alpha;\ \alpha\in I)$, then it commutes with every $P[\mathfrak{C}]$ too. But (if I is infinite) $A = U(z^0_\alpha;\ \alpha\in I)$ with a non-convergent $\Pi_{\alpha\in I} z^0_\alpha$, commutes with all $U(z_\alpha;\alpha\in I)$, and (use Lemma 6.2.3, (I)) with no $P[\mathfrak{C}]$.

Lemma 6.3.4. As A commutes with every $U(z_\alpha;\, \alpha \epsilon I)$ (by Lemma 6.3.1), so

$A U(z_\alpha; \alpha \epsilon I)(\Pi \otimes_{\alpha \epsilon I} f_\alpha^0) = U(z_\alpha; \alpha \epsilon I)\, A(\Pi \otimes_{\alpha \epsilon I} f_\alpha^0) = U(z_\alpha; \alpha \epsilon I)\Phi.$

Thus A maps $\mathfrak{M}[\Pi \otimes_{\alpha \epsilon I} f_\alpha]$ on a set which determines the closed, linear set $\mathfrak{M}[\Phi]$. That is: The range of $A E[\Pi \otimes_{\alpha \epsilon I} f_\alpha^0]$ determines the closed, linear set $\mathfrak{M}[\Phi]$.

Let us use the symbol η (cf. (7), p. 141): A and $E[\Pi \otimes_{\alpha \epsilon I} f_\alpha^0]$ are both $\epsilon\, \mathcal{B}^\#$, therefore $\eta\, \mathcal{B}^\#$, and thus the characterisation of $\mathfrak{M}[\Phi]$ given above gives $\mathfrak{M}[\Phi]\eta\, \mathcal{B}^\#$. So its projection is $E[\Phi]\, \eta\, \mathcal{B}^\#$, too. But this means $E[\Phi] \epsilon\, \mathcal{B}^\#$. (Cf. loc. cit. above.)

Lemma 6.3.6. If a projection F commutes with all $P[\mathfrak{C}]$ ($\mathfrak{C} \epsilon \Gamma$) and $U(z_\alpha;\, \alpha \epsilon I)$ ($|z| = 1$), then $F \epsilon\, \mathcal{B}^\#$.

Proof: Let $\mathfrak{N}$ be the closed, linear set of F. Assume $\Phi \epsilon \mathfrak{N}$. Φ is a limit of a sequence of (finite) linear aggregates of elements of the form $\Psi = \Pi \otimes_{\alpha \epsilon I} f_\alpha$, and so a fortiori of elements $\Psi \epsilon\, \Pi \otimes^{\mathfrak{C}}_{\alpha \epsilon I} \mathfrak{H}_\alpha$, $\mathfrak{C} \epsilon \Gamma$. So $\Phi = F\Phi$ is a limit of a sequence of (finite) linear aggregates of elements $F\Psi$, $\Psi \epsilon\, \Pi \otimes_{\alpha \epsilon I} \mathfrak{H}_\alpha$, $\mathfrak{C} \epsilon \Gamma$. As F and $P[\mathfrak{C}]$ commute, and their closed, linear sets are $\mathfrak{N}$ resp. $\Pi \otimes^{\mathfrak{C}}_{\alpha \epsilon I} \mathfrak{H}_\alpha$, therefore such an $F\Psi\, \epsilon\, \mathfrak{N}$ and at the same time $\epsilon\, \Pi \otimes^{\mathfrak{C}}_{\alpha \epsilon I} \mathfrak{H}_\alpha$. This makes it clear, that $\mathfrak{N}$ is the closed, linear set determined by those $\Psi \epsilon \mathfrak{N}$, for which $\Psi \epsilon\, \Pi \otimes^{\mathfrak{C}}_{\alpha \epsilon I} \mathfrak{H}_\alpha$, $\mathfrak{C} \epsilon \Gamma$, holds too.

As F commutes with $U(z_\alpha;\, \alpha \epsilon I)$, therefore $U(z_\alpha;\, \alpha \epsilon I)$ maps $\mathfrak{N}$ on itself. We have therefore for the above $\Psi \epsilon \mathfrak{N}$, $U(z_\alpha;\, \alpha \epsilon I)\Psi \epsilon \mathfrak{N}$ too, and thus $\mathfrak{M}[\Psi] \subset \mathfrak{N}$. Therefore the following statement holds too: $\mathfrak{N}$ is the closed linear set determined by the $\mathfrak{M}[\Psi]$ of all those $\Psi \epsilon \mathfrak{N}$, for which $\Psi \epsilon\, \Pi \otimes^{\mathfrak{C}}_{\alpha \epsilon I} \mathfrak{H}_\alpha$, $\mathfrak{C} \epsilon \Gamma$, holds too.

Now these $\mathfrak{M}(\Psi)$ are all $\eta\, \mathcal{B}^\#$ by Lemma 6.3.5 (η as above, cf. (7), p. 141). Thus $\mathfrak{N}\, \eta\, \mathcal{B}^\#$ too, and therefore its projection $F\, \eta\, \mathcal{B}^\#$, that is $F \epsilon\, \mathcal{B}^\#$.

We are now in the position to prove:

Theorem IX. $A \epsilon\, \mathcal{B}^\#$ if and only if A commutes with all $P[\mathfrak{C}]$ ($\mathfrak{C} \epsilon \Gamma$) and $U(z_\alpha;\, \alpha \epsilon I)$ ($|z_\alpha| = 1$).

Proof: Denote the set of these $P[\mathfrak{C}]$ and $U(z_\alpha;\, \alpha \epsilon I)$ by $\mathcal{S}$, Then we must prove (cf. (9), p. 388)

$$\mathcal{B}^\# = \mathcal{S}'.$$

As we have rings on both sides, it suffices to show that both sides contain the same projections. (Cf. (9), p. 392.)

Every projection of $\mathcal{B}^\#$ belongs to $\mathcal{S}'$ by Lemma 6.3.1, and every projection of $\mathcal{S}'$ belongs to $\mathcal{B}^\#$ by Lemma 6.3.6. Thus the proof is completed.

6.4. Theorem IX gives a complete characterisation of $\mathfrak{B}^{\#}$, but it is desirable to have a more constructive one, as contained in the Theorem which follows.

Theorem X. (I) If $A \in \mathfrak{B}^{\#}$, then $\Phi \in \Pi\otimes^{\mathfrak{C}}_{\alpha\in I}\mathfrak{H}_\alpha$ implies $A\Phi \in \Pi\otimes^{\mathfrak{C}}_{\alpha\in I}\mathfrak{H}_\alpha$ so that A may be considered as an operator in $\Pi\otimes^{\mathfrak{C}}_{\alpha\in I}\mathfrak{H}_\alpha$ (instead of $\Pi\otimes_{\alpha\in I}\mathfrak{H}_\alpha$), for every $\mathfrak{C} \in \Gamma$. We will denote A, when thus restricted to $\Pi\otimes^{\mathfrak{C}}_{\alpha\in I}\mathfrak{H}_\alpha$, by $A^{\mathfrak{C}}$.

(II) Select for each $\mathfrak{C}_w \in \Gamma_w$ an $\mathfrak{C} \subset \mathfrak{C}_w$, $\mathfrak{C} \in \Gamma$, say $\mathfrak{C} = \mathfrak{C}(\mathfrak{C}_w)$. If a bounded operator $A_{\mathfrak{C}_w}$ is given in each $\Pi\otimes^{\mathfrak{C}(\mathfrak{C}_w)}_{\alpha\in I}\mathfrak{H}_\alpha$, $\mathfrak{C}_w \in \Gamma_w$, then an $A \in \mathfrak{B}^{\#}$ with $A^{\mathfrak{C}(\mathfrak{C}_w)} = A_{\mathfrak{C}_w}$, for all $\mathfrak{C}_w \in \Gamma_w$ exists if and only if the set (of real and $\geqq 0$ numbers) $\mathfrak{S}(|||A_{\mathfrak{C}_w}|||;\ \mathfrak{C}_w \in \Gamma_w)$ is bounded. And this $A \in \mathfrak{B}^{\#}$ is unique, if it exists.

(III) $$|||A||| = \text{l.u.b. } \mathfrak{S}\left(|||A^{\mathfrak{C}}|||;\ \mathfrak{C} \in \Gamma\right) = \\ = \text{l.u.b. } \mathfrak{S}\left(|||A^{\mathfrak{C}(\mathfrak{C}_w)}|||;\ \mathfrak{C}_w \in \Gamma_w\right)$$ [43]).

Proof: Ad (I): As A commutes with the projection of $\Pi\otimes^{\mathfrak{C}}_{\alpha\in I}\mathfrak{H}_\alpha$, $P(\mathfrak{C})$, all statements are immediate.

Ad (II): Necessity: Obviously $|||A^{\mathfrak{C}}||| \leqq |||A|||$, so under our assumptions $|||A_{\mathfrak{C}_w}||| = |||A^{\mathfrak{C}(\mathfrak{C}_w)}||| \leqq |||A|||$, and $|||A|||$ is an upper bound for the set in question.

Sufficiency: Consider an arbitrary $\mathfrak{C} \in \Gamma$. There is a unique $\mathfrak{C}_w \in \Gamma_w$ with $\mathfrak{C} \subset \mathfrak{C}_w$. Now $\mathfrak{C}$, $\mathfrak{C}(\mathfrak{C}_w) \subset \mathfrak{C}_w$, so a $U(z_\alpha;\ \alpha\in I)$ $(|z_\alpha| = 1)$ exists, which maps $\Pi\otimes^{\mathfrak{C}(\mathfrak{C}_w)}_{\alpha\in I}\mathfrak{H}_\alpha$ on $\Pi\otimes^{\mathfrak{C}}_{\alpha\in I}\mathfrak{H}_\alpha$. Thus $A_{\mathfrak{C}} = U(z_\alpha;\ \alpha\in I)\,A_{\mathfrak{C}_w}\left(U(z_\alpha;\ \alpha\in I)\right)^{-1}$ is an operator in $\Pi\otimes^{\mathfrak{C}}_{\alpha\in I}\mathfrak{H}_\alpha$.

$A_{\mathfrak{C}}$ does not depend on the particular choice of the z_α if $\mathfrak{C}$ is fixed: If $U(z_\alpha;\ \alpha\in I)$ and $U(z'_\alpha;\ \alpha\in I)$ both map $\Pi\otimes^{\mathfrak{C}(\mathfrak{C}_w)}_{\alpha\in I}\mathfrak{H}_\alpha$ on $\Pi\otimes^{\mathfrak{C}}_{\alpha\in I}\mathfrak{H}_\alpha$, then $\left(U(z'_\alpha;\ \alpha\in I)\right)^{-1}U(z_\alpha;\ \alpha\in I) = U(z_\alpha\overline{z'_\alpha};\ \alpha\in I)$ maps $\Pi\otimes^{\mathfrak{C}(\mathfrak{C}_w)}_{\alpha\in I}\mathfrak{H}_\alpha$ on itself, and so $\Pi_{\alpha\in I}\, z_\alpha\overline{z'_\alpha}$ must be convergent (by Lemma 6.2.3, (I)). Thus $\left(U(z'_\alpha;\ \alpha\in I)\right)^{-1}U(z_\alpha;\ \alpha\in I) = $ $= U(z_\alpha\overline{z'_\alpha};\ \alpha\in I) = \Pi_{\alpha\in I}\, z_\alpha\overline{z'_\alpha}\cdot 1 = c\cdot 1$ (cf. as above), $U(z_\alpha;\ \alpha\in I) =$ $= cU(z'_\alpha;\ \alpha\in I)$. So

$$U(z_\alpha;\ \alpha\in I)A_{\mathfrak{C}_w}\left(U(z_\alpha;\ \alpha\in I)\right)^{-1} = U(z'_\alpha;\ \alpha\in I)A_{\mathfrak{C}_w}\left(U(z'_\alpha;\ \alpha\in I)\right)^{-1}$$

proving our statement.

If $\mathfrak{C} = \mathfrak{C}(\mathfrak{C}_w)$ then we may choose $z_\alpha = 1$, $U(z_\alpha;\ \alpha\in I) = 1$ and so $A_{\mathfrak{C}}$ coincides with our original $A_{\mathfrak{C}_w}$.

---

[43]) l.u.b. = least upper bound.

Finally, if $U(z_\alpha;\ \alpha\in I)$ maps $\Pi\otimes_{\alpha\in I}^{\mathfrak{C}}\mathfrak{H}_\alpha$ on $\Pi\otimes_{\alpha\in I}^{\mathfrak{D}}\mathfrak{H}_\alpha$, then $\mathfrak{C}\subset\mathfrak{C}_w\in\Gamma_w$ implies $\mathfrak{D}\subset\mathfrak{C}_w$, and if $U(z'_\alpha;\ \alpha\in I)$ maps $\Pi\otimes_{\alpha\in I}^{\mathfrak{C}(\mathfrak{C}_w)}\mathfrak{H}_\alpha$ on $\Pi\otimes_{\alpha\in I}^{\mathfrak{C}}\mathfrak{H}_\alpha$, then $U(z_\alpha z'_\alpha;\ \alpha\in I)=U(z_\alpha;\ \alpha\in I)\,U(z'_\alpha;\ \alpha\in I)$ maps $\Pi\otimes_{\alpha\in I}^{\mathfrak{C}(\mathfrak{C}_w)}\mathfrak{H}_\alpha$ on $\Pi\otimes_{\alpha\in I}^{\mathfrak{D}}\mathfrak{H}_\alpha$. So

$$A_{\mathfrak{D}}=U(z_\alpha z'_\alpha;\ \alpha\in I)\,A_{\mathfrak{C}(\mathfrak{C}_w)}\big(U(z_\alpha z'_\alpha;\ \alpha\in I)\big)^{-1}=$$

$$(*)\qquad =U(z_\alpha;\ \alpha\in I)U(z'_\alpha;\ \alpha\in I)A_{\mathfrak{C}(\mathfrak{C}_w)}\big(U(z'_\alpha;\ \alpha\in I)\big)^{-1}\big(U(z_\alpha;\ \alpha\in I)\big)^{-1}$$

$$=U(z_\alpha;\ \alpha\in I)A_{\mathfrak{C}}\big(U(z_\alpha;\ \alpha\in I)\big)^{-1}.$$

Define now an operator A^0 as follows: $A^0\Phi$ is defined if and only if $\Phi=\Psi_1+\ldots+\Psi_l$, where $\Psi_1\in\Pi\otimes_{\alpha\in I}^{\mathfrak{C}_1}\mathfrak{H}_\alpha,\ldots,\Psi_l\in\Pi\otimes_{\alpha\in I}^{\mathfrak{C}_l}\mathfrak{H}_\alpha$, the $\mathfrak{C}_1,\ldots,\mathfrak{C}_l\in\Gamma$ being mutually different, and then

$$A^0\Phi=A^0(\Psi_1+\ldots+\Psi_l)=A_{\mathfrak{C}_1}\Psi_1+\ldots+A_{\mathfrak{C}_l}\Psi_l.$$

A^0 is clearly linear and commutes with every $P(\mathfrak{C})$. Owing to (*) it commutes with every $U(z_\alpha;\ \alpha\in I)$ too. The domain of A^0 is everywhere dense.

Put $C=\text{l.u.b.}\ \mathfrak{S}(|||A_{\mathfrak{C}_w}|||;\ \mathfrak{C}_w\in\Gamma_w)$. Then each

$$|||A_{\mathfrak{C}}|||=|||A_{\mathfrak{C}(\mathfrak{C}_w)}|||\leqq C,$$

and so

$$\|A^0\Phi\|^2=\|A_{\mathfrak{C}_1}\Psi_1+\ldots+A_{\mathfrak{C}_l}\Psi_l\|^2=\|A_{\mathfrak{C}_1}\Psi_1\|^2+\ldots+\|A_{\mathfrak{C}_l}\Psi_l\|^2\leqq$$

$$\leqq C^2\|\Psi_1\|^2+\ldots+C^2\|\Psi_l\|^2=C^2(\|\Psi_1\|^2+\ldots+\|\Psi_l\|^2)=$$

$$=C^2\|\Psi_1+\ldots+\Psi_l\|^2=C^2\|\Phi\|^2,$$

$$(**)\qquad \|A^0\Phi\|\leqq C\|\Phi\|.$$

Thus A^0 extends by continuity to an everywhere defined operator A (in $\Pi\otimes_{\alpha\in I}\mathfrak{H}_\alpha$). This A is linear, along with A^0. (**) gives, by continuity, $\|A\Phi\|\leqq C\|\Phi\|$ so $A\in\mathfrak{B}_\otimes$ and $|||A|||\leqq C$. Finally A commutes with all $P[\mathfrak{C}]$ and $U(z_\alpha;\ \alpha\in I)$, along with A^0. So Theorem IX gives $A\in\mathfrak{B}^\#$.

Finally $\Phi\in\Pi\otimes_{\alpha\in I}^{\mathfrak{C}(\mathfrak{C}_w)}\mathfrak{H}_\alpha$ gives $A\Phi=A^0\Phi=A_{\mathfrak{C}_w}\Phi$. Thus $A^{\mathfrak{C}(\mathfrak{C}_w)}=A_{\mathfrak{C}_w}$. Therefore this A meets all our requirements.

Uniqueness: Assume, that $A',A''\in\mathfrak{B}^\#$, and $A'^{\mathfrak{C}(\mathfrak{C}_w)}=A''^{\mathfrak{C}(\mathfrak{C}_w)}$ for all $\mathfrak{C}_w\in\Gamma_w$. If $\mathfrak{C}\subset\mathfrak{C}_w$, then a $U(z_\alpha;\ \alpha\in I)$ which maps $\Pi\otimes_{\alpha\in I}^{\mathfrak{C}(\mathfrak{C}_w)}\mathfrak{H}_\alpha$ on $\Pi\otimes_{\alpha\in I}^{\mathfrak{C}}\mathfrak{H}_\alpha$ exists. As this $U(z_\alpha;\ \alpha\in I)$ commutes with A',A''. the assumption $A'^{\mathfrak{C}(\mathfrak{C}_w)}=A''^{\mathfrak{C}(\mathfrak{C}_w)}$ implies $A'^{\mathfrak{C}}=A''^{\mathfrak{C}}$. As $\mathfrak{C}_w\in\Gamma_w$ was arbitrary, this holds for all $\mathfrak{C}\in\Gamma$.

Thus $A'\Phi=A''\Phi$ if $\Phi\in\Pi\otimes_{\alpha\in I}^{\mathfrak{C}}\mathfrak{H}_\alpha$, $\mathfrak{C}\in\Gamma$. Therefore it holds for all $\Phi(\in\Pi\otimes_{\alpha\in I}\mathfrak{H}_\alpha)$, that is $A'=A''$.

Ad (III): $|||A^{\mathfrak{C}}||| \leqq |||A|||$ is obvious, and so

$$\text{l.u.b. } \mathfrak{S}(|||A^{\mathfrak{C}}|||;\ \mathfrak{C} \in \Gamma) \leqq |||A|||.$$

We saw in the sufficiency-proof of (II), that if we put $A_{\mathfrak{C}_w} = A^{\mathfrak{C}(\mathfrak{C}_w)}$, then

$$|||A||| \leqq C = \text{l.u.b. } \mathfrak{S}(|||A_{\mathfrak{C}_w}|||;\ \mathfrak{C}_w \in \Gamma_w) = \\ = \text{l.u.b. } \mathfrak{S}(|||A^{\mathfrak{C}(\mathfrak{C}_w)}|||;\ \mathfrak{C}_w \in \Gamma_w).$$

Finally l.u.b. $\mathfrak{S}(|||A^{\mathfrak{C}(\mathfrak{C}_w)}|||;\ \mathfrak{C}_w \in \Gamma_w) \leqq$ l.u.b. $\mathfrak{S}(|||A^{\mathfrak{C}}|||;\ \mathfrak{C} \in \Gamma)$ is obvious. The three inequalities give together

$$|||A||| = \text{l.u.b. } \mathfrak{S}(|||A^{\mathfrak{C}}|||;\ \mathfrak{C} \in \Gamma) = \text{l.u.b. } \mathfrak{S}(|||A^{\mathfrak{C}(\mathfrak{C}_w)}|||;\ \mathfrak{C}_w \in \Gamma_w)$$

as desired.

Theorems IX, X make it clear, that the elements of $\mathfrak{B}^{\#}$ are characterised by two restrictions:

(I) An $A \in \mathfrak{B}^{\#}$ is reduced by each $\Pi\otimes_w{}^{\mathfrak{C}_w}_{\alpha \in I}\, \mathfrak{H}_\alpha$, $\mathfrak{C}_w \in \Gamma_w$. (Cf. Lemma 6.3.1, „Reduction" is defined in (8), pp. 78—80.)

(II) Within each $\Pi\otimes_w{}^{\mathfrak{C}_w}_{\alpha \in I}\, \mathfrak{H}_\alpha$, $\mathfrak{C}_w \in \Gamma_w$, the $A \in \mathfrak{B}^{\#}$ is reduced by each $\Pi\otimes^{\mathfrak{C}}_{\alpha \in I}\, \mathfrak{H}_\alpha$, $\mathfrak{C} \subset \mathfrak{C}_w$, $\mathfrak{C} \in \Gamma$, and its behaviour in any one of these $\Pi\otimes^{\mathfrak{C}}_{\alpha \in I}\, \mathfrak{H}_\alpha$ determines it in the remaining ones (for a given $\mathfrak{C}_w$ and the $\mathfrak{C} \subset \mathfrak{C}_w$, $\mathfrak{C} \in \Gamma$).

Now (II), as indeed the entire difference between the subdivisions of $\Pi\otimes_{\alpha \in I}\, \mathfrak{H}_\alpha$ into $\Pi\otimes^{\mathfrak{C}}_{\alpha \in I}\, \mathfrak{H}_\alpha$'s resp. $\Pi\otimes_w{}^{\mathfrak{C}_w}_{\alpha \in I}\, \mathfrak{H}_\alpha$'s, is ultimately due to our way of handling the non-convergent but quasi-convergent case in 2.5. (The $U(z_\alpha;\ \alpha \in I)$ which map an $\mathfrak{C} \subset \mathfrak{C}_w$, $\mathfrak{C} \in \Gamma$, on other $\mathfrak{D} \subset \mathfrak{C}_w$, $\mathfrak{D} \in \Gamma$, have non-convergent but quasi-convergent $\Pi_{\alpha \in I}\, z_\alpha$'s, cf. Lemma 6.2.3.) A more complicated procedure in dealing with such infinite products, using generalised-Banach-limits, would have permitted us to avoid this. Compared with our present method, however, it would have been highly artificial and arbitrary, and would have implied serious difficulties in the formulation of an associative law.

Having clarified the resp. roles of the $\mathfrak{C} \in \Gamma$ and $\mathfrak{C}_w \in \Gamma_w$ we proceed to determine their numbers.

LEMMA 6.4.1. (I) If I is finite, then Γ and Γ_w both possess exactly one element, which is the same for both: the set $\mathfrak{C}_0$ of all sequences $(f_\alpha;\ \alpha \in I)$ [44].

[44]) So $\Pi\otimes^{\mathfrak{C}_0}_{\alpha \in I}\, \mathfrak{H}_\alpha = \Pi\otimes_w{}^{\mathfrak{C}_0}_{\alpha \in I}\, \mathfrak{H}_\alpha = \Pi\otimes_{\alpha \in I}\, \mathfrak{H}_\alpha$, $P[\mathfrak{C}_0] = P_w[\mathfrak{C}_0] = 1$. Every $\Pi_{\alpha \in I}\, z_\alpha$ converges, so $U(z_\alpha;\ \alpha \in I) = (\Pi_{\alpha \in I}\, z_\alpha) \cdot 1$. Thus Theorem IX gives $\mathfrak{B}^{\#} = \mathfrak{B}_{\otimes}$, in accordance with (7), p. 135.

(II) If I is infinite, its power being $\aleph^*$, then for each $\mathfrak{C}_w \in \Gamma_w$ the number of the $\mathfrak{C} \subset \mathfrak{C}_w$, $\mathfrak{C} \in \Gamma$, is $2^{\aleph^*}$.

(III) If the number of $\alpha \in I$ with a $\geqq 2$-dimensional $\mathfrak{H}_\alpha$ is finite, then Γ_w possesses exactly one element, if this number is infinite, then power of Γ_w is $\geqq \aleph$ [45]).

Proof: Ad (I): Obvious by Lemma 3.3.5.

Ad (II): Given an $\mathfrak{C}_w \in \Gamma_w$, the number of all $\mathfrak{C} \subset \mathfrak{C}_w$, $\mathfrak{C} \in \Gamma$. is obviously $\leqq$ the number of all combinations of z_α, $\alpha \in I$ $(|z_\alpha|=1)$, that is $\aleph^{\aleph^*} = 2^{\aleph_0 \aleph^*} = 2^{\aleph^*}$.

As $\aleph_0 \aleph^* = \aleph^*$, decompose I into mutually disjoint sets $J_\gamma, \gamma \in \mathrm{L}$, each J_γ having the power $\aleph_0$, and L having the power $\aleph^*$. For any set $\mathrm{L}' \subset \mathrm{L}$ form $z_\alpha^{\mathrm{L}'} \begin{cases} = 1 & \text{if } \alpha \in J_\gamma, \gamma \in \mathrm{L}' \\ = -1 & \text{if } \alpha \in J_\gamma, \gamma \notin \mathrm{L}' \end{cases}$. Choose a sequence $(f_\alpha^0; \alpha \in I) \in \mathfrak{C}_w$ $\|f_\alpha\| = 1$. Then all $(z_\alpha^{\mathrm{L}} f_\alpha^0; \alpha \in \mathrm{L}) \in \mathfrak{C}_w$. If $\mathrm{L}' \neq \mathrm{L}''$, then a γ exists, for which $\gamma \in \mathrm{L}'$ but $\notin \mathrm{L}''$, or conversely. At any rate $\alpha \in J_\gamma$ gives $z_\alpha^{\mathrm{L}'} z_\alpha^{\mathrm{L}''} = -1$, so that $z_\alpha^{\mathrm{L}'} z_\alpha^{\mathrm{L}''} = -1$ occurs infinitely many times. Thus $\Sigma_{\alpha \in I} |(z_\alpha^{\mathrm{L}'} f_\alpha^0, z_\alpha^{\mathrm{L}''} f_\alpha^0) - 1| = \Sigma_{\alpha \in I} |z_\alpha^{\mathrm{L}'} z_\alpha^{\mathrm{L}''} - 1|$ contains infinitely many terms $|-1-1| = 2$, and is therefore divergent. That is, $(z_\alpha^{\mathrm{L}'} f_\alpha^0; \alpha \in I)$ not $\approx (z_\alpha^{\mathrm{L}''} f_\alpha^0; \alpha \in I)$ by Definition 3.3.2.

Summing up: All $\mathfrak{C}(z_\alpha^{\mathrm{L}'} f_\alpha^0; \alpha \in I)$ with $\mathrm{L}' \subset \mathrm{L}$ are $\subset \mathfrak{C}_w, \in \Gamma$, and mutually different. Their number is $2^{\aleph^*}$. So the number of the $\mathfrak{C} \subset \mathfrak{C}_w$, $\mathfrak{C} \in \Gamma$ is $\geqq 2^{\aleph^*}$. Thus it must be $= 2^{\aleph^*}$.

Ad (III): Finite number of $\alpha \in I$ with $\geqq 2$-dimensional $\mathfrak{H}_\alpha$: We want to prove $(f_\alpha;\ \alpha \in I) \underset{w}{\approx} (g_\alpha;\ \alpha \in I)$ for all C_0-sequences. By Lemma 3.3.7, we may assume $\|f_\alpha\| = \|g_\alpha\| = 1$.

If $\mathfrak{H}_\alpha$ is 1-dimensional, then $f_\alpha = c_\alpha g_\alpha$, $|c_\alpha| = 1$, so $|(f_\alpha, g_\alpha)| = 1$, $||(f_\alpha, g_\alpha)| - 1| = 0$. Thus $\Sigma_{\alpha \in I} ||(f_\alpha, g_\alpha)| - 1|$ converges, and so $(f_\alpha;\ \alpha \in I) \underset{w}{\approx} (g_\alpha;\ \alpha \in I)$ by Lemma 6.1.3.

Infinite number of $\alpha \in I$ with $\geqq 2$-dimensional $\mathfrak{H}_\alpha$: Let $\alpha_{i,j}$, $i, j = 1, 2, \ldots$ be an enumerably infinite double-sequence of such α's, and $\varphi_{\alpha_{i,j}}$, $\psi_{\alpha_{i,j}}$ two normalised orthogonal elements of $\mathfrak{H}_{\alpha_{i,j}}$. For each $\alpha \neq \text{all } \alpha_{i,j}$ select an $f_\alpha^0 \in \mathfrak{H}_\alpha$ with $\|f_\alpha^0\| = 1$. For any set $N \subset \mathfrak{S}\ (1, 2, \ldots)$ form

$$g_\alpha^N \begin{cases} = \varphi_{\alpha_{i,j}} & \text{if } \alpha = \alpha_{i,j}, \quad i \in N, \\ = \psi_{\alpha_{i,j}} & \text{if } \alpha = \alpha_{i,j}, \quad i \notin N, \\ = f_\alpha^0 & \text{if } \alpha \neq \text{all } \alpha_{i,j}. \end{cases}$$

Clearly every $(g_\alpha^N;\ \alpha \in I)$ is a C_0-sequence. If $N' \neq N''$, then an i

[45]) $\aleph$ = power of the continuum, $\aleph_0$ = power of any enumerably infinite set.

exists, for which $i \in N$ but $\notin N''$ or conversely. At any rate $g^{N'}_{\alpha_{i,j}}$ and $g^{N''}_{\alpha_{i,j}}$ coincide with $\varphi_{\alpha_{i,j}}$ and $\psi_{\alpha_{i,j}}$ in some order, $(g^{N'}_{\alpha_{i,j}}, g^{N''}_{\alpha_{i,j}}) = 0$ so that $(g^{N'}_{\alpha}, g^{N''}_{\alpha}) = 0$ occurs infinitely many times. Thus $\sum_{\alpha \in I} \big| |(g^{N'}_{\alpha}, g^{N''}_{\alpha})| - 1 \big|$ contains infinitely many terms $= |0-1| = 1$, and is therefore divergent. That is, $(g^{N'}_{\alpha}; \alpha \in I)$ not $\underset{w}{\approx} (g^{N''}_{\alpha}; \alpha \in I)$ by Lemma 6.1.3.

Summing up: All $\mathfrak{C}_w (g^{N}_{\alpha}; \alpha \in I)$ with $N \subset \mathfrak{S}(1, 2, \ldots)$ are $\in \Gamma_w$ and mutually different. Their number is $2^{\aleph_0} = \aleph$. So the number of all $\mathfrak{C}_w \in \Gamma_w$ is $\geqq \aleph$.

We forego an exact determination of the power of Γ_w, which would present no difficulties. Clearly power $(\Gamma) = 2^{\aleph^*} \cdot$ power (Γ_w).

Part IV: Discussion of a special case.

Chapter 7: Discussion of a special case.

7.1. The unitary spaces $\Pi\otimes^{\mathfrak{C}}_{\alpha \in I} \mathfrak{H}_\alpha$ are isomorphic to each other by Theorem V, and each $\Pi \underset{w}{\otimes}{}^{\mathfrak{C}_w}_{\alpha \in I} \mathfrak{H}_\alpha$ contains the same number of $\Pi\otimes^{\mathfrak{C}}_{\alpha \in I} \mathfrak{H}_\alpha$ by Lemma 6.4.1, (II). Therefore the structure of $\Pi\otimes_{\alpha \in I} \mathfrak{H}_\alpha$ (and of its subspaces $\Pi\otimes^{\mathfrak{C}}_{\alpha \in I} \mathfrak{H}_\alpha$, $\Pi \underset{w}{\otimes}{}^{\mathfrak{C}_w}_{\alpha \in I} \mathfrak{H}_\alpha$) can only be investigated further, by considering other objects in $\Pi\otimes_{\alpha \in I} \mathfrak{H}_\alpha$: Operators and rings of operators. This was done in Chapter 6 for the ring $\mathcal{B}^{\#}$, the next things to discuss are therefore subrings of $\mathcal{B}^{\#}$. Considering the restricted form in which the associative law for $\Pi\otimes_{\alpha \in I} \mathfrak{H}_\alpha$ had to be formulated in Theorem VI (cf. also the remarks after this Theorem), structural questions of some interest will necessarily arise in connection with the associative law.

We know from Theorems IX, X, that every $A \in \mathcal{B}^{\#}$ behaves in the same way in each $\Pi\otimes^{\mathfrak{C}}_{\alpha \in I} \mathfrak{H}_\alpha$ within one given $\Pi \underset{w}{\otimes}{}^{\mathfrak{C}_w}_{\alpha \in I} \mathfrak{H}_\alpha$. Therefore we can only expect interesting phenomena, if more than one $\mathfrak{C}_w \in \Gamma_w$ exists. This means, by Lemma 6.4.1, (III): If infinitely many $\alpha \in I$ with $\geqq$ 2-dimensional $\mathfrak{H}_\alpha$ exist. Furthermore we know by Theorem VII, that complications in connection with the associative law will only arise, if L (that is, the number of pieces I_γ, $\gamma \in$ L of I) is infinite. And as I_γ's with one element α are clearly irrelevant, therefore each I_γ must have $\geqq 2$ elements.

Thus the simplest possible example, on which the essential features of an infinite direct product $\Pi\otimes_{\alpha \in I} \mathfrak{H}_\alpha$ may already be observed, is this one:

Let I be enumerably infinite, each $\mathfrak{H}_\alpha$, $\alpha \in I$, 2-dimensional, let each I_γ, $\gamma \in \mathrm{L}$, have exactly 2 elements, and thus L be enumerably infinite.

Specifically:

Let I be the set of all pairs (n, τ), $n = 1, 2, \ldots, \tau = 1, 2$, L be the set of all $n = 1, 2, \ldots, I_n$ the set consisting of $(n, 1)$ and $(n, 2)$ $\big((n, \tau), n$ replace $\alpha, \gamma\big)$, $\underline{\mathfrak{H}_{(n,\tau)}}$ is 2-dimensional.

$\mathcal{B}^\#$ is the ring generated by all $\overline{\mathcal{B}_{(n,\tau)}}$. A subring of $\mathcal{B}^\#$ which is essentially affected by an application of the associative law (in the sense of Theorem VI) to the above I, L and I_n, is the ring generated by all $\overline{\mathcal{B}_{(n,1)}}$, which we will denote by $\mathcal{C}^\#$. So we have:

$$\mathcal{B}^\# = \mathcal{R}\big(\overline{\mathcal{B}_{(n,\tau)}};\ n = 1, 2, \ldots,\ \tau = 1, 2\big),$$
$$\mathcal{C}^\# = \mathcal{R}\big(\overline{\mathcal{B}_{(n,1)}};\ n = 1, 2, \ldots\big).$$

7.2. We now wish to see the effect of the „associative transformation" of our $\Pi\otimes_{\alpha \in I} \mathfrak{H}_\alpha$ into $\Pi\otimes_{\gamma\in \mathrm{L}}\big(\Pi\otimes_{\alpha \in I_\gamma} \mathfrak{H}_\alpha\big)$ on these $\mathcal{B}^\#$ and $\mathcal{C}^\#$. That is: Besides $\Pi\otimes_{\alpha\in I} \mathfrak{H}_\alpha = \Pi\otimes_{\substack{n=1,2,\ldots\\ \tau=1,2}} \mathfrak{H}_{(n,\tau)}$, we wish to form $\Pi\otimes_{\gamma\in L}\big(\Pi\otimes_{\alpha\in I_\gamma} \mathfrak{H}_\alpha\big) = \Pi\otimes_{n=1,2,\ldots}\big(\mathfrak{H}_{(n,1)} \otimes \mathfrak{H}_{(n,2)}\big)$, too, and see what happens to $\mathcal{B}^\#$ and $\mathcal{C}^\#$.

Let us first consider the situation in $\Pi\otimes_{\alpha\in I} \mathfrak{H}_\alpha = \Pi\otimes_{\substack{n=1,2,\ldots\\ \tau=1,2}} \mathfrak{H}_\alpha$. Each $\Pi\otimes^{\mathfrak{C}}_{\substack{n=1,2,\ldots\\ \tau=1,2}} \mathfrak{H}_{(n,\tau)}$ has $\aleph_0$ dimensions, because the $\mathbf{F}$ of Theorem V is clearly enumerably infinite. So each $\Pi\otimes^{\mathfrak{C}}_{\substack{n=1,2,\ldots\\ \tau=1,2}} \mathfrak{H}_{(n,\tau)}$ is a Hilbert space, and as $\mathcal{B}^\#$ coincides in it by Theorem X with the ring of all its bounded operators, we may say, using the terminology of (7), p. 172: $\mathcal{B}^\#$ is a *factor* of class (I_∞) [46]) in each $\Pi\otimes^{\mathfrak{C}}_{\substack{n=1,2,\ldots\\ \tau=1,2}} \mathfrak{H}_{(n,\tau)}$.

As to $\mathcal{C}^\#$, observe first, that the associative law may be applied to $\Pi\otimes_{\alpha\in I} \mathfrak{H}_\alpha = \Pi\otimes_{\substack{n=1,2,\ldots\\ \tau=1,2}} \mathfrak{H}_{(n,\tau)}$ with $\mathrm{L}' = (1, 2)$, $I'_\tau = \mathfrak{S}\big((n, \tau);\ n = 1, 2, \ldots\big)$, establishing its isomorphism with $\Pi\otimes_{\gamma\in \mathrm{L}'}\big(\Pi\otimes_{\alpha\in I'_\gamma} \mathfrak{H}_\alpha\big) = \big(\Pi\otimes_{n=1,2,\ldots} \mathfrak{H}_{(n,1)}\big) \otimes \big(\Pi\otimes_{n=1,2,\ldots} \mathfrak{H}_{(n,2)}\big)$ (cf. Theorem VII.) In particular, we may form for every $\mathfrak{C}$ of $\Pi\otimes_{\substack{n=1,2,\ldots\\ \tau=1,2}} \mathfrak{H}_{(n,\tau)}$ the $\mathfrak{C}_1$ of $\Pi\otimes_{n=1,2,\ldots} \mathfrak{H}_{(n,1)}$ and the $\mathfrak{C}_2$ of $\Pi\otimes_{n=1,2,\ldots} \mathfrak{H}_{(n,2)}$ which correspond to it by Theorem VI, (I);

---

[46]) That is: A *direct factor* cf. (7), pp. 139 and 173.

then $\Pi\otimes^{\mathfrak{C}}_{\substack{n=1,2,\ldots\\ \tau=1,2}} \mathfrak{H}_{(n,\tau)}$ and $(\Pi\otimes^{\mathfrak{C}_1}_{n=1,2,\ldots} \mathfrak{H}_{(n,1)})\otimes(\Pi\otimes^{\mathfrak{C}_2}_{n=1,2,\ldots} \mathfrak{H}_{(n,2)})$ will correspond to each other under the above isomorphism (by Theorem VI, (III), and Theorem VII).

It is evident, that instead of forming a $\overline{\mathcal{B}_{(n,1)}}$ directly in $\Pi\otimes_{\substack{n=1,2,\ldots\\ \tau=1,2}} \mathfrak{H}_{(n,\tau)}$, we might as well form it indirectly: First in $\Pi\otimes_{n=1,2,\ldots} \mathfrak{H}_{(n,1)}$ and then extending again in $(\Pi\otimes_{n=1,2,\ldots} \mathfrak{H}_{(n,1)})\otimes \otimes\, \Pi\otimes_{n=1,2,\ldots} \mathfrak{H}_{(n,2)})$. Thus $\mathcal{C}^{\#} = \mathcal{R}(\overline{\mathcal{B}_{(n,1)}};\ n = 1, 2, \ldots)$ (in the $\Pi\otimes_{\substack{n=1,2,\ldots\\ \tau=1,2}} \mathfrak{H}_{(n,1)}$-sence) obtains from the $\mathcal{B}^{\#}$ of $\Pi\otimes_{n=1,2,\ldots} \mathfrak{H}_{(n,1)}$, by extending it in $(\Pi\otimes_{n=1,2,\ldots} \mathfrak{H}_{(n,1)})\otimes(\Pi\otimes_{n=1,2,\ldots} \mathfrak{H}_{(n,2)})$. Thus it is isomorphic to this $\mathcal{B}^{\#}$, and in $\Pi\otimes^{\mathfrak{C}}_{\substack{n=1,2,\ldots\\ \tau=1,2}} \mathfrak{H}_{(n,\tau)}$ it is isomorphic to this $\mathcal{B}^{\#}$ in $\Pi\otimes^{\mathfrak{C}_1}_{n=1,2,\ldots} \mathfrak{H}_{(n,1)}$. But the $\mathcal{B}^{\#}$ of $\Pi\otimes_{n=1,2,\ldots} \mathfrak{H}_{(n,1)}$ in $\Pi\otimes^{\mathfrak{C}_1}_{n=1,2,\ldots} \mathfrak{H}_{(n,1)}$ is again a factor of class (I_∞) (by the same argument as above for $\Pi\otimes_{\substack{n=1,2,\ldots\\ \tau=1,2}} \mathfrak{H}_{(n,\tau)}$), therefore the same is true for our $\mathcal{C}^{\#}$ in $\Pi\otimes^{\mathfrak{C}}_{\substack{n=1,2,\ldots\\ \tau=1,2}} \mathfrak{H}_{(n,\tau)}$.

7.3. Let us now investigate the situation in $\Pi\otimes_{\gamma\in \mathrm{L}}(\Pi\otimes_{\alpha\in I} \mathfrak{H}_{\alpha_\gamma}) = \Pi\otimes_{n=1,2,\ldots} (\mathfrak{H}_{(n,1)} \otimes \mathfrak{H}_{(n,2)})$. For $\mathcal{B}^{\#\prime}$ (we write $\mathcal{B}^{\#\prime}$ instead of $\mathcal{B}^{\#}$, to emphasize the difference) the argument of 7.2 applies again (using now the 4-dimensional $\mathfrak{H}_{(n,1)} \otimes \mathfrak{H}_{(n,2)}$ in place of the 2-dimensional $\mathfrak{H}_{(n,\tau)}$). $\mathcal{B}^{\#\prime}$ is a factor of class (I_∞) in each $\Pi\otimes^{\mathfrak{D}}_{n=1,2,\ldots} (\mathfrak{H}_{(n,1)} \otimes \mathfrak{H}_{(n,2)})$ which are all Hilbert spaces.

As to $\mathcal{C}^{\#\prime}$ (in place of $\mathcal{C}^{\#}$), we must, of course, modify the definition of $\mathcal{C}^{\#\prime}$ somewhat: For each $\mathcal{B}_{(n,1)}$ (in $\mathfrak{H}_{(n,2)}$) we first extend in $\mathfrak{H}_{(n,1)} \otimes \mathfrak{H}_{(n,2)}$ to $\overline{\mathcal{B}_{(n,1)}}$, and then we extend this in $\Pi\otimes_{n=1,2,\ldots} (\mathfrak{H}_{(n,1)} \otimes \mathfrak{H}_{(n,2)})$ to $\overline{\overline{\mathcal{B}_{(n,1)}}}$. And now we may form (as at the end of § 7.1)

$$\mathcal{C}^{\#\prime} = \mathcal{R}(\overline{\overline{\mathcal{B}}}_{(n,1)};\ n = 1, 2, \ldots).$$

While until now the rings $\mathcal{B}^{\#}$, $\mathcal{C}^{\#}$, $\mathcal{B}^{\#\prime}$ behaved isomorphically in all incomplete direct products, this need not be (and as we will soon see, is not) the case for $\mathcal{C}^{\#\prime}$ in the various $\Pi\otimes^{\mathfrak{D}}_{n=1,2,\ldots} (\mathfrak{H}_{(n,1)} \otimes \mathfrak{H}_{(n,2)})$. We proceed now to discuss this in detail.

Let $\varphi_{(n,\tau),\varkappa}$, $\varkappa = 1, 2$, be a complete normalised orthogonal set in $\mathfrak{H}_{(n,\tau)}$. Then $\varphi_{(n,1),\varkappa} \otimes \varphi_{(n,2),\lambda}$, $\varkappa, \lambda = 1, 2$, is one in $\mathfrak{H}_{(n,1)} \otimes \mathfrak{H}_{(n,2)}$.

Thus the general element of $\mathfrak{H}_{(n,\tau)}$ is

$$f_{(n,\tau)} = \Sigma_{k=1}^{2}\, x_{(n,\tau),\varkappa}\, \varphi_{(n,\tau),\varkappa},$$

while that one of $\mathfrak{H}_{(n,1)} \otimes \mathfrak{H}_{(n,2)}$ is

$$(*) \qquad g_{(n)} = \sum_{\varkappa,\lambda=1}^{2} y_{(n),\varkappa\lambda}\, \varphi_{(n,1),\varkappa} \otimes \varphi_{(n,2)\lambda},$$

We will treat the $x_{(n,\tau),\varkappa}$ as vectors:

$$\xi_{(n,\tau)} = \big((x_{(n,\tau),1},\, x_{(n,\tau),2})\big)\,,$$

and the $y_{(n),\varkappa\lambda}$ as matrices:

$$H_{(n)} = \left(\begin{pmatrix} y_{(n),11} & y_{(n),12} \\ y_{(n),21} & y_{(n),22} \end{pmatrix}\right).$$

Consider now an incomplete direct product $\Pi\otimes^{\mathfrak{D}}_{n=1,2,\ldots}(\mathfrak{H}_{(n,1)} \otimes \mathfrak{H}_{(n,2)})$. It is characterised by any C_0-sequence $(g^0_{(n)};\, n = 1, 2, \ldots) \in \mathfrak{D}$, $g^0_{(n)} \in \mathfrak{H}_{(n,1)} \otimes \mathfrak{H}_{(n,2)}$. Use the expansion $(*)$ for each $g^0_{(n)}$, then the matrices $H^0_{(n)} = \left(\begin{pmatrix} y^0_{(n),11} & y^0_{(n),12} \\ y^0_{(n),21} & y^0_{(n),22} \end{pmatrix}\right)$, $n = 1, 2, \ldots$, obtain. Thus $\Pi\otimes^{\mathfrak{D}}_{n=1,2,\ldots}(\mathfrak{H}_{(n,1)} \otimes \mathfrak{H}_{(n,2)})$ is characterised by this sequence of matrices $H^0_{(n)}$, $n = 1, 2, \ldots$.

Observe now the following points:

(a) We can choose the $g^0_{(n)}$, $n = 1, 2, \ldots$, with $\| g^0_{(n)} \| = 1$ (by Lemma 3.3.7), that is: We may assume, that

$$\sum_{\varkappa,\lambda=1}^{2} | y^0_{(n),\varkappa\lambda} |^2 = 1.$$

(b) From the point of view of isomorphism of the parts of $\mathcal{C}^{\#\prime}$ in the various $\Pi\otimes^{\mathfrak{D}}_{n=1,2,\ldots}(\mathfrak{H}_{(n,1)} \otimes \mathfrak{H}_{(n,2)})$ a permutation of the factors $\mathfrak{H}_{(n,1)} \otimes \mathfrak{H}_{(n,2)}$, $n = 1, 2, \ldots$, does not matter — all our constructions being entirely independent of any ordering of the factors [47]). Therefore any permutation of the $H^0_{(n)}$, $n = 1, 2, \ldots$, is immaterial for our isomorphism-problem of $\mathcal{C}^{\#\prime}$ in $\Pi\otimes^{\mathfrak{D}}_{n=1,2,\ldots}(\mathfrak{H}_{(n,1)} \otimes \mathfrak{H}_{(n,2)})$.

(c) From the same point of view any change of the complete, normalised, orthogonal sets in the various $\mathfrak{H}_{(n,\tau)}$ is immaterial. This is rather obvious, or else it may be proved with the help of Theorem IV.

Replace therefore $\varphi_{(n,\tau),\varkappa}$, $\varkappa = 1, 2$, by $\varphi'_{(n,\tau),\varkappa}$, $\varkappa = 1, 2$, where

$$\varphi_{(n,\tau),\varkappa} = \sum_{\lambda=1}^{2} u_{(n,\tau),\varkappa\lambda}\, \varphi'_{(n,\tau),\lambda}, \qquad \varkappa = 1, 2,$$

the matrices $U_{(n,\tau)} = \left(\begin{pmatrix} u_{(n,\tau),11} & u_{(n,\tau),12} \\ u_{(n,\tau),21} & u_{(n,\tau),22} \end{pmatrix}\right)$ being unitary, but otherwise arbitrary. It is clear, that this replaces each $H^0_{(n)}$ by

$$(\dagger) \qquad H^{0\prime}_{(n)} = V_{(n)}\, H^0_{(n)}\, W_{(n)},$$

[47]) But not from splitting up and recombining factors! We have an unrestricted commutative law, but a very restricted associative one. (Cf. Theorem VI.)

where $V_{(n)}$ = transposed matrix of $U_{(n,1)}$, $W_{(n)} = U_{(n,2)}$. Thus $V_{(n)}$, $W_{(n)}$ are again arbitrary unitary matrices.

Now it is well-known, that every matrix can be carried by (†) into the diagonal form, with diagonal elements $\geqq 0$ [48]). So we may assume that $H^0_{(n)}$ has this form: $\left(\begin{pmatrix} y^0_{(n)} & 0 \\ 0 & z^0_{(n)} \end{pmatrix}\right)$, $y^0_{(n)}, z^0_{(n)} \geqq 0$. By another application of (†) we may interchange $y^0_{(n)}$, $z^0_{(n)}$ $\left(V_{(n)} = W_{(n)} = \left(\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\right)\right)$, so we may assume, that $y^0_{(n)} \geqq z^0_{(n)} \geqq 0$.

By (a) we may assume, that $(y^0_{(n)})^2 + (z^0_{(n)})^2 = 1$. So we have: $y^0_{(n)} = \sqrt{\frac{1+\alpha_n}{2}}$, $z^0_{(n)} = \sqrt{\frac{1-\alpha_n}{2}}$, $0 \leqq \alpha_n \leqq 1$. And by (b) no permutation of the α_n, $n = 1, 2, \ldots$ matters.

(d) We have obtained the normal form

$$(\S) \qquad g^0_{(n)} = \sqrt{\frac{1+\alpha_n}{2}}\,\varphi_{(n,1),1} \otimes \varphi_{(n,2),1} + \sqrt{\frac{1-\alpha_n}{2}}\,\varphi_{(n,1),2} \otimes \varphi_{(n,2),2}.$$

For two such $g^0_{(n)}$, $\overline{g^0_{(n)}}$ with α_n, $\overline{\alpha_n}$ respectively $(0 \leqq \alpha_n, \overline{\alpha_n} \leqq 1)$, we see that $\left(g^0_{(n)}, \overline{g^0_{(n)}}\right) = \frac{1}{2}\left(\sqrt{1+\alpha_n}\sqrt{1+\overline{\alpha_n}} + \sqrt{1-\alpha_n}\sqrt{1-\overline{\alpha_n}}\right)$. So they determine the same equivalence-class $\mathfrak{D}$ and the same $\Pi\otimes^{\mathfrak{D}}_{n=1,2,\ldots}(\mathfrak{H}_{(n,1)} \otimes \mathfrak{H}_{(n,2)})$, if

$$\Sigma_{n=1,2,\ldots}\left|\tfrac{1}{2}\left(\sqrt{1+\alpha_n}\sqrt{1+\overline{\alpha_n}} + \sqrt{1-\alpha_n}\sqrt{1-\overline{\alpha_n}}\right) - 1\right|$$

converges (by Definition 2.3.2). This has the majorant

$$(\S\S) \qquad \Sigma_{n=1,2,\ldots}\left(\sqrt{\frac{1-\alpha_n}{1+\alpha_n}} - \sqrt{\frac{1-\overline{\alpha_n}}{1+\overline{\alpha_n}}}\right)^2 \text{ [49]},$$

therefore the convergence of (§§) suffices.

---

[48]) Given any H, H^*H is Hermitean, so a unitary W exists, so that W^*H^*HW is diagonal, say with the diagonal d_1, d_2. As it is semi-definite, so $d_1, d_2 \geqq 0$. Let K be the diagonal matrix with the diagonal $\sqrt{d_1}$, $\sqrt{d_2}$; then $K^*K = K^2 = W^*H^*HW$, so always $\|Kf\| = \|HWf\|$. Therefore a unitary V with $K = VHW$ exists.

[49]) Clearly

$$\left(\tfrac{1}{2}\left(\sqrt{1+\alpha_n}\sqrt{1+\overline{\alpha_n}} + \sqrt{1-\alpha_n}\sqrt{1-\overline{\alpha_n}}\right)\right)^2 + \\ + \left(\tfrac{1}{2}\left(\sqrt{1-\alpha_n}\sqrt{1+\overline{\alpha_n}} - \sqrt{1+\alpha_n}\sqrt{1-\overline{\alpha_n}}\right)\right)^2 = 1,$$

so $\frac{1}{2}\left(\sqrt{1+\alpha_n}\sqrt{1+\overline{\alpha_n}} + \sqrt{1-\alpha_n}\sqrt{1-\overline{\alpha_n}}\right) \leqq 1$, and consecutively

$$0 \leqq 1 - \tfrac{1}{2}\left(\sqrt{1+\alpha_n}\sqrt{1+\overline{\alpha_n}} + \sqrt{1-\alpha_n}\sqrt{1-\overline{\alpha_n}}\right) \leqq \\ \leqq 1 - \left(\tfrac{1}{2}\left(\sqrt{1+\alpha_n}\sqrt{1+\overline{\alpha_n}} + \sqrt{1-\alpha_n}\sqrt{1-\overline{\alpha_n}}\right)\right)^2 = \\ = \left(\tfrac{1}{2}\left(\sqrt{1-\alpha_n}\sqrt{1+\overline{\alpha_n}} - \sqrt{1+\alpha_n}\sqrt{1-\overline{\alpha_n}}\right)\right)^2 = \\ = \frac{(1+\alpha_n)(1+\overline{\alpha_n})}{4}\left(\sqrt{\frac{1-\alpha_n}{1+\alpha_n}} - \sqrt{\frac{1-\overline{\alpha_n}}{1+\overline{\alpha_n}}}\right)^2 \leqq \left(\sqrt{\frac{1-\alpha_n}{1+\alpha_n}} - \sqrt{\frac{1-\overline{\alpha_n}}{1+\overline{\alpha_n}}}\right)^2.$$

We repeat:

LEMMA 7.3.1. From the point of view of isomorphism of the parts of $\mathcal{C}^{\#'}$ in the various $\Pi\otimes^{\mathfrak{D}}_{n=1,2,\ldots}(\mathfrak{H}_{(n,1)}\otimes\mathfrak{H}_{(n,2)})$ it suffices to consider the equivalence classes $\mathfrak{D}$ of sequences $(g^0_{(n)};\ n=1,2,\ldots)$ of the form (§), with $0\leqq\alpha_n\leqq 1$, $n=1,2,\ldots$. Any permutation of these α_n, $n=1,2,\ldots$ or any replacement by other $\overline{\alpha_n}$, $n=1,2,\ldots$ for which (§§) converges, is immaterial.

7.4. We will now discuss two extreme special cases. Consider first the case $\alpha_1=\alpha_2=\ldots=1$.

LEMMA 7.4.1. (I) The sequence α_n, $n=1,2,\ldots$, which characterises a given $\Pi\otimes^{\mathfrak{D}}_{n=1,2,\ldots}(\mathfrak{H}_{(n,1)}\otimes\mathfrak{H}_{(n,2)})$ can be chosen as $\alpha_1=\alpha_2=\ldots=1$ if and only if $\mathfrak{D}$ is the equivalence class of a sequence $(g^0_{(n)};\ n=1,2,\ldots)$ with $g^0_{(n)}=f^0_{(n,1)}\otimes f^0_{(n,2)}$ $(f_{(n,\tau)}\in\mathfrak{H}_{(n,\tau)})$.

(II) In any such $\Pi\otimes^{\mathfrak{D}}_{n=1,2,\ldots}(\mathfrak{H}_{(n,1)}\otimes\mathfrak{H}_{(n,2)})$ the ring $\mathcal{C}^{\#'}$ is a factor of class (I_∞). (Cf. § 7.2).

Proof: Ad (I): Necessity: If $\alpha_1=\alpha_2=\ldots=1$, then (§) in § 7.3 gives

$$g^0_{(n)}=\varphi_{(n,1)}\otimes\varphi_{(n,2)}.$$

Sufficiency: We have $g^0_{(n)}=f^0_{(n,1)}\otimes f^0_{(n,2)}$. As $\sum_{n=1,2,\ldots}\big|\|g^0_{(n)}\|-1\big|$ converges, so $g^0_{(n)}=0$, $\big|\|g^0_{(n)}\|-1\big|=1$ can occur for a finite number of $n=1,2,\ldots$ only. With these exceptions $g^0_{(n)}\neq 0$, $f^0_{(n,1)}, f^0_{(n,2)}\neq 0$. For the exceptional n's we may change $f^0_{(n,1)}, f^0_{(n,2)}$ (use Lemma 3.3.5), so as to have always $f^0_{(n,1)}, f^0_{(n,2)}\neq 0$. Now Lemma 3.3.7 permits us to replace these $g^0_{(n)}=f^0_{(n,1)}\otimes f^0_{(n,2)}$ by

$$\frac{1}{\|g^0_{(n)}\|}g^0_{(n)}=\frac{1}{\|f^0_{(n,1)}\|\cdot\|f^0_{(n,2)}\|}(f^0_{(n,1)}\otimes f^0_{(n,2)})=\frac{1}{\|f^0_{(n,1)}\|}f^0_{(n,1)}\otimes\frac{1}{\|f^0_{(n,2)}\|}f^0_{(n,2)}.$$

In other words: We may assume $\|f^0_{(n,1)}\|=\|f^0_{(n,2)}\|=1$.

We now could choose the $\varphi_{(n,\tau),\varkappa}$ with

$$\varphi_{(n,1),1}=f^0_{(n,1)},\quad \varphi_{(n,2),1}=f^0_{(n,2)}.$$

Then clearly $\alpha_n=1$, that is $\alpha_1=\alpha_2=\ldots=1$.

Character of $\mathcal{C}^{\#'}$: Assume $\alpha_1=\alpha_2=\ldots=1$, that is $g^0_{(n)}=f^0_{(n,1)}\otimes f^0_{(n,2)}$, $\|f^0_{(n,1)}\|=\|f^0_{(n,2)}\|=1$ (cf. above). Apply the associative law (as described in 7.1) in the formulation of Theorem VI.

$(f^0_{(n,\tau)};\ n=1,2,\ldots,\tau=1,2)$ is clearly a C_0-sequence for $\Pi\otimes_{\substack{n=1,2\\ \tau=1,2,\ldots}}\mathfrak{H}_{(n,\tau)}$; let $\mathfrak{C}$ be its equivalence-class. Then (the $\mathfrak{C}_n$ being inessential, as all $I_n=\mathfrak{S}((n,1),(n,2))$ are finite) $\mathfrak{C}_0=\mathfrak{D}$.

So $\Pi\otimes^{\mathfrak{D}}_{n=1,2,\ldots}(\mathfrak{H}_{(n,1)}\otimes\mathfrak{H}_{(n,2)})$ is isomorphic to $\Pi\otimes^{\mathfrak{C}}_{\substack{n=1,2,\ldots\\ \tau=1,2}}\mathfrak{H}_{(n,\tau)}$. $\mathcal{C}^{\#\prime}$ being the ring generated by all $\overline{\overline{\mathcal{B}_{(n,1)}}}$, $n=1,2,\ldots$, considered in $\Pi\otimes^{\mathfrak{D}}_{n=1,2,\ldots}(\mathfrak{H}_{(n,1)}\otimes\mathfrak{H}_{(n,2)})$, this isomorphism carries it into the ring generated by all $\overline{\overline{\mathcal{B}_{(n,1)}}}$, $n=1,2,\ldots$, in $\Pi\otimes^{\mathfrak{C}}_{\substack{n=1,2,\ldots\\ \tau=1,2}}\mathfrak{H}_{(n,\tau)}$ — which is the $\mathcal{C}^{\#}$ of 7.2.

So our $\mathcal{C}^{\#\prime}$ is isomorphic to the $\mathcal{C}^{\#}$ of 7.2, and therefore it is a factor of class (I_∞).

7.5. Consider next thecase $\alpha_1=\alpha_2=\ldots=0$.

LEMMA 7.5.1. In any $\Pi\otimes^{\mathfrak{D}}_{n=1,2,\ldots}(\mathfrak{H}_{(n,1)}\otimes\mathfrak{H}_{(n,2)})$ with $\alpha_1=\alpha_2=\ldots=0$, the ring $\mathcal{C}^{\#\prime}$ is a factor of class (II_1). (Cf. (7), p. 172.)

Proof: We proceed in the inverse direction: We will analyse one of the examples of factors of class (II_1), given in (7), pp. 192—209, and show that it is isomorphic to $\mathcal{C}^{\#\prime}$ in the said $\Pi\otimes^{\mathfrak{D}}_{n=1,2,\ldots}(\mathfrak{H}_{(n,1)}\otimes\mathfrak{H}_{(n,2)})$.

(I) Let S be the set of all (enumerably infinite) sequences $x=(\alpha_m;\ m=1,2,\ldots)$ where each $\alpha_m=0,1$. Let $\mathfrak{G}$ be the set of those $x=(\alpha_m,\ m=1,2,\ldots)\in S$ for which $\alpha_m\neq 0$ occurs for a finite number of m's only.

Define in S: If $x=(\alpha_m;\ m=1,2,\ldots)$, $y=(\beta_m;\ m=1,2,\ldots)$ then $x\oplus y=(\gamma_m;\ m=1,2,\ldots)$ where

$$\gamma_m=\alpha_m+\beta_m \pmod 2 \qquad (\gamma_m=0,1).$$

Under this definition of „composition" $x\oplus y$, S is clearly a (commutative) group, with the „unit" $0=(0;\ m=1,2,\ldots)$, and $\mathfrak{G}$ is an (enumerably infinite) subgroup of S.

For S (but not for $\mathfrak{G}$!) we use the mapping

$$\Xi:\qquad x=(\alpha_m;\ m=1,2,\ldots)\rightarrow\xi(x)=\sum_{m=1}^{\infty}\frac{\alpha_m}{2^m}$$

of S on the numerical interval $0\leqq\xi\leqq 1$. Except for the Ξ image of $\mathfrak{G}$, the set of all dyadically rational numbers, which is a set of Lebesgue-measure 0, this mapping is one-to-one. So the common (exterior) Lebesgue-measure in $0\leqq\xi\leqq 1$ is mapped by the inverse of Ξ on a Lebesgue-measure in S, in the sense of (7), Definition 12.1.2 on p. 192. We will denote it by μ^* (and for „measurable" sets by μ, cf. loc. cit. above).

We will now consider $\mathfrak{G}$ (with the „composition" $a\oplus b$ for $a,b\in\mathfrak{G}$) as the „group" and S (with the „mappings" $x\rightarrow a\oplus x$ for $a\in\mathfrak{G}$, $x\in S$) as the „space" described in (7), pp. 192—195. In the sense of Definition 12.1.5, p. 195, eod. (we replace the notations ab, ax used there by $a \oplus b$, $a \oplus x$) $\mathfrak{G}$ is an m-group and ergodic in S. m-group character: Ad (I) loc. cit.: If $a = (\alpha_m;\ m = 1, 2, \ldots)$ and $\alpha_m = 0$ for all $m \geqq m_0$, then the mapping $x \to a \oplus x$ of S corresponds by Ξ to a mapping of $0 \leqq \xi \leqq 1$ of this nature: Every interval $\frac{k}{2^{m_0}} < \xi < \frac{k+1}{2^{m_0}}$ $(k = 0, 1, \ldots, 2^{m_0} - 1)$ undergoes a translation as a whole. So the common Lebesgue-measure is left invariant in $0 \leqq \xi \leqq 1$, and corresponding μ^* in S. Ad (II): Obvious. Ad (III): If $a \neq 0$, then clearly every $a \oplus x \neq x$. The ergodicity will be established in (IV) below.

(II) Form for these S, $\mathfrak{G}$ the spaces $\mathfrak{H}_S$ and $\mathfrak{H}_{\mathfrak{G}S}$ of all (complex-valued) functions $f(x)$ resp. $F(x, a)$ $(x \in S,\ a \in \mathfrak{G})$ which are μ-measurable in x for each $a \in \mathfrak{G}$, and with a finite $\int_S |f(x)|^2\, dx$ resp. $\Sigma_{a \in \mathfrak{G}} \int_S |F(x, a)|^2 dx$ $\left(\int_S \ldots dx \text{ in the } \mu\text{-sense}\right)$, so that

$$(f, g) = \int_S f(x)\overline{g(x)}\, dx \text{ resp. } (F, G) = \Sigma_{a \in \mathfrak{G}} \int_S F(x, a)\,\overline{G(x, a)}\, dx.$$

(Cf. (7), p. 194.) Form the bounded operators

$$\overline{U_{a_0}} F(x, a) = F(x \oplus a_0, a \oplus a_0) \qquad (a_0 \in \mathfrak{G}),$$
$$\overline{\mathrm{L}_{\varphi(x)}}\, F(x, a) = \varphi(x) F(x, a) \ \ (\varphi(x) \text{ bounded and } \mu\text{-measurable})$$

(pp. 198—199, loc. cit.) and the ring $\mathcal{M}$ which they generate (p. 200, loc. cit.).

In forming $\mathcal{M}$ we need not to use all these $\varphi(x)$ and $a \in \mathfrak{G}$. We may obviously restrict ourselves, in forming $\mathcal{M}$, to (bounded) Baire-functions $\varphi(x)$, then by continuity to continuous functions $\varphi(x)$ of $\xi(x)$, and then again by continuity to functions $\varphi(x)$ of this form:

$$\varphi(x) = c_k \text{ for } \frac{k}{2^{m_0}} \leqq \xi(x) < \frac{k+1}{2^{m_0}} \quad (k = 0, 1, \ldots, 2^{m_0} - 1)$$

for any $m_0 = 1, 2, \ldots$. But $\varphi(x) = \Sigma_{k=0}^{2^{m_0}-1} c_k \varphi_k^{m_0}(x)$ where $\varphi_k^{m_0}(x) \begin{cases} = 1 \text{ if } \frac{k}{2^{m_0}} \leqq \xi < \frac{k}{2^{m_0}} \\ = 0 \end{cases}$ otherwise so we may even restrict ourselves to the $\varphi_k^{m_0}(x)$.

If $\frac{k}{2^{m_0}} = \Sigma_{m=1}^{m_0} \frac{\gamma_m}{2^m}$, then $\varphi_k^{m_0}(x) = 1$ if in $x = (\beta_m;\ m = 1, 2, \ldots)$ $\beta_m = \gamma_m$ for all $m \leqq m_0$, otherwise it is $= 0$. Put

$$\psi_l(x) = (-1)^{\beta_l}, \text{ where } x = (\beta_m;\ m = 1, 2, \ldots),$$

then we have $\varphi_k^{m_0}(x) = \Pi_{l=1}^{m_0} \frac{1}{2}(1 + (-1)^{\gamma_l} \psi_l(x))$. Thus the further restriction to the $\psi_l(x)$, $l = 1, 2, \ldots$, is legitimate.

Put

$$a_l = (\delta_{lm};\ m = 1, 2, \ldots) \quad \left(\delta_{lm} \begin{cases} = 1 \text{ for } l = m \\ = 0 \text{ for } l \neq m \end{cases}\right),$$

clearly $a_l \in \mathfrak{G}$, and if $a = (\alpha_m;\ m = 1, 2, \ldots) \in \mathfrak{G}$, then $a = a_{l_1} \oplus \ldots \oplus a_{l_\nu}$ where $l_1, \ldots l_\nu$ are those m for which $\alpha_m \neq 0$. So it suffices to use the $\overline{U}_{a_l}$, $l = 1, 2, \ldots$, only (instead of all $\overline{U}_{a_0}$, $a_0 \in \mathfrak{G}$).

So we have proved:

$$\mathcal{M} = \mathcal{R}(\overline{U}_{a_l}, \mathbf{L}_{\psi_l(x)};\ l = 1, 2, \ldots).$$

(III) For each $a_0 = (\alpha_m;\ m = 1, 2, \ldots) \in \mathfrak{G}$ form

$$\omega_{a_0}(x) = \psi_{l_1}(x) \cdots \psi_{l_\nu}(x)$$

where the $l_1, \ldots, l_\nu$ are those m for which $\alpha_m \neq 0$. Define, if $b_0 \in \mathfrak{G}$ too,

$$F_{a_0 b_0}(a, x) \begin{cases} = \omega_{a_0}(x) & \text{if } a = b_0, \\ = 0 & \text{if } a \neq b_0. \end{cases}$$

One verifies immediately, that the $F_{a_0 b_0}(a, x)$, $a_0, b_0 \in \mathfrak{G}$, are mutually orthogonal, and as $\|F_{a_0 b_0}(a, x)\| = \|\omega_{a_0}(x)\| = 1$, they are normalised, too.

If a $G(a, x) \in \mathfrak{H}_{\mathfrak{G}S}$ is orthogonal to all $F_{a_0 b_0}(a, x)$, $a_0, b_0 \in \mathfrak{G}$, then we have $\int_S G(b_0, x)\overline{f(x)}\,dx = 0$ for all $f(x) = \omega_{a_0}(x)$. The considerations of (II) extend this to all bounded, μ-measurable $f(x)$. Put

$$f(x) = \operatorname{sgn} G(b_0, x) \begin{cases} = \dfrac{G(b_0, x)}{|G(b_0, x)|} & \text{if } G(b_0, x) \neq 0, \\ = 0 & \text{if } G(b_0, x) = 0, \end{cases}$$

then $\int_S |G(b_0, x)|\,dx = 0$ obtains. So $G(b_0, x) = 0$, except for an x-set of μ-measure 0, for each $b_0 \in \mathfrak{G}$. So the normalised, orthogonal set $F_{a_0 b_0}(a, x)$, $a_0, b_0 \in \mathfrak{G}$, is complete, too.

If $a_0 = (\alpha_m;\ m = 1, 2, \ldots)$, then one verifies easily, that

$$\overline{U}_{a_l} F_{a_0 b_0}(a, x) = F_{a_0 b_0}(a \oplus a_l, x \oplus a_l) = (-1)^{\alpha_l} F_{a_0 (b_0 \oplus a_l)}(a, x),$$
$$\mathbf{L}_{\psi_l(x)} F_{a_0 b_0}(a, x) = \psi_l(x) F_{a_0 b_0}(a, x) = F_{(a_0 \oplus a_l) b_0}(a, x).$$

(IV) Literally the same argument as above shows, that the $\omega_{a_0}(x)$, $a_0 \in \mathfrak{G}$ form a normalised, orthogonal, and complete set of functions in the space $\mathfrak{H}_S$ and that for the operator

$$U_{c_0} f(x) = f(x \oplus c_0)$$

we have $(a_0 = (\alpha_m; m = 1, 2, \ldots))$

$$U_{a_l} \omega_{a_0}(x) = \omega_{a_0}(x \oplus a_l) = (-1)^{\alpha_l} \omega_{a_0}(x).$$

If the μ-measurable set $T \subset S$ differs for each $c_0 \in \mathfrak{G}$ from its image by $x \to x \oplus c_0$ by a set of μ-measure 0 only (depending on c_0), then form $f_T(x) \begin{cases} = 1 & \text{for } x \in T \\ = 0 & \text{for } x \in T \end{cases}$. Then $f_T \in \mathfrak{H}_S$ and $U_{c_0} f_T = f_T$ in $\mathfrak{H}_S$. Now write (in $\mathfrak{H}_S$)

$$f_T = \Sigma_{a_0 \in \mathfrak{G}} u_{a_0} \omega_{a_0} \qquad \text{(the } u_{a_0} \text{ are complex numbers),}$$

so

$$U_{a_l} f_T = \Sigma_{a_0 \in \mathfrak{G}} u_{a_0} (-1)^{\alpha_l} \omega_{a_0}.$$

So $(-1)^{\alpha_l} u_{a_0} = u_{a_0}$ and therefore $u_{a_0} = 0$ if ever $\alpha_l \neq 0$ occurs for $a_0 = (\alpha_m; m = 1, 2, \ldots)$, that is if $a_0 \neq 0$. So $f_T = u_0 \omega_0$, $f_T(x) = u_0 \omega_0(x) = u_0$, and thus $u_0 = 0$ or 1, $\mu(T) = 0$ or $\mu(S - T) = 0$. Thus $\mathfrak{G}$ is ergodic in S.

(V) Let us now return to (III). As $a_0 = (\alpha_m; m = 1, 2, \ldots)$, $b_0 = (\beta_m; m = 1, 2, \ldots)$ we prefer to denote the $F_{a_0 b_0}$ by $F_{\alpha_1 \beta_1 \alpha_2 \beta_2 \ldots}$. So we have:

The $F_{\alpha_1 \beta_1 \alpha_2 \beta_2 \ldots}$ ($\alpha_1, \beta_1, \alpha_2, \beta_2, \ldots = 0, 1$, but only a finite number of them is $\neq 0$) form a complete, normalised, orthogonal set in $\mathfrak{H}_{\mathfrak{G} S}$. Besides

$$U_{\alpha_l} F_{\alpha_1 \beta_1 \alpha_2 \beta_2 \ldots \alpha_l \beta_l \ldots} = (-1)^{\alpha_l} F_{\alpha_1 \beta_1 \alpha_2 \beta_2 \ldots \alpha_l (1-\beta_l) \ldots},$$

$$L_{\psi_l(x)} F_{\alpha_1 \beta_1 \alpha_2 \beta_2 \ldots \alpha_l \beta_l \ldots} = F_{\alpha_1 \beta_1 \alpha_2 \beta_2 \ldots (1-\alpha_l) \beta_l \ldots}.$$

(We write $1 - \alpha$ in all places, where we should write $\alpha + 1$, since these numbers are to be reduced mod 2.)

(VI) Consider next $\Pi \otimes^{\mathfrak{D}}_{n=1,2,\ldots} (\mathfrak{H}_{(n,1)} \otimes \mathfrak{H}_{(n,2)})$, where $\mathfrak{D}$ is the equivalence class of $g_1^0, g_2^0, \ldots$ with

$$g_n^0 = \frac{1}{\sqrt{2}} \varphi_{(n,1),1} \otimes \varphi_{(n,2),1} + \frac{1}{\sqrt{2}} \varphi_{(n,1),2} \otimes \varphi_{(n,2),2}.$$

Apply Lemma 4.1.4 and Theorem V: For every $n = 1, 2, \ldots$ the space $\mathfrak{H}_{(n,1)} \otimes \mathfrak{H}_{(n,2)}$ has 4 dimensions, so let every K_n be the 4-element set of all pairs (α, β), $\alpha, \beta = 0, 1$, and let the pair $(0, 0)$ play the role assigned to 0 loc. cit. above.

Put

$$\varphi_{n,(0,0)} = \frac{1}{\sqrt{2}}(\varphi_{(n,1),1} \otimes \varphi_{(n,2),1} + \varphi_{(n,1),2} \otimes \varphi_{(n,2),2}),$$

$$\varphi_{n,(0,1)} = \frac{1}{\sqrt{2}}(\varphi_{(n,1),1} \otimes \varphi_{(n,2),1} - \varphi_{(n,1),2} \otimes \varphi_{(n,2),2}),$$

$$\varphi_{n,(1,0)} = \frac{1}{\sqrt{2}}(\varphi_{(n,1),2} \otimes \varphi_{(n,2),1} + \varphi_{(n,1),1} \otimes \varphi_{(n,2),2}),$$

$$\varphi_{n,(1,1)} = \frac{1}{\sqrt{2}}(\varphi_{(n,1),2} \otimes \varphi_{(n,2),1} - \varphi_{(n,1),1} \otimes \varphi_{(n,2),2});$$

one verifies easily, that this is a complete, normalised, orthogonal set in $\mathfrak{H}_{(n,1)} \otimes \mathfrak{H}_{(n,2)}$, and has $\varphi_{n,(0,0)} = g_n^0$, as required. So the $\Pi\otimes_{n=1,2,\ldots} \varphi_{n,\beta(n)}$ $\big(\beta(n) = (\alpha_n, \beta_n)$, $\alpha_n, \beta_n = 0, 1$ for every $n = 1, 2, \ldots$ and $\beta(n) = (0, 0)$, that is $\alpha_n = \beta_n = 0$, except for a finite number of n's$\big)$ form a complete, normalised, orthogonal set in $\Pi\otimes^{\mathfrak{D}}_{n=1,2,\ldots} (\mathfrak{H}_{(n,1)} \otimes \mathfrak{H}_{(n,2)})$. We write

$$\Phi_{\alpha_1\beta_1\alpha_2\beta_2\ldots} = \Pi\otimes_{n=1,2,\ldots} \varphi_{n,\beta(n)} = \Pi\otimes_{n=1,2,\ldots} \varphi_{n,(\alpha_n,\beta_n)}.$$

Consider now the two operators U^n, L^n in $\mathfrak{H}_{(n,1)}$, defined by

$$U^n \varphi_{(n,1),1} = \varphi_{(n,1),1}, \qquad \mathrm{L}^n \varphi_{(n,1),1} = \varphi_{(n,1),2},$$
$$U^n \varphi_{(n,1),2} = -\varphi_{(n,1),2}, \qquad \mathrm{L}^n \varphi_{(n,1),2} = \varphi_{(n,1),1}.$$

One verifies easily, that the operators $\overline{U^n}$, $\overline{\mathrm{L}^n}$ in $\mathfrak{H}_{(n,1)} \otimes \mathfrak{H}_{(n,2)}$ map $\varphi_{n,(0,0)}, \varphi_{n,(0,1)}, \varphi_{n,(1,0)}, \varphi_{n,(1,1)}$ on $\varphi_{n,(0,1)}, \varphi_{n,(0,0)}, -\varphi_{n,(1,1)}, -\varphi_{n,(1,0)}$ resp. $\varphi_{n,(1,0)}, \varphi_{n,(1,1)}, \varphi_{n,(0,0)}, \varphi_{n,(0,1)}$, that is, $\varphi_{n,(\alpha,\beta)}$ on $(-1)^\alpha \varphi_{n,(\alpha,1-\beta)}$ resp. $\varphi_{n,(1-\alpha,\beta)}$. Therefore we have for $\overline{\overline{U^n}}$, $\overline{\overline{\mathrm{L}^n}}$ in $\Pi\otimes^{\mathfrak{D}}_{n=1,2,\ldots} (\mathfrak{H}_{(n,1)} \otimes \mathfrak{H}_{(n,2)})$

$$\overline{\overline{U^n}} \Phi_{\alpha_1\beta_1\ldots\alpha_n\beta_n\ldots} = (-1)^{\alpha_n} \varphi_{\alpha_1\beta_1\ldots\alpha_n(1-\beta_n)\ldots},$$
$$\overline{\overline{\mathrm{L}^n}} \Phi_{\alpha_1\beta_1\ldots\alpha_n\beta_n\ldots} = \Phi_{\alpha_1\beta_1\ldots(1-\alpha_n)\beta_n\ldots}.$$

Observe finally, that in $\mathfrak{H}_{(n,1)}$ the 4 operators $1, U^n, \mathrm{L}^n, U^n\mathrm{L}^n$ are linearly independent, and as $\mathfrak{H}_{(n,1)}$ is 2-dimensional, this is the maximum number of linearly independent operators in $\mathfrak{H}_{(n,1)}$. So all of them are linear aggregates of these; therefore $\mathcal{B}_{(n,1)} = \mathcal{R}(U^n, \mathrm{L}^n)$. Consequently

$$\overline{\overline{\mathcal{B}}}_{(n,1)} = \mathcal{R}\big(\overline{\overline{U^n}}, \overline{\overline{\mathrm{L}^n}}\big) \text{ in } \Pi\otimes^{\mathfrak{D}}_{n=1,2,\ldots} (\mathfrak{H}_{(n,1)} \otimes \mathfrak{H}_{(n,2)}).$$

Now we have in $\Pi\otimes^{\mathfrak{D}}_{n=1,2,\ldots} (\mathfrak{H}_{(n,1)} \otimes \mathfrak{H}_{(n,2)})$

$$\mathcal{C}^{\#'} = \mathcal{R}\big(\overline{\overline{\mathcal{B}}}_{(n,1)};\, n = 1, 2, \ldots\big) = \mathcal{R}\big(\overline{\overline{U^n}}, \overline{\overline{\mathrm{L}^n}};\, n = 1, 2, \ldots\big).$$

(VII) Compare $\mathfrak{H}_{\mathfrak{G}S}$ and $\Pi\otimes^{\mathfrak{D}}_{n=1,2,\ldots}(\mathfrak{H}_{(n,1)}\otimes\mathfrak{H}_{(n,2)})$. The $F_{\alpha_1\beta_1\alpha_2\beta_2\ldots}$ resp. the $\Phi_{\alpha_1\beta_1\alpha_2\beta_2}$ (with the same restrictions on the $\alpha_1, \beta_1, \alpha_2, \beta_2, \ldots$) are complete, normalised, orthogonal sets in these two spaces. So an isomorphism of $\mathfrak{H}_{\mathfrak{G}S}$ and $\Pi\otimes^{\mathfrak{D}}_{n=1,2,\ldots}(\mathfrak{H}_{(n,1)}\otimes\mathfrak{H}_{(n,2)})$ exists, which carries each $F_{\alpha_1\beta_1\alpha_2\beta_2\ldots}$ into the corresponding $\Phi_{\alpha_1\beta_1\alpha_2\beta_2\ldots}$. (V) and (VII) establish therefore, that it carries $\overline{U}_{a_l}$ into $\overline{\overline{U}}^l$ and $\overline{\mathrm{L}_{\psi_l(x)}}$ into $\overline{\overline{\mathrm{L}}}^l$. Therefore it carries $\mathcal{M} = \mathcal{R}(\overline{U}_{a_l}, \overline{\mathrm{L}_{\psi_l(x)}};\ l = 1, 2, \ldots)$ (cf. end of (II)) into

$$\mathcal{C}^{\#\prime} = \mathcal{R}(\overline{\overline{U}}^l, \overline{\overline{\mathrm{L}}}^l;\ l = 1, 2, \ldots) \text{ (cf. end of (VI)).}$$

Now $\mathcal{M}$ is a factor of class (II_1) by (7), p. 206. (This is Lemma 13.1.2 loc. cit.: Every one-point set has the common Lebesgue-measure 0.) As $\mathcal{C}^{\#\prime}$ is (spatially) isomorphic to $\mathcal{M}$, the same is true for $\mathcal{C}^{\#\prime}$. This completes the proof.

7.6. Lemmata 7.4.1 and 7.5.1 show, how essentially different the ring

$$\mathcal{C}^{\#\prime} = \mathcal{R}(\overline{\overline{\mathcal{B}}}_{(n,1)};\ n = 1, 2, \ldots)$$

is in the various incomplete direct products

$$\Pi\otimes^{\mathfrak{D}}_{n=1,2,\ldots}(\mathfrak{H}_{(n,1)}\otimes\mathfrak{H}_{(n,2)}).$$

The two cases considered, $\alpha_1 = \alpha_2 = \ldots = 1$ and $\alpha_1 = \alpha_2 = \ldots = 0$ are only two extremes, and Lemma 7.3.1 describes, how other sequences $\alpha_1, \alpha_2, \ldots$ (all $\geqq 0, \leqq 1$) could be used. We wish only to mention the choise $\alpha_n \begin{cases} = 1 \text{ for } n \text{ even} \\ = 0 \text{ for } n \text{ odd} \end{cases}$ in which case a factor of class (II_∞) (cf. (7) p. 172) results, as can be shown without much trouble.

We surmise, that $\mathcal{C}^{\#\prime}$ is a factor in every $\Pi\otimes^{\mathfrak{D}}_{n=1,2,\ldots}\mathfrak{H}_{(n,1)}$ (that is: for every choice of $\alpha_1, \alpha_2, \ldots$). Its class can never be (I_n), $n = 1, 2, \ldots$, because $\mathcal{C}$ has clearly no finite linear bases. We know, that it may be (I_∞), (II_1) and, as observed above, (II_∞) too. Thus the only question which remains is this: Does class (III_∞) occur for any choice of the $\alpha_1, \alpha_2, \ldots$?

The question, whether factors of class (III_∞) exist at all, is as yet unsolved (cf. (7), p. 208), and we do not wish to formulate any hypothesis concerning it. But we are rather inclined to surmise, that the above $\mathcal{C}^{\#\prime}$ will not be a factor of class (III_∞), whatever the choice of the $\alpha_1, \alpha_2, \ldots$.

(Received October 8th, 1937.)